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ON THE DIVERGENCE OF A CERTAIN RANDOM SERIES

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The divergence of the stochastic series $\sum_{n=1}^{\infty} S_n^+/n$ is investigated, where S_n^+ is the positive part of the sum of the first n components of a sequence of independent, identically distributed random variables $\{X_i, i = 1, 2, \cdots\}$. It is shown that if $P(X_1 = 0) \neq 1$ then either this series or the companion series $\sum_{n=1}^{\infty} S_n^-/n$ diverges almost surely. If $EX_1^2 < \infty$ and $EX_1 = 0$ then necessarily both of these series diverge. The method of proof also yields the almost sure divergence of $\sum_{n=1}^{\infty} S_n/n$. These results are extended to the series $\sum_{n=1}^{\infty} S_n^+/n^{1+p}$ for $0 \leqq p < \frac{1}{2}$ by a slightly different method of proof which does not, however, yield the divergence of $\sum_{n=1}^{\infty} S_n/n^{1+p}$.

1. Introduction. Let $X_1, X_2, \cdots$ be independent, identically distributed random variables with partial sums $S_n = \sum_{i=1}^{n} X_i$. Further, let $S_n^+ = \max(S_n, 0)$ and $S_n^- = \max(-S_n, 0)$. In [4] an adaptive sequential algorithm of interest in Learning Theory was developed. It was claimed that the algorithm converged to a "good" set of strategies with finite loss. In [5] it was shown that the finite loss claim is not valid, but that the algorithm does converge to a "good" set of strategies under rather restrictive assumptions. In investigating this question when these assumptions are not satisfied, it was established in [5] that a "good" set of strategies does not exist if the series

$$Y_+ = \sum_{n=1}^{\infty} \frac{S_n^+}{n}$$

diverges. Let $Y_- = \sum_{n=1}^{\infty} S_n^-/n$. In Section 2 of this note we establish that if $P(X_1 = 0) \neq 1$ then a certain trichotomy always holds which implies that $P(Y_+ = \infty) = 1$ or $P(Y_- = \infty) = 1$. We also establish that if $EX_1^2 < \infty$ and $EX_1 = 0$ then both equalities hold, thus the series of interest diverges almost surely.

These results will be seen to depend on the divergence of the series $\sum_{n=1}^{\infty} n^{-1}$. Because S_n^+ and S_n^- increase with n "on the average" one would expect at least one of the series

$$Y_+(p) = \sum_{n=1}^{\infty} \frac{S_n^+}{n^{1+p}} \qquad \text{and} \qquad Y_-(p) = \sum_{n=1}^{\infty} \frac{S_n^-}{n^{1+p}}$$

to diverge for positive values of p. In Section 3 we show by a slightly different method of proof that if $P(X_1 = 0) \neq 1$ then for $0 \leqq p < \frac{1}{2}$, $P(Y_+(p) = \infty) = 1$ or $P(Y_-(p) = \infty) = 1$. Again, both diverge if $EX_1^2 < \infty$ and $EX_1 = 0$.

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2. Divergence of Y_+. Let $\bar{Y} = \limsup_m \sum_{n=1}^{m} S_n/n$, $\underline{Y} = \liminf_m \sum_{n=1}^{m} S_n/n$ and $Y = \bar{Y} I_{[\bar{Y}=\underline{Y}]}$, where I_A denotes the set indicator function of A.

THEOREM 1. *If $P(X_1 = 0) \neq 1$, then exactly one of the following holds:*

(i) $P(Y = -\infty) = 1$
(ii) $P(Y = \infty) = 1$
(iii) $P(\bar{Y} = \infty, \underline{Y} = -\infty) = 1$.

Thus, in particular, $\sum_{n=1}^{\infty} S_n/n$ diverges almost surely.

PROOF. Write $\sum_{n=1}^{m} S_n/n = U_m + V_m$, where $U_m = (1 + \frac{1}{2} + \cdots + 1/m)X_1$. Note that

$$[\limsup V_m < \infty, \lim U_m = -\infty] \subset [Y = -\infty]$$
$$[\liminf V_m > -\infty, \lim U_m = \infty] \subset [Y = \infty]$$
$$[\limsup V_m = \infty, \liminf V_m = -\infty, \lim U_m = +\infty] \subset [\bar{Y} = \infty]$$
$$[\limsup V_m = \infty, \liminf V_m = -\infty, \lim U_m = -\infty] \subset [\underline{Y} = -\infty].$$

Now, by the Hewitt–Savage Zero-One Law ([2] pages 63, 64), the events on the right-hand sides of these inclusions have probability zero or one. If $P(X_1 \leqq 0) = 1$, clearly (i) holds, while (ii) holds if $P(X_1 \geqq 0) = 1$. If both $P(X_1 > 0)$ and $P(X_1 < 0)$ are positive, then so are $P(\lim U_m = \infty)$ and $P(\lim U_m = -\infty)$. Since the sequences $\{U_m\}$ and $\{V_m\}$ are independent it follows that at least one of the first three events on the left of the above inclusions has positive probability and the third and forth have positive probability together. Consequently, the events on the right can be combined into three mutually exclusive events $[Y = \infty]$, $[Y = -\infty]$ and $[\bar{Y} = \infty, \underline{Y} = -\infty]$ at least one of which (and hence exactly one of which) has probability 1.

This theorem has the following immediate corollary.

COROLLARY 1. *If $P(X_1 = 0) \neq 1$, then either $P(Y_+ = \infty)$ or $P(Y_- = \infty) = 1$ or both.*

Examples are easily constructed in which one or the other or both equalities hold. Clearly, if X_1 is symmetrically distributed, both hold. On the other hand, if EX_1 exists then it is easily argued by means of the strong law of large numbers that exactly one of the equalities holds if $EX_1 \neq 0$. If $EX_1 = 0$, the situation is not clear. However, it is resolved if the variance exists as the next theorem shows.

THEOREM 2. *If $EX_1^2 < \infty$ and $EX_1 = 0$ then $P(Y_+ = \infty) = P(Y_- = \infty) = 1$.*

PROOF. By the Hewitt–Savage Zero-One Law, $[Y_+ = \infty]$ and $[Y_- = \infty]$ have probabilities 0 or 1. Thus, if the conclusion of the theorem is false, necessarily either $P(Y = \infty) = 1$ or $P(Y = -\infty) = 1$. If $T_m = \sum_{n=1}^{m} S_n/n$ and $\{r_m\}$ is any sequence of positive constants it follows that either $\lim_m P(r_m T_m > 0) = 1$ or $\lim_m P(r_m T_m < 0) = 1$.

Now, write $T_m = \sum_{k=1}^m c_{k,m} X_k$, where

$$c_{k,m} = \frac{1}{k} + \cdots + \frac{1}{m},$$

and consider

$$\frac{T_m}{(\sum_{k=1}^m c_{k,m}^2)^{\frac{1}{2}}} = \sum_{k=1}^m Y_{k,m}, \tag{1}$$

where $Y_{k,m} = d_{k,m} X_k$ and $d_{k,m} = c_{k,m}/(\sum_{k=1}^m c_{k,m}^2)^{\frac{1}{2}}$. It will follow that the sequence of random variables (1) is asymptotically normal with mean zero and variance EX_1^2 if the normal convergence criterion

$$\lim_m \sum_{k=1}^m \int_{|y| \geqq \varepsilon} y^2 F_{k,m}(dy) = 0 \tag{2}$$

is satisfied, where $F_{k,m}$ is the cdf of $Y_{k,m}$ ([3] page 103). However, it is easily established that $d_{1,m} > d_{2,m} > \cdots > d_{m,m}$ for all m and $\lim_m d_{1,m} = 0$. Thus, letting F_k denote the cdf of X_k, we have

$$\begin{aligned} \sum_{k=1}^m \int_{|y| \geqq \varepsilon} y^2 F_{k,m}(dy) &= \sum_{k=1}^m d_{k,m}^2 \int_{|d_{k,m} x| \geqq \varepsilon} x^2 F_k(dx) \\ &\leqq \sum_{k=1}^m d_{k,m}^2 \int_{|d_{1,m} x| \geqq \varepsilon} x^2 F_k(dx) \\ &= \int_{|d_{1,m} x| \geqq \varepsilon} x^2 F_1(dx) . \end{aligned}$$

This last expression tends to 0 as $m \to \infty$ and condition (2) is established. It follows that $\lim_m P(r_m T_m > 0) = \lim_m P(r_m T_m < 0) = \frac{1}{2}$ when $r_m = (\sum_{k=1}^m c_{k,m}^2)^{-\frac{1}{2}}$. This establishes the theorem.

REMARKS. Theorem 2 can be generalized to some extent using the general central limit theorem developed, for example, in [3]. All that is required is that there exist a sequence $\{r_m\}$ of positive constants such that $\{r_m T_m\}$ converges in law and the resulting infinitely divisible limit law have support on both sides of the origin. Baxter and Shapiro [1] give necessary and sufficient conditions for the support to be one-sided and by violating one of their conditions or by selecting the support to contain the origin (see [7]), examples can easily be constructed for which the conclusion of Theorem 2 is valid even when $EX_1^2 = \infty$.

However, as we now indicate, this method cannot be used to weaken the moment conditions of Theorem 2 to $E|X_1| < \infty$ and $EX_1 = 0$. Suppose that $P(X_1 = -2) = \frac{1}{2}$ and that X_1 has a density function asymptotically of the form $f(x) \sim x^{-2} \log^{-2} x$ as $x \to \infty$ normalized to guarantee $P(X_1 \geqq 0) = \frac{1}{2}$ and $EX_1^+ = 1$. It follows that $EX_1 = 0$. Now, let b_m be defined implicitly by the relation

$$\sum_{k=1}^m \frac{c_{k,m}}{b_m} \left(\log \frac{b_m}{c_{k,m}}\right)^{-1} = 1 .$$

It can be verified that if $T_m/b_m = \sum_{k=1}^m Y_{k,m}$ as in (1), then the conditions for the convergence in law of T_m/b_m given in ([3] Theorem 4, page 124) are satisfied. Moreover, $M(u) \equiv 0$, $N(u) \equiv 0$, $\sigma^2 = 0$ and $\gamma(\tau) \equiv 1$, where the functions $M(u)$, $N(u)$ and the constant σ^2 are determined by the Lévy representation of the characteristic function of the limiting random variable and

$$\gamma(\tau) = \lim_{m\to\infty} \sum_{k=1}^{m} \int_{|x|<\tau} xF_{k,m}(dx) .$$

It follows that $T_m/b_m \to 1$ in law and in probability; but this information is insufficient to perform the classification in Theorem 1. The almost sure convergence of T_m/b_m remains an open question.

3. Divergence of $Y_+(p)$.

THEOREM 3. *If $P(X_1 = 0) \neq 1$ then for $0 \leqq p < \frac{1}{2}$ either $P(Y_+(p) = \infty) = 1$ or $P(Y_-(p) = \infty) = 1$ (or both).*

PROOF. For $p \geqq 0$, let $R_n = S_n/n^p$. Then

$$Y_+(p) = \sum_{n=1}^{\infty} \frac{R_n^+}{n} = \sum_{n=1}^{\infty} \frac{R_n}{n} I_{(0,1)}(R_n) + \sum_{n=1}^{\infty} \frac{R_n}{n} I_{[1,\infty]}(R_n) .$$

But

$$\sum_{n=1}^{\infty} \frac{R_n}{n} I_{[1,\infty)}(R_n) \geqq \sum_{n=1}^{\infty} \frac{1}{n} I_{[1,\infty)}(R_n) .$$

Thus, denoting the series on the right-hand side of this inequality by $W_+(p)$, it follows from the Hewitt-Savage Zero-One Law that if $P(W_+(p) = \infty) > 0$, then $P(Y_+(p) = \infty) = 1$.

Let $W_-(p) = \sum_{n=1}^{\infty} n^{-1} I_{(-\infty,-1]}(R_n)$ and $W(p) = \sum_{n=1}^{\infty} n^{-1} I_{(-1,1)}(R_n)$. By the same argument, $P(W_-(p) = \infty) > 0$ implies $P(Y_-(p) = \infty) = 1$. But, $W_+(p) + W_-(p) + W(p) = \sum_{n=1}^{\infty} n^{-1} = \infty$. Thus, at least one of these series is infinite with positive probability. Now, if $P(W(p) = \infty) > 0$ then $EW(p) = \infty$. However,

$$EW(p) = \sum_{n=1}^{\infty} \frac{1}{n} P(|R_n| < 1)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| < n^p) .$$

By a theorem of B. Rosén ([6] page 324), for $0 \leqq p < \frac{1}{2}$ there exists a constant C (depending on p put not on n) such that $P(|S_n| < n^p) \leqq Cn^{p-\frac{1}{2}}$. It follows that $EW(p) < \infty$; thus $P(W(p) = \infty) = 0$ for these values of p. Consequently, $P(W_+(p) = \infty) > 0$ or $P(W_-(p) = \infty) > 0$ and the theorem is proved.

COROLLARY 2. *If $EX_1^2 < \infty$ and $EX_1 = 0$, then for $0 \leqq p < \frac{1}{2}$, $P(Y_+(p) = \infty) = P(Y_-(p) = \infty) = 1$.*

PROOF. The proof of Theorem 2 goes through with $c_{k,m} = k^{-(1+p)} + \cdots + m^{-(1+p)}$.

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