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LIMITING BEHAVIOR OF MAXIMA IN STATIONARY GAUSSIAN SEQUENCES[1]

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Let $\{X_n, n \geqq 1\}$ be a real-valued, stationary Gaussian sequence with mean zero and variance one. Let $M_n = \max_{1\leqq i\leqq n} X_i$, $r_n = E(X_{n+1}X_1)$; $c_n = (2\ln n)^{\frac{1}{2}}$ and $b_n = c_n - \frac{1}{2}[\ln(4\pi \ln n)]/c_n$. Define $U_n = 2c_n(M_n - c_n)/\ln\ln n$ and $V_n = c_n(M_n - b_n)$. If $r_n = O(1/\ln n)$ as $n \to \infty$ then

(i) $p(\liminf_{n\to\infty} U_n = -1) = p(\limsup_{n\to\infty} U_n = 1) = 1$, and
(ii) $E\{\exp(tV_n)\} \to E\{\exp(tX)\}$

as $n \to \infty$ for all t sufficiently small where X is a random variable with distribution function $e^{-e^{-x}}$; $-\infty < x < \infty$.

1. Statement of results. Let $\{X_n, n > 1\}$ be a real-valued, stationary Gaussian sequence with $EX_n = 0$ and $EX_n^2 = 1$. Let $\{r_n, n \geqq 0\}$, $r_n = E(X_{n+1}X_1)$ be the covariance sequence and $M_n = \max_{1\leqq i\leqq n} X_i$. The convergence of M_n, suitably normalized, as $n \to \infty$ has been of considerable interest. Berman ([1], Theorem 3.1) showed that $r_n \ln n \to 0$ as $n \to \infty$ is sufficient for $(M_n - b_n)c_n$ to converge in distribution to X where X has distribution function $e^{-e^{-x}}$, $-\infty < x < \infty$; and b_n and c_n are constants defined as follows.

$$(1.1) \qquad c_n = (2\ln n)^{\frac{1}{2}}, \qquad b_n = c_n - [\ln(4\pi \ln n)]/2c_n .$$

Convergence of M_n/c_n and $M_n - c_n$, called "relative stability" and "stability" of M_n respectively, has been studied by Gnedenko [3], Berman [1] and Pickands [5]. Pickands shows that as $n \to \infty$, $r_n \to 0$ is sufficient for $M_n/c_n \to 1$ a.s. and $r_n \ln n \to 0$ is sufficient for $M_n - c_n \to 0$ a.s.

Pickands ([7] Theorem 1.1) also considered the problem of the rate at which $M_n - c_n$ converges to zero. In this direction we prove the following theorem.

THEOREM 1. *If*

$$(1.2) \qquad r_n = O(1/\ln n) \qquad \textit{as} \quad n \to \infty$$

then

$$(1.3) \qquad \liminf_{n\to\infty} 2(M_n - c_n)c_n/\ln\ln n = -1 \quad \text{a.s.}$$

and

$$(1.4) \qquad \limsup_{n\to\infty} 2(M_n - c_n)/\ln\ln n = +1 \quad \text{a.s.}$$

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Pickands uses the condition $r_n n^\gamma \to 0$ as $n \to \infty$ for some $0 < \gamma < 1$ instead of (1.2).

An example is given in ([5] page 193) which shows that r_n can be chosen to tend to zero so slowly that on some subsequence $\{n_k\}$, $M_{n_k} - \alpha_{n_k}$ converges in distribution to a normal variable, the α_n being constants with $c_n - \alpha_n \to \infty$ as $n \to \infty$. Consequently it is deduced that $r_n \to 0$ is not sufficient for Theorem 1. Theorem 2 exhibits a specific rate at which r_n may tend to zero that is sufficient to violate (1.3) and thus illustrates the sharpness of condition (1.2).

THEOREM 2. *If*

(1.5) r_n *is non-increasing and*

(1.6) $$r_n \ln n / \ln \ln n \to \infty \qquad \text{as } n \to \infty,$$

then

(1.7) $$\liminf_{n\to\infty} (M_n - c_n)c_n / \ln \ln n = -\infty \quad \text{a.s.}$$

The convergence of moments of suitably normalized maxima was considered by Pickands ([6] Theorem 2.1) for independent variables. Theorem 3 shows the convergence of moment generating functions for maxima of dependent variables.

THEOREM 3. *Let X be a random variable with distribution function $e^{-e^{-x}}$, $-\infty < x < \infty$. For all stationary Gaussian sequences $\{X_n, n \geqq 1\}$ satisfying*

(1.8) $$r_n \ln n \to 0 \qquad \text{as } n \to \infty$$

(1.9) $$\lim_{n\to\infty} E(\exp(tY_n)) = E(\exp(tX))$$

for all t sufficiently small where $Y_n = c_n(M_n - b_n)$.

The following two lemmas are used as tools in the proofs of Theorems 1 and 3 and may be of independent interest.

LEMMA 1. (1.2) *Implies that*

(1.10) $$\exp(tA^2)P(M_n \leqq b_n - A/c_n) \to 0 \qquad \text{as } A \to \infty$$

uniformly in n for all t sufficiently small.

LEMMA 2. *Under condition* (1.2)

(1.11) $$p(M_n \geqq c_n + ((1-\varepsilon)\ln \ln n)/2c_n \ \text{ i.o.}) = 1$$

for all $\varepsilon > 0$.

Lemma 2 is a special case of Theorem B of Pathak and Qualls [4]. The result (1.11) was obtained independently of Pathak and Qualls at about the same time and uses methods similar to those of Lemma 1.

As corollaries of Theorem 3 we obtain the following numerical estimates. For any random variable X let $\sigma^2(X) = EX^2 - (EX)^2$. Then

COROLLARY 1. $\lim_{n\to\infty} (\ln n)\sigma^2(M_n) = (\pi^2 - 6)/12$.

COROLLARY 2. $E(M_n^k)/b_n^k = 1 + O(1/\ln n)$ *as* $n \to \infty$ *for all* $k \geqq 1$.

Section 2 contains proofs of Theorems 1, 2 and 3 and Lemma 1. For proof of Lemma 2 we refer to [4]. Section 3 has comment on nonstationary sequences.

2. Proofs. PROOF OF LEMMA 1. Define $\delta_x = \sup_{i \geqq x} r_i$. Condition (1.2) together with the stationarity of the sequence implies that $0 < \delta_1 < 1$. If we let $L(n) = [n^\gamma]$ ($[\cdot]$ denoting integral part), $0 < \gamma < 1$, then (1.2) implies that $\delta_{L(n)} \ln n$ is bounded as $n \to \infty$. In the following we set

$$\varphi(u) = (2\pi)^{-\frac{1}{2}} \exp(-u^2/2)\,; \qquad \Phi(u) = \int_{-\infty}^{u} \varphi(x)\,dx, \quad -\infty < u < \infty\,.$$

In order to find an upper bound for $p(M_n \leqq b_n - A/c_n)$ we select some variables out of $X_1 \cdots X_n$ as follows. Group the variables $X_1 \cdots X_n$ consecutively into $[n/L(n)]$ blocks of $L(n)$ variables each. Excluding the variables in every other block, we will be left with $mL(n)$ variables where $m = [(n/L(n)] + 1)/2]$. Thus we have selected $X_{2iL(n)+j}$, $i = 0, 1, \cdots, m-1; j = 1, 2, \cdots, L(n)$. Rename these as $U_1, \cdots, U_{mL(n)}$. Clearly,

$$P(M_n \leqq b_n - A/c_n) \leqq P(\max_{1 \leqq i \leqq mL(n)} U_i \leqq b_n - A/c_n)\,.$$

Next consider the variables

$$Z_{ij} = (1-\delta_1)^{\frac{1}{2}} Y_{ij} + (\delta_1 - \delta_{L(n)})^{\frac{1}{2}} W_i + \delta_{L(n)}^{\frac{1}{2}} V$$

for $i = 1, 2, \cdots, m$, $j = 1, 2 \cdots L(n)$, where the Y_{ij}'s, W_i's and V are mutually independent Gaussian variables with zero mean and unit variance. We see that the covariance matrix of $U_1 \cdots U_{mL(n)}$ is bounded above by that of $Z_{11}, Z_{12} \cdots, Z_{1L(n)}, Z_{21} \cdots, Z_{m,L(n)}$. By Slepian's Lemma ([8] Lemma 1, page 468)

$$\begin{aligned} & p(\max_{1 \leqq i \leqq mL(n)} U_i \leqq b_n - A/c_n) \\ (2.1) \qquad & \leqq p[(1-\delta_1)^{\frac{1}{2}} Y_{ij} + (\delta_1 - \delta_{L(n)})^{\frac{1}{2}} W_i + \delta_{L(n)}^{\frac{1}{2}} V \leqq b_n - A/c_n \ \ \forall i, j] \\ & = \int_{-\infty}^{\infty} p[(1-\delta_1)^{\frac{1}{2}} Y_{ij} + (\delta_1 - \delta_{L(n)})^{\frac{1}{2}} W_i \leqq b_n - A/c_n \\ & \qquad - \delta_{L(n)}^{\frac{1}{2}} u \ \ \forall i, j]\varphi(u)\,du\,. \end{aligned}$$

If we split the integration in (2.1) into ranges $(-\infty, A_n]$ and $[A_n, \infty)$ where $A_n = A/(2c_n \delta_{L(n)}^{\frac{1}{2}})$, then the right-hand side of (2.1) may be seen to be at most

$$(2.2) \qquad \Phi(-A_n) + p\{(1-\delta_1)^{\frac{1}{2}} Y_{ij} + (\delta_1 - \delta_{L(n)})^{\frac{1}{2}} W_i \leqq b_n - A/2c_n \ \ \forall i, j\}\,.$$

First, for all $A \geqq 0$

$$\begin{aligned} \exp(tA^2)\Phi(-A_n) &= \exp(tA^2)(1 - \Phi(A_n)) \\ &\leqq \exp(tA^2) A_n^{-1} \varphi(A_n) \\ &= (A_n 2\pi)^{-1} \exp\{tA^2 - A^2/8c_n^2 \delta_{L(n)}\}\,. \end{aligned}$$

The last expression tends to zero as $A \to \infty$ uniformly in n for sufficiently smal values of t since $c_n^2 \delta_{L(n)}$ is bounded as $n \to \infty$. Result (1.10) will therefor follow if we show that $\exp(tA^2) \times$ {second term in 2.2} tends to zero as $A \to \infty$ uniformly in n.

Define $V_j = \{(1-\delta_1)^{\frac12} Y_{ij} + (\delta_1 - \delta_{L(n)})^{-\frac12} W_1\}(1-\delta_{L(n)})^{-\frac12} j = 1, 2 \cdots L(n)$. Because of the independence of Y_{ij}'s and W_i's, the second term in (2.2) is equal to

$$\{p(\max_{1\le j\le L(n)} V_j \leqq E_n)\}^m \tag{2.3}$$

where $E_n = (1-\delta_{L(n)})^{-\frac12}(b_n - A/2c_n)$. We know that $EV_j = 0$; $EV_j^2 = 1$ and $EV_jV_k = (\delta_1 - \delta_{L(n)})/(1-\delta_{L(n)})$ for $j \neq k$. Consider the joint normal variables $\xi_1 \cdots \xi_{L(n)}$ with zero mean, unit variance and equal correlation δ_1. By Slepian's Lemma ([8] Lemma 1) (2.3) is at most

$$\{p(\max_{1\le i\le L(n)} \xi_i \leqq E_n)\}^m \,. \tag{2.4}$$

Define

$$Q(A, n) = \exp(tA^2)\{p(\max_{1\le i\le L(n)} \xi_i \leqq E_n)\}^m \,. \tag{2.5}$$

We now show that given $\eta > 0$, there exist ν_0 and A_0 (both depending on η alone) such that for all $n \geqq \nu_0$ and $A \geqq A_0$

$$Q(A, n) < \eta \,.$$

The result will follow by observing that $Q(A, n) \to 0$ as $A \to \infty$ for every fixed n. We use two distinct comparisons to bound (2.4). The first comparison is used to bound $Q(A, n)$ for $0 \leqq A \leqq 2(1-\rho)b_nc_n$, the second for $A > 2(1-\rho)b_nc_n$ (cf. (2.19)).

We first state a result of Berman [1] that is used repeatedly.

LEMMA (Berman (1964)). *Let $\{\chi_n, n \geqq 1\}$ and $\{\zeta_n, n > 1\}$ be stationary Gaussian sequences satisfying $E\chi_n = E\zeta_n = 0$; $E\chi_n^2 = E\zeta_n^2 = 1$; $E\chi_{n+1}\chi_1 = \rho_n$ and $E\zeta_{n+1}\zeta_1 = 0$. For every real number a and every positive integer n,*

$$\begin{aligned} &|p\{\max_{1\le i\le n} \chi_i \leqq a\} - p\{\max_{1\le i\le n} \zeta_i \leqq a\}| \\ &\qquad \leqq \textstyle\sum_{j=1}^{n-1} |\rho_j|(n-j)(2\pi)^{-1}(1-\rho_j^2)^{-\frac12} \exp\{-a^2/1 + |\rho_j|\} \,. \end{aligned} \tag{2.6}$$

Using this result, (2.4) is at most

$$\{\Phi^{L(n)}(E_n) + \textstyle\sum_{j=1}^{L(n)-1} (\delta_1(L(n)-j)/(1-\delta_1)^{\frac12} 2\pi) \exp(-E_n^2/(1+\delta_1))\}^m \tag{2.7}$$

The sum in (2.7) is bounded above by

$$(1-\delta_1)^{-\frac12} L^2(n) \exp(-E_n^2/(1+\delta_1)) \,. \tag{2.8}$$

Let $0 \leqq A \leqq 2(1-\rho)b_nc_n$ for ρ, $0 < \rho < 1$ to be chosen later. Then $E_n < \rho b_n$ and (2.8) is less than

$$h(n) = (1-\delta_1^2)^{-\frac12} L^2(n) \exp(-\rho^2 b_n^2/(1+\delta_1)) \,.$$

Using the definitions of b_n and $L(n)$, we see that $h(n) = 0((\ln n)^{\rho^2/(1+\delta_1)} n^{-2(-\gamma+\rho^2/(1+\delta_1))})$. Thus for $0 \leqq A \leqq 2(1-\rho)b_nc_n$, (2.7) is bounded above by

$$\{\Phi^{L(n)}(E_n) + h(n)\}^m = \Phi^{mL(n)}(E_n)\{1 + h(n)\Phi^{-L(n)}(E_n)\}^m \,. \tag{2.9}$$

Recall that $-\ln \Phi(x) \sim \varphi(x)/x$ as $x \to \infty$. Since $A \leqq 2(1-\rho)b_nc_n$, $b_n - A/2c_n \to \infty$ as $n \to \infty$ and

$$\begin{aligned} \Phi^{L(n)}(E_n) \geqq \Phi^{L(n)}(\rho b_n) &= \exp\{L(n) \ln \Phi(\rho b_n)\} \\ &\geqq \exp\{-c' L(n)\varphi(\rho b_n)/\rho b_n\} \end{aligned} \tag{2.10}$$

for some $c' > 1$ and n large. By definition of b_n and $L(n)$, $-c'L(n)\varphi(\rho b_n)/\rho b_n = o(n^{\gamma-\rho^2})$ as $n \to \infty$. If we select $\rho^2 > \gamma$, the right hand side of (2.10) tends to 1. Thus for sufficiently largen n (independent of A), the expression in the brackets of (2.9) is less than or equal to $\{1 + 2h(n)\}^m$. We may choose ρ and γ, $\rho^2 > \gamma$ such that $\{1 + 2h(n)\}^m \to 1$ as $n \to \infty$. (For example select $\gamma \leqq (1 - \delta_1)/4$ and $\rho^2 = 1 - (1 - \delta_1)/4 - (1 - \delta_1)^2/8$. Then $\rho^2 > \gamma$ and $2(\rho^2/(1 + \delta_1) - \gamma) > 1 - \gamma$ so $\{1 + 2h(n)\}^m \to e^0 = 1$ since $m = O(n^{1-\gamma})$ as $n \to \infty$.) We can find ν_1 such that for all $n \geqq \nu_1$ and $0 \leqq A \leqq 2(1 - \rho)b_n c_n$, (2.9) is no bigger than $2\Phi^{mL(n)}(E_n)$. Define

$$f(A, n) = 2 \exp(tA^2)\Phi^{mL(n)}(E_n) \tag{2.11}$$

so that $Q(A, n) \leqq f(A, n)$ for all $n \geqq \nu_1$ and $0 \leqq A \leqq 2(1 - \rho)b_n c_n$. To find the uniform rate at which $f(A, n) \to 0$ as $A \to \infty$, we consider

$$\begin{aligned} \frac{d}{dA} f(A, n) &= 2 \exp(tA^2)\Phi^{mL(n)-1}(E_n) \\ &\quad \times \left\{2tA\Phi(E_n) - \frac{mL(n)}{2c_n(1 - \delta_{L(n)})^{\frac{1}{2}}}\varphi(E_n)\right\}. \end{aligned} \tag{2.12}$$

The last factor in (2.12) is bounded by

$$\begin{aligned} 2tA &- \frac{mL(n)}{2n} \exp\left\{\frac{Ab_n(1 - A/4b_n c_n)}{2c_n(1 - \delta_{L(n)})} - \frac{b_n^2\delta_{L(n)}}{(1 - \delta_{L(n)})}\right\} \\ &\leqq 2tA - \frac{mL(n)}{2n} \exp\{Ab_n(1 + \rho)/4c_n - b_n^2\delta_{L(n)}\} \end{aligned} \tag{2.13}$$

for all $0 \leqq A \leqq 2(1 - \rho)b_n c_n$. The derivative of the right-hand side of (2.13) with respect to A is

$$2t - \frac{mL(n)}{2n} \cdot \frac{(1 + \rho)b_n}{4c_n} \exp\{b_n(1 + \rho)A/4c_n - b_n^2\delta_{L(n)}\}.$$

For all $A \geqq 0$ and t sufficiently small, this is negative since $b_n^2\delta_{L(n)}$ is bounded as $n \to \infty$. Thus the right-hand side of (2.13) is at most $-mL(n)/2n$. Substituting in (2.12) we get

$$\begin{aligned} \frac{d}{dA} f(A,n) &< -2 \exp(tA^2)\Phi^{mL(n)-1}(E_n) \cdot \frac{mL(n)}{2n} \\ &\leqq -\frac{mL(n)}{n} f(A, n) < f(A, n) \end{aligned}$$

for large n since $mL(n)/n \to \frac{1}{2}$ as $n \to \infty$. Therefore for all $n \geqq \nu_2$ (independent of A) and $0 \leqq A \leqq 2(1 - \rho)b_n c_n$,

$$\frac{d}{dA} f(A, n)/f(A, n) < -1.$$

Integrating both sides w.r.t. A we get

$$f(A', n) \leqq f(0, n)e^{-A'}$$

for all $n \geqq \nu_2$ and $0 \leqq A' \leqq 2(1-\rho)b_n c_n$. But

$$f(0, n) = 2\Phi^{mL(n)}(b_n(1-\delta_{L(n)})^{-\frac{1}{2}}) < 2$$

so for all $n \geqq \max(\nu_1, \nu_2)$ and $0 \leqq A \leqq 2(1-\rho)b_n c_n$,

$$Q(A, n) \leqq f(A, n) \leqq 2e^{-A} . \tag{2.14}$$

Going back to (2.4), we see that another upper bound for (2.4) is

$$\{p(\xi_1 \leqq E_n)\}^m = \Phi^m(E_n) .$$

Define

$$g(A, n) = \exp(tA^2)\Phi^m(E_n) . \tag{2.15}$$

We will show that the derivative w.r.t. A of $g(A, n)$ is negative for all $A > 4b_n c_n$ and that the maximum of $g(A, n)$ for $2(1-\rho)b_n c_n \leqq A \leqq 4b_n c_n$ tends to zero as $n \to \infty$. First

$$\begin{aligned}\frac{d}{dA} g(A, n) &= \exp(tA^2)\Phi^{m-1}(E_n) \\ &\quad \times \{2tA\Phi(E_n) - m\varphi(E_n)/2c_n(1-\delta_{L(n)})^{\frac{1}{2}}\} .\end{aligned} \tag{2.16}$$

Now for all $A > 4b_n c_n$ (hence $(A/2c_n) - b_n > 0$),

$$\begin{aligned}A\Phi(E_n) &= A\{1 - \Phi(-E_n)\} \\ &\leqq A(1-\delta_{L(n)})^{\frac{1}{2}}\varphi(E_n)/((A/2c_n) - b_n) \\ &= (1-\delta_{L(n)})^{\frac{1}{2}}\varphi(E_n)/(\tfrac{1}{2}c_n^{-1} - b_n A^{-1}) \\ &< 4c_n(1-\delta_{L(n)})^{\frac{1}{2}}\varphi(E_n) .\end{aligned}$$

Substituting in (2.16) we get for all $A > 4b_n c_n$,

$$\begin{aligned}\frac{d}{dA} g(A, n) &< (1-\delta_{L(n)})^{\frac{1}{2}} c_n \varphi_n(E_n) \exp(tA^2)\Phi^{m-1}(E_n) \\ &\quad \times \{8t - m/2c_n^2(1-\delta_{L(n)})\} ,\end{aligned}$$

and this is negative for small t.

Next, if we set

$$\mathscr{G}(n) = \max\{g(A, n) \leftarrow 2(1-\rho)b_n c_n \leqq A \leqq 4b_n c_n\} ,$$

then

$$\begin{aligned}\mathscr{G}(n) &\leqq \exp(t(4b_n c_n)^2)\Phi^m((b_n - (1-\rho)b_n)/(1-\delta_{L(n)})^{\frac{1}{2}}) \\ &\leqq \exp\{64t(\ln n)^2\}\Phi^m(\rho b_n/(1-\delta_{L(n)})^{\frac{1}{2}}) .\end{aligned} \tag{2.17}$$

By Cramér ([2], page 374), there exists a c'', $0 < c'' < 1$ such that for all n large, the right-hand side of (2.17) is bounded by

$$\exp\{64t(\ln n)^2 - c''m\varphi(\rho' b_n)/\rho' b_n\} \qquad \text{where } \rho' = \rho(1-\delta_{L(n)})^{-\frac{1}{2}} .$$

By definition of b_n, this expression is no bigger than

$$\exp\{64t(\ln n)^2 - (\text{const.})(m/b_n)n^{-\rho'^2}\} . \tag{2.18}$$

We select ρ and γ, such that (2.18) tends to zero as $n \to \infty$. (If $\gamma \leqq (1 - \delta_1)/4$ and $\rho^2 = 1 - (1 - \delta_1)/4 - (1 - \delta_1)^2/8$ we have

$$1 - \gamma - \rho^2/1 - \delta_{L(n)} > 1 - (1 - \delta_1)/4 - (1 - \delta_{L(n)})^{-1}(1 - (1 - \delta_1)/4) + (1 - \delta_{L(n)})^{-1}(1 - \delta_1)^2/8$$

which is positive for n large since $\delta_{L(n)} \to 0$ as $n \to \infty$ and then (2.18) tends to zero as $n \to \infty$.)

Now by definitions of Q, f and g (cf. (2.5), (2.11), (2.15)) we have

$$(2.19 \qquad Q(A, n) \leqq 2e^{-A} \quad \forall n \geqq \max(\nu_1, \nu_2) \text{ and } 0 \leqq A \leqq 2(1 - \rho)b_n c_n$$
$$\leqq \mathscr{G}(n) \quad \forall A > 2(1 - \rho)b_n c_n .$$

Hence given $\eta > 0$, choose A_0 so large that $2e^{-A} < \eta$ for all $A \geqq A_0$ and choose $\nu_0 > \max(\nu_1 \nu_2)$ so large that $\mathscr{G}(n) < \eta \ \forall n \geqq \nu_0$. Then (2.19) gives

$$Q(A, n) < \eta$$

for all $n \geqq \nu_0$ (independent of A) and $A \geqq A_0$ (independent of n).

The observation that $Q(A, n) \to 0$ as $A \to \infty$ for every fixed n is clear since

$$Q(A, n) \leqq \exp(tA^2)\Phi^m(E_{n,A}) = \exp(tA^2)\{1 - \Phi(-E_{n,A})\}^m$$

where $E_{n,A} = E_n = (b_n - A/2c_n)(1 - \delta_{L(n)})^{-\frac{1}{2}}$. Let A be so large that $(A/2c_n) - b_n > 1$; then the above expression is at most

$$\exp(tA^2)\varphi^m(-E_{n,A})(1 - \delta_{L(n)})^{m/2} < \exp\left\{tA^2 - \frac{m}{2}(A/2c_n - b_n)^2\right\}$$

and it tends to zero as $A \to \infty$ for t sufficiently small. Thus there exists a_n such that for all $A \geqq a_n$, $Q(A, n) \leqq \eta$. Let $a^* = \max_{0 \leqq i \leqq \nu_0} a_i$; then

$$Q(A, n) \leqq \eta \qquad \forall A \geqq a^* \qquad \text{and for all } n.$$

This completes the proof of Lemma 1.

PROOF OF THEOREM 1. We will prove that (1.2) implies

$$\limsup_{n\to\infty} 2c_n(M_n - c_n)/\ln \ln n = +1 \quad \text{a.s.} \tag{2.20}$$

and

$$\liminf_{n\to\infty} 2c_n(M_n - c_n)/\ln \ln = -1 \quad \text{a.s.} \tag{2.21}$$

We first consider the lim sup. According to Lemma 2, for $\varepsilon > 0$

$$P(2c_n(M_n - c_n)/\ln \ln n > 1 - \varepsilon \text{ i.o}) = 1 .$$

In other words

$$\limsup_{n\to\infty} 2c_n(M_n - c_n)/\ln \ln n \geqq 1 \quad \text{a.s.}$$

With no conditions on the covariance sequence, Theorem 2.1 of [7] shows that

$$\limsup_{n\to\infty} 2c_n(M_n - c_n)/\ln \ln n \leqq 1 \quad \text{a.s.}$$

Hence (2.20) is proved.

To prove (2.21), we first show that

$$\liminf_{n\to\infty} 2c_n(M_n - c_n)/\ln\ln n \geqq -1 \quad \text{a.s.}$$

By Lemmas 3.3 and 3.4 of [7], it is sufficient to show that the inequalities

$$M_{n(\varepsilon,m)} \leqq a(n(\varepsilon, m), \varepsilon) \qquad m = 1, 2, \cdots \tag{2.22}$$

hold only finitely often with probability one $\forall \varepsilon > 0$, where $n(\varepsilon, m) = [e^{\varepsilon m}]$ and

$$a(n, \varepsilon) = c_n - \left(\frac{1}{2} + \varepsilon\right)\ln\ln n/c_n = b_n - \varepsilon \ln\ln n/c_n\,.$$

We look at

$$\begin{aligned}\sum_{n=1}^{\infty} P\{M_{n(\varepsilon,m)} \leqq a(n(\varepsilon, m), \varepsilon)\}\\ \leqq m_0 + \sum_{m_0}^{\infty} P\{M_{n(\varepsilon,m)} \leqq a(n(\varepsilon, m), \varepsilon)\}\,.\end{aligned} \tag{2.23}$$

By Lemma 1, if (1.2) holds then for some $t > 0$ we can choose m_0 so large that the right-hand side of (2.23) is at most

$$m_0 + \sum_{m_0}^{\infty} \exp\{-t\varepsilon^2(\ln\ln n(\varepsilon\, m))^2\} \leqq m_0 + \sum_{m_0}^{\infty} \exp\{-t\varepsilon^2(\ln \varepsilon m)^2\}\,.$$

The above sum is finite for all $\varepsilon > 0$ and (2.22) follows by the Borel–Cantelli Lemma. To complete the proof of Theorem 1 we need to show that for all $\varepsilon > 0$

$$P(M_n \leqq c_n - (1 - \varepsilon)\ln\ln n/(2c_n) \;\text{ i.o.}) = 1\,,$$

or

$$P(M_n \leqq b_n + \varepsilon \ln\ln n/(2c_n) \quad \text{finitely often}) = 0.$$

The last probability is

$$\begin{aligned}P(\textstyle\bigcup_n \bigcap_{k=n}^{\infty} \{M_k > b_k + \varepsilon \ln\ln k/c_k\})\\ \leqq \textstyle\sum_n P(\bigcap_{k=n}^{\infty} \{M_k > b_k + \varepsilon \ln\ln k/c_k\})\\ = \textstyle\sum_n \lim_{N\to\infty} P(\bigcap_{k=n}^{N} \{M_k > b_k + \varepsilon \ln\ln k/c_k\})\\ = \textstyle\sum_n \lim_{N\to\infty} P(M_n > b_n + \varepsilon \ln\ln N/c_N)\,.\end{aligned}$$

By (2.6)

$$\begin{aligned}P(M_N > b_N + \varepsilon \ln\ln N/c_N)\\ \leqq P(M_N^* > b_N + \varepsilon \ln\ln N/c_N)\\ + \sum_{j=1}^{N} \frac{|r_j|(N-j)}{2\pi(1-r_j^2)^{\frac{1}{2}}} \exp - \frac{(b_N + \varepsilon \ln\ln nN/c_N)^2}{1+|r_j|}\end{aligned}$$

where M_N^* is maximum of N independent standard normal variables. The first term in the right-hand side above tends to zero as $N \to \infty$ by Theorem 3.1 of [1] and the second term tends to zero as $N \to \infty$ by arguments similar to those in Lemma 3.1 of [1]. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. We assume that

$$r_n \quad \text{is non-increasing}\,; \tag{2.24}$$

$$r_n \ln n/\ln\ln n \to \infty \qquad \text{as } n \to \infty \tag{2.25}$$

and we will show that

$$\liminf_{n\to\infty} (M_n - c_n)c_n/\ln\ln n = -\infty \quad \text{a.s.} \tag{2.26}$$

Define $A_n(K) = \{(M_n - c_n)c_n < -K\ln\ln n\}$. To show that $P(A_n(K) \text{ i.o.}) = 1$ it will be sufficient to show that $\lim_{n\to\infty} P(A_n^c(K)) = 0 \ \forall K$.

Let $\{Z_n, n > 1\}$ be a sequence of independent, Gaussian variables with zero mean and unit variance. Let $M_n^* = \max_{1\leq i\leq n} Z_i$ and let U be a standard normal variable independent of $\{Z_n, n \geq 1\}$. The covariance matrix of $\{(1 - r_n)^{\frac{1}{2}}Z_i + r_n^{\frac{1}{2}}U, 1 \leq i \leq n\}$ is dominated above by the covariance matrix of $x_1 \cdots x_n$, for all n since $\{r_n\}$ is non-increasing. Hence by Slepian's Lemma ([8] Lemma 1), we have for all K and n,

$$\begin{aligned} P(A_n^c(K)) &\leq P(c_n((1 - r_n)^{\frac{1}{2}}M_n^* + r_n^{\frac{1}{2}}U - c_n) \geq -K\ln\ln n) \\ &= P((1 - r_n)^{\frac{1}{2}}c_n(M_n^* - c_n) \geq c_n^2 r_n(1 + (1 - r_n)^{\frac{1}{2}})^{-1} \\ &\qquad -K\ln\ln n - c_n r_n^{\frac{1}{2}}U) \\ &\leq P((1 - r_n)^{\frac{1}{2}}c_n(M_n^* - c_n) \geq c_n^2 r_n/4 - K\ln\ln n) \\ &\qquad + (1 - \Phi(c_n r_n^{\frac{1}{2}}/4)). \end{aligned}$$

Clearly the second term in the right-hand side above tends to zero as $n \to \infty$. Condition (2.25) implies that the first term tends to zero by a classical result that

$$\limsup_{n\to\infty} (M_n^* - c_n)c_n/\ln\ln n = \tfrac{1}{2} \quad \text{a.s.} \tag{2.27}$$

((2.27) is also special case of Theorem 1). Theorem 2 is proved.

Proof of Theorem 3. We will show that if (1.8) holds then for t sufficiently small,

$$\lim_{n\to\infty} \exp(tY_n) = \exp(tX) \tag{2.28}$$

where $Y_n = c_n(M_n - b_n)$. The random variable X is defined in the statement of Theorem 3.

Under the condition (1.8) we know that Y_n converge in distribution to X ([1], Theorem 3.1). (2.28) follows if for small t

$$\int_{|x|\geq A} \exp(tx)\, dF_n(x) \to 0 \qquad \text{as } A \to \infty, \tag{2.29}$$

uniformly in n where $F_n(x) = P(c_n(M_n - b_n) \leq x)$, $x \in (-\infty, \infty)$.

First we prove that the integral for $x > A$ in (2.29) tends to zero without restricting the covariance sequence at all. For any $t < 1$, write

$$\begin{aligned} \int_A^\infty \exp(tx)\, dF_n(x) &= -\int_A^\infty \exp(tx)\, d(1 - F_n(x)) \\ &= \exp(tA)P(M_n > b_n + A/c_n) \\ &\qquad + t\int_A^\infty \exp(tx)P(M_n > b_n + x/c_n)\, dx\,. \end{aligned} \tag{2.30}$$

The first term in (2.30) tends to zero as $A \to \infty$ uniformly in n since for all $A \geqq 0$

$$\begin{aligned}\exp(tA)P(M_n > b_n + A/c_n) &= \exp(tA)P(\text{at least one of } x_1 \cdots x_n > b_n + x/c_n) \\ &\leqq \exp(tA)\cdot n \cdot (1 - \Phi(b_n + A/c_n)) \\ &\leqq \exp(tA)\cdot n\cdot \varphi(b_n + A/c_n)/(b_n + A/c_n) \\ &= c_n(b_n + A/c_n)^{-1}\exp((t - b_n/c_n)A - A^2/2c_n^2)\,.\end{aligned}$$

(Recall that $b_n/c_n \to 1$ as $n \to \infty$.)

For the second term in (2.30), set $t' = (1 + t)/2$. Then $t < t' < 1$ and by the preceding discussion for a given $\delta > 0$, there exists $A_0(t')$ such that for all $x \geqq A_0(t')$

$$\exp(t'x)P(M_n > b_n + x/c_n) < \delta$$

or

$$\exp(tx)P(M_n > b_n + x/c_n) < \delta \exp(-(t' - t)x)\,.$$

Thus

$$\int_A^\infty \exp(tx)P(M_n > b_n + x/c_n)\,dx < \delta(t' - t)^{-1}\exp(-(t' - t)A)$$

for all $A \geqq A_0(t')$. The right-hand side above tends to zero as $A \to \infty$ uniformly in n and one part of (2.29) is proved.

Now we consider

$$\begin{aligned}\int_{-\infty}^{-A}\exp(tx)\,dF_n(x) &= \exp(-tA)P(M_n \leqq b_n - A/c_n) \\ &\quad - t\int_A^\infty \exp(-tx)P(M_n \leqq b_n - x/c_n)\,dx\,.\end{aligned}$$

We will be done if we show that for all $t > 0$ the following expression tends to zero as $A \to \infty$ uniformly in n

$$\exp(tA)P(M_n \leqq b_n - A/c_n) + t\int_A^\infty \exp(tx)\cdot P(M_n \leqq b_n - x/c_n)\,dx\,. \tag{2.31}$$

For any $t_0 > 0$.

$$\begin{aligned}\exp(tx)P(M_n \leqq b_n - A/c_n) &= \exp(-t_0x(x - t/t_0)) \\ &\quad\times \exp(t_0x^2)P(M_n \leqq b_n - x/c_n)\,.\end{aligned}$$

Suppose (1.8) holds so that Lemma 1 is valid for all $t \leqq t_0$. Then there exists $A_0 > t/t_0 + 1$ such that for all $x \geqq A_0$ and for all n, $\exp(t_0x^2)P(M_n \leqq b_n - x/c_n) < 1$ so

$$\exp(tx)P(M_n \leqq b_n - x/c_n) \leqq \exp(t_0x)\,.$$

The expression in (2.31) is at most $\exp(-t_0A) + (t/t_0)\exp(-t_0A)$ for all $A \geqq A_0$. This completes the proof of Theorem 3.

Proof of Corollary 1. By Theorem 3, $EY_n = c_n(EM_n - b_n) \to EX$ and $EY_n^2 = c_n^2E(M_n - b_n)^2 \to EX^2$ as $n \to \infty$. Thus

$$\begin{aligned}\sigma^2(Y_n) &= EY_n^2 - (EY_n)^2 \\ &= c_n^2\{EM_n^2 - 2b_nEM_n + b_n^2 - (EM_n)^2 + 2b_nEM_n - b_n^2\} \\ &= c_n^2\sigma^2(M_n)\,.\end{aligned}$$

But $\sigma^2(M_n) \to \sigma^2(X)$ as $n \to \infty$ and the result follows since $\sigma^2(X) = (\pi^2/6) - 1$ by Cramér ([2], page 376).

PROOF OF COROLLARY 2. That

$$b_n^{-k}E(M_n^k) = 1 + O(1/\ln n) \qquad \text{as } n \to \infty$$

for all $k \geqq 1$ will be proved by induction. By Theorem 3 we have, as $n \to \infty$,

$$EM_n = b_n + (E(X)/c_n) + o(1/c_n)$$

and

$$E(M_n^2) = b_n^2 + (2b_nE(X)/c_n) + o(1) + EX^2/(2\ln n)\,.$$

Hence the result is true for $k = 1$ and $k = 2$. Let it be true for all $k \leqq l$. Consider

$$E(M_n - b_n)^{l+1} = E(M_n^{l+1}) + \textstyle\sum_{j=1}^{l+1} \binom{l+1}{j}(-b_n)^jE(M_n^{l+1-j})\,.$$

By Theorem 3, the left-hand side above is equal to

$$(E(X^{l+1})/c_n^{l+1}) + o(1/c_n^{l+1}) \qquad \text{as } n \to \infty\,.$$

Therefore as $n \to \infty$,

$$E(M_n^{l+1}) = -\textstyle\sum_{j=1}^{l+1} \binom{l+1}{j}(-b_n)^jE(M_n^{l+1-j}) + (E(X^{l+1})/c_n^{l+1}) + o(1/c_n^{l+1})$$

or

$$\begin{aligned}(E(M_n^{l+1})/b_n^{l+1}) = &-\textstyle\sum_{j=1}^{l+1} \binom{l+1}{j}(-1)^j(E(M_n^{l+1-j})/b_n^{l+1-j}) \\ &+ (E(X^{l+1})/(c_nb_n)^{l+1}) + o(1/c_n^{l+1})\,.\end{aligned}$$

By the induction hypothesis the right-hand side above is equal to

$$-(1 + O(1/\ln n)) \textstyle\sum_{j=1}^{l+1} \binom{l+1}{j}(-1)^j + O(1/(\ln n)^{l+1})\,.$$

But $-\sum_{j=1}^{l+1} (-1)^j = 1$ and the result follows.

We also note that Corollary 2 implies that

$$E|M_n|^k/b_n^k = 1 + O(1/\ln n) \qquad \text{as } n \to \infty$$

since

$$\begin{aligned}E|M_n|^k &\leqq E(M_n^k) + 2k \textstyle\int_0^\infty x^{k-1}P(M_n \leqq -x)\,dx \\ &\leqq E(M_n^k) + 2k \textstyle\int_0^\infty x^{k-1}\Phi(-x)\,dx \\ &= E(M_n^k) + \text{constant}.\end{aligned}$$

3. **Discussion.** REMARK 1. The existence of covariance sequences satisfying (1.2) or (1.6) can be seen by observing that $1/\ln |x|$ and $\ln \ln |x|/\ln |x|$ are non-negative, even and convex on $[K, \infty)$, for some constant K. To satisfy Pólya's criterion we only need to choose the constant K properly and extend the functions on $[0, K)$ by a suitable straight line. Such functions are then covariance functions of real valued, stationary, Gaussian processes $\{X_i(s), s \geqq 0\}$. Restricting these processes to the integers will provide examples of Gaussian sequences satisfying required conditions.

REMARK 2. The assumption of stationarity is not crucial in the proofs of Theorems 1 and 3. For a nonstationary sequence $\{X_n, n \geqq 1\}$, let $EX_n = 0$; $EX_n^2 = 1$ and $r_{m,n} = E(X_mX_n)$. The assumption (1.8) e.g. can be replaced by

(a) $\sup_{m \neq n} r_{m,n} \neq 1$, and

(b) $r_{m,n} \ln |m - n| \to 0$ as $|m - n| \to \infty$ uniformly in m and n.

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