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ON THE LAW OF THE ITERATED LOGARITHM[1]

BY HENRY TEICHER

Rutgers University

A triumvirate of sufficient conditions is given for unbounded, independent random variables to obey the Law of the Iterated Logarithm (LIL). As special cases, new results for weighted i.i.d. random variables and the Hartman–Wintner theorem are obtained. Necessity of finite variance for the two-sided LIL is shown to carry over for a large class of weighted i.i.d. random variables and the Marcinkiewicz–Zygmund example is generalized, simplified and clarified.

1. Introduction. A classical result of Kolmogorov [11] asserts that for bounded independent random variables $\{X_n, n \geqq 1\}$ with $|X_n| = o(s_n/(\log_2 s_n^2)^{\frac{1}{2}})$ where $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $s_n^2 = \sum_1^n \sigma_i^2 \to \infty$, the Law of the Iterated Logarithm holds for $\{X_n\}$, that is,

$$P\left\{\limsup_{n\to\infty} \frac{\sum_1^n X_i}{s_n(2\log_2 s_n^2)^{\frac{1}{2}}} = 1\right\} = 1 .$$

Marcinkiewicz and Zygmund [12] noted that if $\{X_n\}$ are independent with

$$P\left\{X_n = \pm\frac{\exp\{\lambda n/\log n\}}{(\log n)^{\frac{1}{2}}}\right\} = \tfrac{1}{2} ,$$

then the Law of the Iterated Logarithm (LIL) fails provided the positive parameter λ is not too small. In the important case of independent, identically distributed (i.i.d.) random variables, Hartman and Wintner [10] proved that existence of a second moment suffices for the LIL and this has been shown to be necessary by Strassen [14] in the sense that

$$P\left\{\limsup \frac{|\sum_1^n X_j|}{(n\log_2 n)^{\frac{1}{2}}} = \infty\right\} = 1$$

when the variance of X_1 is infinite.

Despite numerous investigations [3], [4], [5], [6], [7], [8], [13], [15] the general case of unbounded independent random variables has proved extremely elusive. Here, the main finding, Theorem 1 and Corollary 1 thereof, furnishes sufficient conditions that are no more stringent than finite variance in the i.i.d. case. Theorem 2 extends Strassen's necessity to weighted i.i.d. random variables while Theorem 3 yields the Hartman–Wintner theorem, its generalization [2] to weighted

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i.i.d. random variables plus new results in the latter case. Moreover, the Marcinkiewicz–Zygmund phenomenon is generalized, simplified and clarified.

The conclusions in the weighted i.i.d. case $\{\sigma_n Y_n,\ n \geqq 1 : Y_n$ i.i.d. random variables with mean 0 and variance $\sigma_Y^2 \leqq \infty$; σ_n nonzero constants with $s_n^2 = \sum_1^n \sigma_i^2 \to \infty\}$ may be summarized as follows: Set $\gamma_n = n\sigma_n^2/s_n^2$.

(i) If (*) $\sigma_n^2 = o(s_n^2/\log_2 s_n^2)$ and (**) $\gamma_n = O((\log_2 s_n^2)^\beta)$, some $\beta < 1$ the LIL holds if $\sigma_Y^2 < \infty$ (Theorem 3). Under (*) and $\gamma_n/\log_2 s_n^2 \uparrow$, conditions (32) and (33) ensure the LIL (Theorem 4). When (*) obtains, $\sigma_Y^2 < \infty$ is necessary for a two-sided LIL, i.e., for both $\{\sigma_n Y_n\}$ and $\{-\sigma_n Y_n\}$ to obey the LIL (Theorem 2).

(ii) If γ_n grows faster than $C \log_2 s_n^2$, something beyond a finite second moment is necessary for the two-sided LIL.

(iii) If $\gamma_n = cn$, for some c in (0, 1) then $s_n^2 = s_1^2/(1-c)^{n-1}$ and conversely. In this case the LIL fails whenever Y is bounded (Theorem 5). A necessary condition for the two-sided LIL is that $E \exp\{c\delta^{-1}Y^2\} < \infty$, $\delta > 8$.

(iv) For Marcinkiewicz–Zygmund growth, i.e. $\sigma_n^2 = (\log n)^{-1} \exp\{2\lambda n/\log n\}$, (*) is violated, $\gamma_n \sim 2\lambda n/\log n \sim \log s_n^2$ and the LIL again fails for bounded random variables provided λ is sufficiently large. Here, in contrast to case (iii), it is the constant $2^{\frac{1}{2}}$ in the LIL that goes astray rather than the order of magnitude (Theorem 5). A necessary condition for the two-sided LIL is $E \exp\{(2\lambda\delta^{-1})^{\frac{1}{2}}|Y|\} < \infty$, $\delta > 8$. Since the Hartman condition (27) is satisfied, the LIL holds when Y is normally distributed. Thus, Marcinkiewicz–Zygmund growth "separates" the normal distribution from that of any bounded random variable and in particular from coin-tossing, i.e., $P\{Y_n = \pm 1\} = \frac{1}{2}$.

It may be noted under (iii) and (iv) that the two-sided LIL fails if either Y is too regular (i.e. bounded) or insufficiently regular (Y in case (iv) or Y^2 in case (iii) lacking an analytic characteristic function).

2. Mainstream. Let $\{X_n, n \geqq 1\}$ denote independent random variables with $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $s_n^2 = \sum_{i=1}^n \sigma_i^2 \to \infty$ and distribution functions $\{F_n, n \geqq 1\}$. In contradistinction to the Law of Large Numbers and Central Limit Theorem, the LIL is not symmetric in $\{X_n\}$, that is, the LIL for $\{X_n\}$ does not entail that for $\{-X_n\}$. When, however, the classical LIL obtains for both $\{X_n\}$ and $\{-X_n\}$, the Borel–Cantelli lemma together with the elementary inequality $|X_n| \leqq |\sum_1^n X_i| + |\sum_1^{n-1} X_i|$ signals the necessity of

$$\sum_{n=1}^\infty P\{|X_n| > \delta s_n(\log_2 s_n^2)^{\frac{1}{2}}\} < \infty\,, \qquad \delta > 2(2)^{\frac{1}{2}}\,. \tag{1}$$

This serves as a point of departure and the first theorem stipulates two conditions which conjoined with (1) for a fixed δ are sufficient for the iterated logarithm law.

It seems worth noting at the outset that

$$\frac{1}{s_n^2} \sum_{j=1}^n \int_{[|x| > \varepsilon s_n(\log_2 s_n^2)^{-\frac{1}{2}}]} x^2\, dF_j(x) = o(1)\,, \qquad \varepsilon > 0 \tag{2}$$

is equivalent to (4) of Theorem 1 which therefore is more stringent than the classical Lindeberg condition for asymptotic normality of $s_n^{-1}\sum_1^n X_i$. Clearly, (4) implies (2) while the converse implication results from the fact that for *arbitrary* $\delta > 0$, the left side of (4) which equals

$$\frac{1}{s_n^2}\left(\sum_{j:s_j\leqq\delta s_n}+\sum_{j:s_j>\delta s_n}\right)\int_{[|x|>\varepsilon s_j(\log_2 s_j^2)^{-\frac12}]}x^2\,dF_j(x)$$
$$\leqq \delta^2+\frac{1}{s_n^2}\sum_j\int_{[|x|>\varepsilon\delta s_n(\log_2 s_n^2)^{-\frac12}]}x^2\,dF_j(x)\to\delta^2$$

under (2).

THEOREM 1. *If* $\{X_n, n \geqq 1\}$ *are independent random variables with* $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $s_n^2 = \sum_1^n \sigma_i^2 \to \infty$, *satisfying for some* $\delta > 0$

$$\sum_{n=1}^\infty P\{|X_n| > \delta s_n(\log_2 s_n^2)^{\frac12}\} < \infty \tag{3}$$

$$\frac{1}{s_n^2}\sum_{j=1}^n\int_{[|x|>\varepsilon s_j(\log_2 s_j^2)^{-\frac12}]}x^2\,dF_j(x) = o(1)\,, \qquad \textit{all } \varepsilon>0 \tag{4}$$

$$\sum_{n=1}^\infty \frac{1}{s_n^2(\log_2 s_n^2)}\int_{[\varepsilon s_n(\log_2 s_n^2)^{-\frac12}<|x|\leqq\delta s_n(\log_2 s_n^2)^{\frac12}]}x^2\,dF_n(x) < \infty\,, \qquad \textit{all } \varepsilon>0\,, \tag{5}$$

then the Law of the Iterated Logarithm holds for $\{X_n\}$, *that is,*

$$P\left\{\limsup_{n\to\infty}\frac{\sum_1^n X_j}{s_n(2\log_2 s_n^2)^{\frac12}} = 1\right\} = 1\,. \tag{6}$$

Since all three requirements involve only $|X_n|$, they imply the LIL for $\{-X_n\}$ as well. The two series appearing in (5) and (3) may be reduced to one via

COROLLARY 1. *If* $\{X_n\}$ *are independent random variables with* $EX_n = 0$, $EX_n^2 = \sigma_n^2$, $s_n^2 = \sum_1^n \sigma_i^2 \to \infty$, *satisfying* (4) *and*

$$\sum_{n=1}^\infty\frac{1}{(s_n^2\log_2 s_n^2)^{\alpha/2}}\int_{[|x|>\varepsilon s_n(\log_2 s_n^2)^{-\frac12}]}|x|^\alpha\,dF_n(x)<\infty\,, \qquad \textit{all } \varepsilon>0 \tag{7}$$

for some α *in* $(0, 2]$ *then the* LIL (6) *holds for* $\{X_n\}$ *and* $\{-X_n\}^2$.

PROOF. Clearly the series of (7) exceeds the series obtained from (7) by restricting the range of integration to $(\varepsilon s_n(\log_2 s_n^2)^{-\frac12}, \delta s_n(\log_2 s_n^2)^{\frac12}]$ and this, in turn, dominates the series of (5) multiplied by $\delta^{\alpha-2}$. Also for $\varepsilon < \delta$, (7) dominates

$$\sum_{n=1}^\infty\frac{1}{(s_n^2\log_2 s_n^2)^{\alpha/2}}\int_{[|x|\geqq\delta s_n(\log_2 s_n^2)^{\frac12}]}|x|^\alpha\,dF_n(x)\geqq\delta^\alpha\sum_{n=1}^\infty P\{|X_n|\geqq\delta s_n(\log_2 s_n^2)^{\frac12}\}$$

for all $\delta > 0$ so that (3) likewise obtains. □

PROOF OF THEOREM. Condition (4) implies

$$\varphi_n(\varepsilon) = \max_{m\geqq n} s_m^{-2}\sum_{j=1}^m\int_{[x^2>\varepsilon^2 s_j^2(\log_2 s_j^2)^{-1}]}x^2\,dF_j(x) = o(1)\,, \qquad \varepsilon>0$$

and hence permits the choice of integers $n_{k+1} > n_k$ such that $\varphi_n(k^{-2}) < k^{-2}$ for $n \geqq n_k$, $k \geqq 1$. Define $\varepsilon_n' = k^{-2}$, $n_k \leqq n < n_{k+1}$, $k \geqq 1$. Then $\varepsilon_n' \downarrow 0$ and for

$n_k \leqq n < n_{k+1}$,

$$(4)' \qquad \frac{1}{s_n^2} \sum_{j=1}^n \int_{[x^2 > \varepsilon_j^2 s_j^2/\log_2 s_j^2]} x^2\, dF_j(x) \leqq \frac{1}{s_n^2} \sum_{j=1}^n \int_{[x^2 > \varepsilon_n^2 s_j^2/\log_2 s_j^2]} x^2\, dF_j(x)$$
$$\leqq \varphi_n(\varepsilon_n) \leqq \varphi_{n_k}(\varepsilon_n) < k^{-2} = o(1)$$

as $n \to \infty$ provided $\varepsilon_n = \varepsilon_n'$. Proceeding in a similar spirit with the tail of the series of (5) there is a sequence $\varepsilon_n'' = o(1)$ such that

$$\sum_{n=1}^\infty (s_n^2 \log_2 s_n^2)^{-1} \int_{[\varepsilon_n s_n (\log_2 s_n^2)^{-\frac{1}{2}} < |x| \leqq \delta s_n (\log_2 s_n^2)^{\frac{1}{2}}]} x^2\, dF_n(x)$$
$$= \sum_{k=1}^\infty \sum_{n > n_k}^{n_{k+1}} \leqq \cdots \leqq \sum_{k=1}^\infty k^{-2} < \infty$$

when $\varepsilon_n = \varepsilon_n''$.

Consequently, $\varepsilon_n = \max(\varepsilon_n', \varepsilon_n'') = o(1)$ and both (4) and (5) hold with ε replaced by ε_j and ε_n respectively.

Define truncation constants $\{b_n, n \geqq 1\}$ by

$$b_n = \frac{\varepsilon_n s_n}{(\log_2 s_n^2)^{\frac{1}{2}}}$$

and, denoting by I_A the indicator function of the set A, let

$$X_n' = X_n I_{[|X_n| \leqq b_n]}, \qquad X_n''' = X_n I_{[|X_n| > \delta s_n (\log_2 s_n^2)^{\frac{1}{2}}]}, \qquad X_n'' = X_n - X_n' - X_n'''$$
$$S_n' = \sum_1^n X_j', \qquad S_n'' = \sum_1^n X_j'', \qquad S_n''' = \sum_1^n X_j'''.$$

Then recalling that $EX_n = 0$,

$$\sigma_n^2 - \sigma_{X_n'}^2 = EX_n^2 I_{[|X_n| > b_n]} + E^2 X_n I_{[|X_n| \leqq b_n]} \leqq 2EX_n^2 I_{[|X_n| > b_n]},$$

whence (4)′ ensures that $\operatorname{Var}(S_n') \sim \operatorname{Var}(\sum_1^n X_j)$ and Kolmogorov's theorem [11] yields

$$(8) \qquad \limsup \frac{\sum_{j=1}^n (X_j' - EX_j')}{s_n (2 \log_2 s_n^2)^{\frac{1}{2}}} = 1$$

with probability one.

Moreover, Kronecker's lemma and the strengthened (or ε_n) version of (5) guarantee that

$$(9) \qquad \frac{1}{s_n (\log_2 s_n^2)^{\frac{1}{2}}} \sum_{j=1}^n (X_j'' - EX_j'')$$

converges almost certainly to zero.

Finally, (3) implies that $S_n''' = O(1)$ with probability one and furthermore

$$(10) \qquad \begin{aligned} |ES_n'''| &\leqq \sum_{j=1}^n \int_{[|x| > \delta s_j (\log_2 s_j^2)^{\frac{1}{2}}]} |x|\, dF_j(x) \\ &\leqq \sum_{j=1}^n \int_{[\delta s_j (\log_2 s_j^2)^{\frac{1}{2}} < |x| \leqq s_n (\log_2 s_n^2)^{-\frac{1}{2}}]} |x|\, dF_j(x) \\ &\quad + \sum_{j=1}^n \int_{[|x| > s_n (\log_2 s_n^2)^{-\frac{1}{2}}]} |x|\, dF_j(x) \\ &= o(s_n (\log_2 s_n^2)^{\frac{1}{2}}) \end{aligned}$$

by virtue of adaptations permitting (3) to be applied to the first sum in the middle of (10) and (4) to the second. The theorem is an immediate consequence of (8), (9), (10) and the remark just prior to (10). □

Corollary 1 appears to be most powerful for α arbitrarily close to but less than two. Curiously, the case $\alpha = 2$, ascribed to Petrov [13] by [3] does not contain the Hartman–Wintner theorem whereas the choices $1 \leqq \alpha < 2$ do.[2] No ground seems to be gained in Theorem 1 by replacing (5) with a condition of the form of (7) but with range of integration as in (5) and α in [1], [2].

COROLLARY 2. *Let* $\{Y_n, n \geqq 1\}$ *be* i.i.d. *random variables with distribution* F, *mean zero, variance* $\sigma_Y^2 < \infty$ *and let* $\{\sigma_n, n \geqq 1\}$ *be (nonzero) constants satisfying* $s_n^2 = \sum_1^n \sigma_i^2 \to \infty$. *If for some* $\delta > 0$

$$\sum_{n=1}^{\infty} P\left\{Y_1^2 > \frac{\delta s_n^2(\log_2 s_n^2)}{\sigma_n^2}\right\} < \infty \tag{11}$$

$$\frac{1}{s_n^2} \sum_{j=1}^n \sigma_j^2 \int_{[y^2 > \varepsilon s_j^2/\sigma_j^2 \log_2 s_j^2]} y^2\, dF(y) = o(1)\,, \qquad \varepsilon > 0 \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{s_n^2(\log_2 s_n^2)} \int_{[\varepsilon s_n^2/\sigma_n^2 \log_2 s_n^2 < y^2 \leqq \delta(s_n^2/\sigma_n^2) \log_2 s_n^2]} y^2\, dF(y) < \infty\,, \qquad \varepsilon > 0 \tag{13}$$

then the LIL *holds for* $\{\sigma_n Y_n, n \geqq 1\}$, *that is,*

$$P\left\{\limsup_{n\to\infty} \frac{\sum_{j=1}^n \sigma_j Y_j}{s_n(2 \log_2 s_n^2)^{\frac{1}{2}}} = \sigma_Y\right\} = 1\,. \tag{14}$$

Since (12) is automatic when the integral therein is $o(1)$, Corollary 1 yields

COROLLARY 3. *Let* $\{Y_n, n \geqq 1\}$ *be* i.i.d. *random variables with finite variance and let* $\{\sigma_n, n \geqq 1\}$ *be nonzero constants satisfying* $s_n^2 \to \infty$ *and*

$$\sigma_n^2 = o\left(\frac{s_n^2}{\log_2 s_n^2}\right).$$

Then if for some α *in* $(0, 2]$

$$\sum_{n=1}^{\infty} \left(\frac{\sigma_n^2}{s_n^2 \log_2 s_n^2}\right)^{\alpha/2} \int_{[y^2 \geqq \varepsilon s_n^2/\sigma_n^2 \log_2 s_n^2]} |y|^\alpha\, dF(y) < \infty\,, \qquad \varepsilon > 0 \tag{15}$$

the LIL *obtains for* $\{\pm\sigma_n(Y_n - EY_1), n \geqq 1\}$, *that is,* (14) *holds with* Y_n *replaced by* $\pm(Y_n - EY_1)$.

3. Weighted i.i.d. case. Let Q denote the class of sequences $\{\sigma_n Y_n, n \geqq 1\}$ *where* $\sigma_n \neq 0$, $n \geqq 1$ $s_n^2 = \sum_{j=1}^n \sigma_j^2 \to \infty$, and where $\{Y_n\}$ are i.i.d. random variables with mean zero, variance $\sigma_Y^2 \leqq \infty$ and distribution F.

As will become apparent, the status of the LIL in Q may be conveniently described in terms of

$$\gamma_n = \frac{n\sigma_n^2}{s_n^2}\,, \qquad n \geqq 1\,. \tag{16}$$

[2] The case $\alpha = 2$ of Corollary 1 in conjunction with Kronecker's lemma reveals that the solitary condition (*) $\sum s_n^{-2} \int_{[x^2 > \varepsilon s_n^2(\log_2 s_n^2)^{-1}]} x^2\, dF_n < \infty$, $\varepsilon > 0$ suffices for the LIL when $EX_n = 0$, $s_n^2 \to \infty$. However, (*), which is clearly unnecessarily stringent, is implied by, hence less stringent than, condition (1) of the theorem of the appendix of [6].

In Q, the necessary condition (1) for the two-sided LIL becomes

$$\sum_{n=1}^{\infty} P\left\{Y_1^2 > \delta \frac{n \log_2 s_n^2}{\gamma_n}\right\} < \infty\,, \qquad \delta > 8\,. \tag{17}$$

Now if

$$\varphi_n \equiv 3\left(\frac{n \log_2 s_n^2}{\gamma_n}\right)^{\frac{1}{2}} = o(n^{\frac{1}{2}}) \tag{18}$$

and $\varphi_n \uparrow$, then letting $\varphi(\cdot)$ denote a monotone extension of φ_n to $(0, \infty)$ its inverse function $\varphi^{-1}(y) = y^2 g(y)$ where $g(y) \to \infty$ as $y \to \infty$ and (17) entails

$$E\varphi^{-1}(|Y_1|) = EY_1^2 g(|Y_1|) < \infty\,. \tag{19}$$

It may be noted that φ_n is automatically increasing if $s_{n-1} \cdot s_{n+1} \leqq s_n^2$, that is, if $\log s_n^2$ is concave.

When γ_n increases more rapidly than $\log_2 s_n^2$, (17) asserts that something beyond a finite second moment is necessary for the two-sided LIL in Q. The next theorem, generalizing that of Strassen [14], stipulates that nothing less than finite variance will do even when $\gamma_n = O(1)$ or more generally when

$$\gamma_n = o\left(\frac{n}{\log_2 s_n^2}\right).$$

THEOREM 2. *Let* $\{\sigma_n, n \geqq 1\}$ *be nonzero constants satisfying* $s_n^2 = \sum_1^n \sigma_i^2 \to \infty$,

$$\sigma_n^2 = o\left(\frac{s_n^2}{\log_2 s_n^2}\right).$$

If $\{Y, Y_n, n \geqq 1\}$ *are* i.i.d. *with* $EY = 0$, $EY^2 = \infty$ *then*

$$P\left\{\limsup \frac{|\sum_{j=1}^n \sigma_j Y_j|}{s_n(\log_2 s_n^2)^{\frac{1}{2}}} = \infty\right\} = 1\,. \tag{20}$$

The proof, which parallels Feller's [7], requires an extension of his key lemma[3].

LEMMA 1. *Let* $\{X_n, n \geqq 1\}$ *be independent, symmetric random variables and* $\{a_n\}$, $\{c_n\}$ *sequences of positive real numbers. Let* $X_n' = X_n I_{[|X_n| \leqq c_n]}$, $S_n' = \sum_1^n X_j'$, $S_n = \sum_1^n X_j$. *Then*

$$P\left\{\limsup \frac{S_n'}{a_n} > 1\right\} = 1$$

implies

$$P\left\{\limsup \frac{S_n'}{a_n} \geqq 1\right\} = 1\,. \tag{21}$$

[3] Professor Harry Kesten has kindly informed the author that a somewhat more general version of Lemma 1 was communicated to him about 5 years ago by his colleague Professor Larry Brown. Essentially the same lemma with $a_n \to \infty$ appears in [5] but (i) existence of the subsequence n_k seems in doubt when S_n/a_n obeys a central limit theorem (ii) it is not clear that the events of (36), (37) are 0–1 events (it may be possible to utilize a result of Lévy but this would certainly require some discussion).

PROOF. By hypothesis, $N_m = \inf\{j \geqq m : S_j' > a_j\}$ ($= \infty$ otherwise) is a finite stopping time for all $m \geqq 1$. If, whenever, $n \geqq m$

$$P\{S_n \geqq S_n', N_m = n\} = P\{S_n \leqq S_n', N_m = n\}, \tag{22}$$

then

$$P\{\bigcup_{j=m}^{\infty} [S_j > a_j]\} \geqq P\{\bigcup_{j=m}^{\infty} [S_j' > a_j, S_j \geqq S_j']\} = P\{S_{N_m} \geqq S'_{N_m}, N_m < \infty\}$$
$$= \sum_{n=m}^{\infty} P\{S_n \geqq S_n', N_m = n\} \geqq \tfrac{1}{2}$$

implying

$$P\left\{\limsup \frac{S_n}{a_n} \geqq 1\right\} \geqq P\{S_n > a_n, \text{ i.o.}\} \geqq \tfrac{1}{2}$$

and hence (21) by the Kolmogorov zero-one law.

To prove (22), set $X_j^* = X_j I_{[|X_j| \leqq c_j]} - X_j I_{[|X_j| > c_j]}$ and note that by symmetry and independence, the joint distributions of $(X_1, \cdots, X_n)$ and $(X_1^*, \cdots, X_n^*)$ are identical for all n. Hence, if $n > m$

$$P\{S_n \geqq S_n', N_m = n\}$$
$$= P\{\textstyle\sum_1^n X_j I_{[|X_j| > c_j]} \geqq 0, \sum_{j=1}^n X_j I_{[|X_j| \leqq c_j]} > a_n, S_h' \leqq a_h, m \leqq h < n\}$$
$$= P\{\textstyle\sum_1^n X_j^* I_{[|X_j^*| > c_j]} \geqq 0, \sum_1^n X_j^* I_{[|X_j^*| \leqq c_j]} > a_n, \sum_1^h X_j^* I_{[|X_j^*| \leqq c_j]} \leqq a_h,$$
$$m \leqq h < n\}$$
$$= P\{S_n \leqq S_n', N_m = n\}.$$

and equality also holds for $n = m$, mutatis mutandis. □

PROOF OF THEOREM 2. Let $\{Y_n^*, n \geqq 1\}$ denote the symmetrized $\{Y_n\}$ and for $c > 0$ set $Y_n' = Y_n^* I_{[|Y_n^*| \leqq c]}$, $X_n' = \sigma_n Y_n'$. If $\sigma_c^2 = EY_n'^2$, then $s_n'^2 = \sum_{j=1}^n \sigma_{X_j'}^2 = \sigma_c^2 s_n^2$ whence $\{X_n'\}$ satisfy Kolmogorov's condition for the LIL, implying

$$P\left\{\limsup \frac{\sum_{j=1}^n \sigma_j Y_j'}{s_n(\log_2 s_n^2)^{\frac{1}{2}}} > \sigma_c\right\} = 1.$$

By the lemma

$$P\left\{\limsup \frac{\sum_{j=1}^n \sigma_j Y_j^*}{s_n(\log_2 s_n^2)^{\frac{1}{2}}} \geqq \sigma_c\right\} = 1 \tag{23}$$

and since $\sigma_c \to \infty$, as $c \to \infty$, (23) holds with σ_c replaced by $+\infty$ which, in turn, yields (20). □

Recall that γ_n was introduced in (16) to designate $n\sigma_n^2/s_n^2$. The case $\gamma_n = O(1)$ was treated in Theorem 1 of [2] whereas, in general $\gamma_1 = 1$, $0 < \gamma_n < n$, $n > 1$. It follows from (16) via $(1 - x)^{-1} > e^x$, $x > 0$ that

$$s_n^2 = s_{n-1}^2\left(1 - \frac{\gamma_n}{n}\right)^{-1}, \qquad \frac{s_n^2}{s_1^2} = \prod_{j=2}^n \left(1 - \frac{\gamma_j}{j}\right)^{-1} > \exp[\textstyle\sum_{j=2}^n \gamma_j/j]. \tag{24}$$

Moreover, $\gamma_n = O(\log s_n^2)$ provided

$$\gamma_n/n \quad \text{is non-increasing, all large} \quad n \tag{25}$$

or more generally

$$\liminf_{n\to\infty} \frac{1}{\gamma_n} \sum_{j=2}^{n} \gamma_j/j > 0 . \tag{26}$$

In fact, under (25), for some integer n_0 and all large n

$$\sum_{j=2}^{n} \gamma_j/j \geqq \sum_{j=n_0}^{n} \gamma_j/j \geqq (n - n_0)\frac{\gamma_n}{n} \geqq \tfrac{1}{2}\gamma_n$$

implying (26) which, in turn, yields, via (24), for some $c > 0$ and all large n

$$\gamma_n \leqq c \sum_{j=2}^{n} \frac{\gamma_j}{j} < c[\log s_n^2 - \log s_1^2] .$$

Of course, if $\gamma_n/n \to 1$ (or merely $\gamma_n/n > \bar{\gamma} > 0$), then (26) holds and $\gamma_n = O(\log s_n^2)$.

If the Hartman condition [9] (for the LIL when Y_n is normal) prevails, that is,

$$\frac{\gamma_n}{n} = \frac{\sigma_n^2}{s_n^2} \leqq 1 - \frac{1}{\delta} , \qquad \text{some } \delta > 1 \tag{27}$$

then s_n^2 grows at most geometrically since via (24) for some $c_1 > 0$,

$$s_n^2 \leqq c_1 \delta^n , \qquad \log_2 s_n^2 \leqq (1 + o(1)) \log n . \tag{28}$$

Moreover, from

$$\left(1 - \frac{\gamma_n}{n}\right)^{-1} = \sum_{i=0}^{k-1} \left(\frac{\gamma_n}{n}\right)^i + \frac{(\gamma_n/n)^k}{1 - \gamma_n/n} \leqq \exp\left\{\sum_{i=1}^{k-1} \left(\frac{\gamma_n}{n}\right)^i + \delta\left(\frac{\gamma_n}{n}\right)^k\right\} \tag{29}$$

there follows, in view of (24),

$$c_1 \exp\left\{\sum_2^n \frac{\gamma_j}{j}\right\} < s_n^2 \leqq c_1 \exp\left\{\min\left[\delta \sum_2^n \frac{\gamma_j}{j} , \sum_2^n \frac{\gamma_j}{j} + \delta \sum_2^n \left(\frac{\gamma_j}{j}\right)^2\right]\right\}$$

and the upper bound collapses to $c_2 \exp\{\sum_2^n \gamma_j/j\}$ when $\sum_{j=2}^{\infty} (\gamma_j/j)^2$ converges.

Recall that (i) the subclass Q of weighted i.i.d. random variables was defined at the beginning of Section 3 and (ii) if γ_n grows faster than $\log_2 s_n^2$, finite variance for Y is insufficient for the LIL. The complementary condition is almost sufficient according to

THEOREM 3. *If* $\{\sigma_n Y_n\} \in Q$ *where* $\sigma_n^2 = o(s_n^2/\log_2 s_n^2)$ *and* $\gamma_n = O((\log_2 s_n^2)^\beta)$, *some* $\beta > 1$, *then the* LIL (14) *holds provided* $\sigma_Y^2 < \infty$.

PROOF. According to Corollary 3 of Theorem 1 it suffices to verify (15) for some α in $(0, 2]$. Define

$$r_n = \frac{\varepsilon s_n^2}{\sigma_n^2 \log_2 s_n^2} = \frac{\varepsilon n}{\gamma_n \log_2 s_n^2} .$$

The hypothesis entails $\gamma_n/n = o(1)$ and *a fortiori* the Hartman condition (27) so that the ensuing Lemma 2 is applicable for $\alpha < 2$. By hypothesis and this lemma, for suitable constants C, C_1, C_ε

$$\sum_{j=1}^{n}\left(\frac{\gamma_j}{j\log_2 s_j^2}\right)^{\alpha/2} \leqq C\sum_{j=1}^{n}\frac{1}{j^{\alpha/2}(\log_2 s_j^2)^{(\alpha/2)(1-\beta)}} \leqq \frac{C_1 n^{(2-\alpha)/2}}{(\log_2 s_n^2)^{(\alpha/2)(1-\beta)}}$$
$$= C_1\left(\frac{n}{\gamma_n \log_2 s_n^2}\right)^{(2-\alpha)/2}\frac{\gamma_n^{(2-\alpha)/2}}{(\log_2 s_n^2)^{\alpha-\alpha\beta/2-1}}$$
$$\leqq C_\varepsilon r_n^{(2-\alpha)/2}(\log_2 s_n^2)^{\beta+1-\alpha}.$$

Hence, for any $\varepsilon > 0$,

$$\sum_{j=1}^{\infty}\left(\frac{\sigma_j^2}{s_j^2\log_2 s_j^2}\right)^{\alpha/2}\int_{[y^2\geqq r_j]}|y|^\alpha\,dF(y)$$
$$= \sum_{j=1}^{\infty}\left(\frac{\gamma_j}{j\log_2 s_j^2}\right)^{\alpha/2}\sum_{n=j}^{\infty}\int_{[r_n\leqq y^2<r_{n+1}]}|y|^\alpha\,dF$$
$$\leqq C_\varepsilon\sum_{n=1}^{\infty} r_n^{(2-\alpha)/2}(\log_2 s_n^2)^{\beta+1-\alpha}\int_{[r_n\leqq y^2<r_{n+1}]}|y|^\alpha\,dF \leqq C_\varepsilon EY^2 < \infty$$

provided $1+\beta \leqq \alpha < 2$, so that (15) obtains. □

COROLLARY 1. *If* $s_n^2 \to \infty$, $\gamma_n = O(1)$ *and* $\{Y, Y_n\}$ *are* i.i.d. *random variables with* $EY = 0$, *the* LIL *holds for* $\{\sigma_n Y_n\}$ *and* $\{-\sigma_n Y_n\}$ *if and only if* $EY^2 < \infty$.

PROOF. The hypothesis implies (27) whence (28) ensures

$$\frac{\sigma_n^2\log_2 s_n^2}{s_n^2} = \frac{\gamma_n\log_2 s_n^2}{n} \leqq \frac{C\log n}{n} = o(1)$$

and the conclusions follow from Theorems 2 and 3. □

It remains to verify the following generalization of Lemma 2 of [2]:

LEMMA 2. *If* $\{\sigma_n\}$ *satisfy* (27), $s_n^2 \to \infty$ *and*

$$\gamma_n = o((\log s_n^2)\log_2 s_n^2) \tag{30}$$

then for any $\mu_1 > 0$ *and* μ_2,

$$\sum_{j=1}^{n}\frac{1}{j^{1-\mu_1}(\log_2 s_j^2)^{\mu_2}} = O\left(\frac{n^{\mu_1}}{(\log_2 s_n^2)^{\mu_2}}\right).$$

PROOF. Under (27), recalling (24) and employing (29) with $k = 1$,

$$\log s_n^2 = \log s_{n-1}^2 - \log\left(1-\frac{\gamma_n}{n}\right) \leqq \left(1+\frac{\delta\gamma_n}{n\log s_{n-1}^2}\right)\log s_{n-1}^2$$

implying

$$\log_2 s_n^2 \leqq \left(1+\frac{\delta\gamma_n}{n(\log s_{n-1}^2)\log_2 s_{n-1}^2}\right)\log_2 s_{n-1}^2.$$

Therefore, noting that these entail $\log_i s_n^2 = (1+o(1))\log_i s_{n-1}^2$, $i = 1, 2$,

$$1-\left(\frac{n-1}{n}\right)^{\mu_1}\left(\frac{\log_2 s_n^2}{\log_2 s_{n-1}^2}\right)^{\mu_2} \geqq 1-\left(1-\frac{1}{n}\right)^{\mu_1}\left(1+\frac{\delta\gamma_n}{n(\log s_{n-1}^2)\log_2 s_{n-1}^2}\right)^{\mu_2}$$
$$\geqq 1-\left[1-\frac{\mu_1}{n}+O\left(\frac{1}{n^2}\right)\right]\left[1+o\left(\frac{1}{n}\right)\right]$$
$$= \frac{\mu_1+o(1)}{n}$$

for $\mu_2 \geqq 0$; the same conclusion is obvious when $\mu_2 < 0$ so that for all μ_2

$$\frac{n^{\mu_1}}{(\log_2 s_n^2)^{\mu_2}} - \frac{(n-1)^{\mu_1}}{(\log_2 s_{n-1}^2)^{\mu_2}} \geqq \frac{\mu_1(1+o(1))}{n^{1-\mu_1}(\log_2 s_n^2)^{\mu_2}}$$

whence for all large n

$$\sum_{j=1}^{n} \frac{1}{j^{1-\mu_1}(\log_2 s_j^2)^{\mu_2}} \leqq \frac{2}{\mu_1} \frac{n^{\mu_1}}{(\log_2 s_n^2)^{\mu_2}}.$$

which is tantamount to the lemma. □

To go beyond the confines of Theorem 3, it seems appropriate to utilize (Corollary 2 of) Theorem 1 as opposed to the weaker Corollary 3. To this end, suppose for all $\varepsilon > 0$, some $\delta > 0$, and all large k that

$$(31) \qquad a_k = a_k(\varepsilon) = \max\left\{n : \frac{\varepsilon n}{\gamma_n \log_2 s_n^2} \leqq \frac{\delta(k+1)\log_2 s_{k+1}^2}{\gamma_{k+1}}\right\} > C_\varepsilon k, \quad C_\varepsilon > 1.$$

For numerous choices of γ_n, (31) is automatic and conclusions for γ_n outside the domain of Theorem 3 flow from

THEOREM 4. *Let* $\{\sigma_n Y_n\} \in Q$ *where* $\sigma_n^2 = o(s_n^2/\log_2 s_n^2)$, $\gamma_n/\log_2 s_n^2 \uparrow$ *and suppose that*

$$(32) \qquad \sum_{n=k}^{a_k} \frac{\gamma_n}{n \log_2 s_n^2} = O\left(h\left(\left(\frac{\delta k \log_2 s_k^2}{\gamma_k}\right)^{\frac{1}{2}}\right)\right), \qquad \varepsilon > 0$$

for some positive δ *and non-decreasing function* h *on* $(0, \infty)$. *If*

$$(33) \qquad EY^2 h(|Y|) < \infty$$

the LIL (14) *obtains.*

PROOF. As noted just prior to Corollary 3, $\sigma_n^2 = o(s_n^2/\log_2 s_n^2)$ guarantees (12) of Corollary 2 of Theorem 1. Moreover, the hypothesis entails the validity of the last inequality of (31) whence (all constants C being strictly positive)

$$\sum_{k}^{a_k} \frac{\gamma_n}{n \log_2 s_n^2} \geqq \frac{C\gamma_k \log(a_k/k)}{\log_2 s_k^2} \geqq \frac{C_1 \gamma_k}{\log_2 s_k^2}, \qquad k \geqq k_0.$$

Hence, setting $r_k = \delta k(\log_2 s_n^2)/\gamma_k$,

$$\begin{aligned} \infty > EY^2 h(|Y|) &\geqq \sum_{k=1}^{\infty} r_k h(r_k^{\frac{1}{2}}) P\{r_k < Y^2 \leqq r_{k+1}\} \\ &\geqq C_2 \sum_{k=1}^{\infty} r_k \sum_{n=k}^{a_k} \frac{\gamma_n}{n \log_2 s_n^2} P\{r_k < Y^2 \leqq r_{k+1}\} \\ &\geqq C_3 \sum_{k=1}^{\infty} kP\{r_k < Y^2 \leqq r_{k+1}\} + O(1) \\ &= C_3 \sum_{n=1}^{\infty} P\left\{Y^2 > \frac{\delta n \log_2 s_n^2}{\gamma_n}\right\} + O(1) \end{aligned}$$

so that (11) likewise holds.

Finally, (13) obtains since for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n \log_2 s_n^2} \int_{[\varepsilon n/\gamma_n \log_2 s_n^2 < y^2 \leqq r_n]} y^2\, dF(y)$$

$$\leqq C_4 + \sum_{k=1}^{\infty} \left(\int_{[r_k < y^2 \leqq r_{k+1}]} y^2\, dF\right) \sum_{n=k}^{a_k} \frac{\gamma_n}{n \log_2 s_n^2}$$

$$\leqq C_\varepsilon \sum_{k=1}^{\infty} h(r_k^{\frac{1}{2}}) \int_{[r_k < y^2 \leqq r_{k+1}]} y^2\, dF \leqq C_\varepsilon E Y^2 h(|Y|) < \infty$$

and the conclusion follows from Corollary 2. □

For example, if $s_n^2 = \exp\{n^\alpha\}$, $0 < \alpha < 1$ so that $\gamma_n \sim \alpha n^\alpha$, then $\sum_k^{a_k} \gamma_n (n \log_2 s_n^2)^{-1} \leqq Ck^\alpha (\log k)^{(3\alpha-1)/(1-\alpha)}$ and $h(k) = k^{2\alpha/(1-\alpha)}(\log k)^{(2\alpha-1)/(1-\alpha)}$. Thus, the LIL holds if $EY^{2/(1-\alpha)}/(\log |Y|)^{(1-2\alpha)/(1-\alpha)} < \infty$ whereas the necessary condition for the 2-sided LIL is tantamount to $EY^{2/(1-\alpha)}/(\log |Y|)^{1/(1-\alpha)} < \infty$. (See note added in proof.)

THEOREM 5. (i) *If* $\sigma_n^2 = c_0^2 b^{2n}$, $b > 1$ *so that* $s_n^2 = \sum_1^n \sigma_i^2$ *grows geometrically,* $\{\sigma_n Y_n, n \geqq 1\}$ *disobeys the* LIL *for all bounded*[4] i.i.d. *random variables* $\{Y_n\}$ *in the sense that the strong law of large numbers*

$$P\left\{\lim_{n\to\infty} \frac{\sum_{j=1}^n \sigma_j Y_j}{c_n s_n} = 0\right\} = 1 \tag{34}$$

holds for any numerical sequence $c_n \to \infty$ *and in particular for* $c_n = (\log_2 s_n^2)^{\frac{1}{2}}$.

(ii) *In the case* $\sigma_n^2 = (\log n)^{-1} \exp\{2\lambda n/\log n\}$, $n > 1$, $\lambda > 0$ *of Marcinkiewicz-Zygmund growth,* $\{\sigma_n Y_n, n > 1\}$ *contravenes the* LIL *for any sequence of bounded* i.i.d. *random variables* $\{Y_n\}$ *in the sense that whenever* $\lambda \sigma_Y^2 > C^2 = \inf\{M^2 \colon P\{|Y_1| \leqq M\} = 1\}$,

$$P\left\{\limsup_{n\to\infty} \frac{\sum_{j=2}^n \sigma_j Y_j}{s_n (\log_2 s_n^2)^{\frac{1}{2}}} < 2^{\frac{1}{2}} \sigma_Y\right\} = 1\,. \tag{35}$$

Moreover, for Y_1 *non-degenerate and* λ *as in* (41)

$$P\left\{\limsup_{n\to\infty} \frac{\sum_{j=2}^n \sigma_j (Y_j - EY_j)}{s_n (\log_2 s_n^2)^{\frac{1}{2}}} > 0\right\} = 1\,. \tag{36}$$

PROOF. Apropos of (i), note that with probability one

$$\frac{1}{s_n} \left|\sum_{j=1}^n \sigma_j Y_j\right| \leqq C \frac{b^n - 1}{b - 1} \left(\frac{b^2 - 1}{b^{2n} - 1}\right)^{\frac{1}{2}} \leqq C \left(\frac{(b+1)}{b-1}\right)^{\frac{1}{2}}, \qquad n \geqq 1$$

so that (34) is immediate.

In case (ii), it is easy to verify and shown in [12] that

$$s_n^2 = \sum_{j=2}^n (\log j)^{-1} \exp\left\{\frac{2\lambda j}{\log j}\right\} \sim \frac{1}{2\lambda} \exp\left\{\frac{2\lambda n}{\log n}\right\}.$$

Now,

$$\sum_{j=2}^n \frac{\exp\{\lambda j/\log j\}}{(\log j)^{\frac{1}{2}}} \leqq (\log n)^{\frac{1}{2}} \sum_{j=2}^n \frac{\exp\{\lambda j/\log j\}}{\log j} \sim \frac{(\log n)^{\frac{1}{2}}}{\lambda} \exp\left\{\frac{\lambda n}{\log n}\right\}$$

---

[4] If $\sigma_n > 0$ and Y_1 is merely bounded above, the same argument devoid of absolute values shows that the LIL fails (where, in case (ii), $\lambda \sigma_Y^2 > \inf\{M^2 \colon P\{Y_1 \leqq M\} = 1\}$).

and for any β in (0, 1), the first sum is at least

$$(\beta \log n)^{\frac{1}{2}} \sum_{j>n^\beta}^{n} \frac{\exp\{\lambda j/\log j\}}{\log j} \sim \frac{(\beta \log n)^{\frac{1}{2}}}{\lambda}\left[\exp\left\{\frac{\lambda n}{\log n}\right\} - \exp\left\{\frac{\lambda n^\beta}{\beta \log n}\right\}\right]$$

and so letting $n \to \infty$ and then $\beta \to 1$

$$\sum_{j=2}^{n} (\log j)^{-\frac{1}{2}} \exp\left\{\frac{\lambda j}{\log j}\right\} \sim \frac{(\log n)^{\frac{1}{2}}}{\lambda} \exp\left\{\frac{\lambda n}{\log n}\right\}. \tag{37}$$

Hence, with probability one

$$\frac{|\sum_{j=2}^{n} \sigma_j Y_j|}{s_n(2 \log_2 s_n^2)^{\frac{1}{2}}} \leqq \frac{C \sum_{j=2}^{n} |\sigma_j|}{2^{\frac{1}{2}} s_n (\log n)^{\frac{1}{2}}(1 + o(1))} \leqq \frac{C(1 + o(1))(2\lambda)^{\frac{1}{2}}}{\lambda(2)^{\frac{1}{2}}} = \frac{C(1 + o(1))}{\lambda^{\frac{1}{2}}},$$

proving (35).

Next, suppose for convenience that $EY_1 = 0$ and note for any ε in (0, 1), setting $M = \max(\lambda^{-1} \log C\varepsilon^{-1}(3/\lambda)^{\frac{1}{2}}, 2|\log p_a|^{-1})$ and $r = [M \log n]$ (where p_a will be defined later) that (37) ensures that

$$|\sum_{2}^{n-r-1} \sigma_j Y_j| \leqq \frac{C(\log n)^{\frac{1}{2}}}{\lambda} \exp\left\{\frac{\lambda(n - M \log n)}{(1 + o(1)) \log n}\right\}$$

implying for all large n that

$$|(s_n^2 \log_2 s_n^2)^{-\frac{1}{2}} \sum_{2}^{n-r-1} \sigma_j Y_j| \leqq C\left(\frac{2}{\lambda}\right)^{\frac{1}{2}} e^{-\lambda M}(1 + o(1)) < \varepsilon. \tag{38}$$

Furthermore, if $m = n - [\gamma \log n]$, it follows from (37) that

$$\begin{aligned} a \sum_{m+1}^{n} \frac{\exp\{\lambda j/\log j\}}{(\log j)^{\frac{1}{2}}} &- C \sum_{n-r}^{m} \frac{\exp\{\lambda j/\log j\}}{(\log j)^{\frac{1}{2}}} \\ &= \frac{(\log n)^{\frac{1}{2}} \exp\{\lambda n/\log n\}}{\lambda} \\ &\quad \times [(a + o(1))(1 - e^{-\lambda\gamma}) - (C + o(1))(e^{-\lambda\gamma} - e^{-\lambda M})] \\ &= \left(\frac{2}{\lambda}\right)^{\frac{1}{2}} (\log n)^{\frac{1}{2}} \frac{\exp\{\lambda n/\log n\}}{(2\lambda)^{\frac{1}{2}}} [a - (a + C)e^{-\lambda\gamma} + Ce^{-M\lambda} + o(1)] \\ &> \delta s_n (\log_2 s_n^2)^{\frac{1}{2}} \end{aligned} \tag{39}$$

provided n is large and

$$a - (a + C)e^{-\lambda\gamma} > \delta(\lambda/2)^{\frac{1}{2}}.$$

The latter inequality and *a fortiori* that connecting the first and last terms of (39) will hold for some $\delta > 0$, e.g. $\delta = \delta_0 = \lambda^{-\frac{1}{2}}[a - (a + C)e^{-\lambda\gamma}]$ provided

$$\lambda > \gamma^{-1} \log(1 + C/a). \tag{40}$$

Consequently, defining the event $B_n = \{\sum_{n-r}^{n} \sigma_j Y_j > \delta_0 s_n(\log_2 s_n^2)^{\frac{1}{2}}\}$, its probability

$$P\{B_n\} \geqq P\{Y_j \geqq a, n - [\gamma \log n] + 1 \leqq j \leqq n\} \geqq p_a^{\gamma \log n} = d_n \quad \text{(say)}$$

where for some a in $(0, C)$

$$p_a = P\{Y_1 \geqq a\}\varepsilon(0, 1) .$$

Define $n_k = [2Mk \log k]$. Then $\{B_{n_k}\}$ are independent events (for all large k) and

$$\begin{aligned}\log \frac{d_{n_{k+1}}}{d_{n_k}} &= \gamma(\log p_a) \log \frac{n_{k+1}}{n_k} \\ &\geqq \gamma(\log p_a) \log \left\{\frac{2(k+1)M \log (k+1) - 1}{2kM \log k}\right\} \\ &= \gamma(\log p_a) \log \left\{\left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k \log k} + O\left(\frac{1}{k^2}\right)\right) - \frac{1}{2kM \log k}\right\} \\ &= -\frac{1}{k} - \frac{1 - 1/2M}{k \log k} + O\left(\frac{1}{k^2}\right)\end{aligned}$$

choosing $\gamma = |\log p_a|^{-1}$. Then

$$d_{n_k} \geqq \frac{\text{const.}}{k(\log k)^{1-1/2M}}, \qquad \sum_k P\{B_{n_k}\} \geqq \sum_k d_{n_k} = \infty$$

and so the Borel–Cantelli lemma and (38) ensure that with probability one

$$\limsup_n \frac{\sum_{j=2}^n \sigma_j Y_j}{s_n(\log_2 s_n^2)^{\frac{1}{2}}} \geqq \limsup_n \frac{\sum_{n-r}^n \sigma_j Y_j}{s_n(\log_2 s_n^2)^{\frac{1}{2}}} + \liminf_n \frac{\sum_2^{n-r-1} \sigma_j Y_j}{s_n(\log_2 s_n^2)^{\frac{1}{2}}} \geqq \delta_0 - \varepsilon$$

provided

(41) $$\lambda > |\log P\{Y_1 - EY_1 \geqq a > 0\}| \log (1 + C/a)$$

with a chosen so that $\lambda < \infty$, recalling (40). Since ε is arbitrary and $\delta_0 > 0$, (36) is established. □

COROLLARY 1. *For $\sigma_n^2 = (\log n)^{-1} \exp\{2\lambda n/\log n\}$ of Marcinkiewicz–Zygmund growth, $\{Y_n\}$ non-degenerate* i.i.d. *random variables whose support is contained in $[-C, C]$ and $\lambda > \max[4C^2/\sigma_Y^2, |\log P\{Y_1 - EY_1 \geqq a > 0\}| \log (1 + C/a)]$*

$$\limsup_n \frac{\sum_{j=2}^n \sigma_j(Y_j - EY_j)}{s_n(2 \log_2 s_n^2)^{\frac{1}{2}}\sigma_Y} = A_\lambda \varepsilon(0, 1)$$

with probability one.

The proof of (36) of Theorem 4 is clearly influenced by that of Lemma 6 of [1]. Although possible, it seems less than equiprobable that A_λ of Corollary 1 depends on F (apart perhaps from C).

Obviously, whenever

$$\sum_{j=1}^n |\sigma_j| = o(s_n(\log_2 s_n^2)^{\frac{1}{2}}) ,$$

$\{\sigma_n Y_n, n \geqq 1\}$ fails to satisfy the LIL for all bounded i.i.d. random variables $\{Y_n\}$. This would appear to be the case with many sequences σ_n^2 between Marcinkiewicz–Zygmund and geometric magnitude.

4. An interesting example. Consider the sequence $\{\sigma_n^2, n \geqq \exp\{e^e\}\}$ defined by

$$s_n^2 = \exp\{\prod_{i=1}^3 (\log_i n)^{\alpha_i}\}, \qquad \alpha_i \geqq 0,\ i = 1, 2, 3 .$$

It will be supposed that $\alpha_1 \geqq 1$; otherwise $s_n^2 = O(n^{(\alpha_1+1)/2})$ and Corollary 1 of Theorem 3 applies. Since

$$\prod_{i=1}^3 \left[\frac{\log_i (n-1)}{\log_i n}\right]^{\alpha_i} - 1 = \frac{-\alpha_1}{n \log n} - \frac{\alpha_2}{n(\log n)\log_2 n} - \frac{\alpha_3}{n(\log n)(\log_2 n)\log_3 n} + O\left(\frac{1}{n^2 \log n}\right)$$

necessarily

$$\frac{\sigma_n^2}{s_n^2} = 1 - \exp\left\{\prod_{i=1}^3 (\log_i n)^{\alpha_i}\left[\prod_{i=1}^3 \left(\frac{\log_i (n-1)}{\log_i n}\right)^{\alpha_i} - 1\right]\right\}$$
$$= \frac{\alpha_1 \prod_{i=2}^3 (\log_i n)^{\alpha_i}}{n(\log n)^{1-\alpha_1}} + \frac{\alpha_2(\log_3 n)^{\alpha_3}}{n \prod_{i=1}^2 (\log_i n)^{1-\alpha_i}} + \frac{\alpha_3}{n \prod_{i=1}^3 (\log_i n)^{1-\alpha_i}} + O\left(\frac{\prod_{i=2}^3 (\log_i n)^{\alpha_i}}{n^2(\log n)^{1-\alpha_1}}\right)$$

so that according to (18)

$$\tfrac{1}{9}\varphi^2(n) = \frac{s_n^2 \log_2 s_n^2}{\sigma_n^2} = \frac{n}{\alpha_1(\log_3 n)^{\alpha_3} \prod_{i=1}^2 (\log_i n)^{\alpha_i - 1}} + \text{lower order terms.}$$

Now if $\psi(n) = \frac{1}{4}\alpha_1 n^2(\log_3 n)^{\alpha_3} \prod_{i=1}^2 (\log_i n)^{\alpha_i-1}$ it is readily verified that $\varphi(\psi(n)) \sim n/2 < n$ whence $\varphi^{-1}(n) \geqq \psi(n)$ for all large n. Consequently, recalling (18), (19), the 2-sided LIL fails in Q whenever

$$EY^2(\log_3 |Y|)^{\alpha_3} \prod_{i=1}^2 (\log_i |Y|)^{\alpha_i-1} = \infty . \tag{42}$$

If $EY^2 < \infty$ this requires either $\alpha_1 > 1$ or $\alpha_1 = 1 < \alpha_2$ or $\alpha_1 = \alpha_2 = 1$, $\alpha_3 > 0$. In the particular case $\alpha_1 = \alpha_2 = 1$, (42) reduces to

$$EY^2(\log_3 |Y|)^{\alpha_3} = \infty . \tag{43}$$

Now $\sigma_n^2 = o(s_n^2/\log_2 s_n^2)$ and

$$\gamma_n = \frac{n\sigma_n^2}{s_n^2} \sim \alpha_1(\log n)^{\alpha_1-1} \prod_{i=2}^3 (\log_i n)^{\alpha_i} \sim \alpha_1 \frac{\log s_n^2}{\log n}$$

so that $\gamma_n = o(n(\log_2 s_n^2)^{-1})$, $\gamma_n(\log_2 s_n^2)^{-1} \uparrow$ and $a_k \sim (\delta/\varepsilon)\alpha_1^2 k(\log_2 k)^2$ implying $\log a_k = \log k + 2\log_3 k + O(1)$.

Thus, if $\alpha_1 = 1, 0 < \alpha_2 < 1$, necessarily $\gamma_n = O((\log_2 s_n^2)^\beta)$, $\beta < 1$ and Theorem 3 guarantees the LIL in Q. (See note added in proof.) Since

$$(\log_2 a_k)^{\alpha_2-1} \prod_{i\neq 2} (\log_i a_k)^{\alpha_i} - (\log_2 k)^{\alpha_2-1} \prod_{i\neq 2} (\log_i k)^{\alpha_i}$$
$$= (\log_2 k)^{\alpha_2-1} \prod_{i\neq 2} (\log_i k)^{\alpha_i}$$
$$\times \left[\left(1 + \frac{2\log_3 k + O(1)}{\log k}\right)^{\alpha_1}\left(1 + o\left(\frac{\log_3 k}{\log k}\right)\right)^{\alpha_3+\alpha_2-1} - 1\right]$$
$$= O((\log k)^{\alpha_1-1}(\log_2 k)^{\alpha_2-1}(\log_3 k)^{\alpha_3+1})$$

it follows that

$$\sum_{n=k}^{a_k} \frac{\gamma_n}{n \log_2 s_n^2} \leqq C \sum_k^{a_k} n^{-1}(\log_3 n)^{\alpha_3} \prod_{i=1}^2 (\log_i n)^{\alpha_i - 1}$$
$$\leqq C'[(\log_2 a_k)^{\alpha_2-1} \prod_{i\neq 2} (\log_i a_k)^{\alpha_i} - (\log_2 k)^{\alpha_2-1} \prod_{i\neq 2} (\log_i k)^{\alpha_i}]$$
$$= O((\log k)^{\alpha_1-1}(\log_2 k)^{\alpha_2-1}(\log_3 k)^{\alpha_3+1}) .$$

Consequently, for $\alpha_1 = 1 \leqq \alpha_2$ or $\alpha_1 > 1$, (32) and therefore the LIL holds provided

$$EY^2(\log |Y|)^{\alpha_1-1}(\log_2 |Y|)^{\alpha_2-1}(\log_3 |Y|)^{\alpha_3+1} < \infty .$$

This should be compared with (42) and (when $\alpha_1 = \alpha_2 = 1$) with (43).

Note added in proof. The writer has recently shown that the necessary condition is also sufficient. Likewise in the example of Section 4 when $\alpha_1 > 1$.

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Statistics Center
Rutgers University
New Brunswick, New Jersey 08903