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A NOTE ON THE STRONG CONVERGENCE OF σ-ALGEBRAS

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A quantity $\int |E_{\mathscr{B}} f| \, dP$ (or equivalently $\int |u - P(A : \mathscr{B})| \, dP$, $0 < u < 1$) associated with a σ-algebra $\mathscr{B}$ is shown to act as a criterion for a type of convergence of σ-algebras. This quantity also defines an ordering of σ-algebras, so that upper and lower limits can be defined in terms of this quantity. Another criterion for the convergence of σ-algebras is described based on the existence of these limits.

1. Introduction. The purpose of this article is to establish a parallelism, with respect to convergence and ordering, between a σ-algebra $\mathscr{B}$ and the L^1-norm $\int |E_{\mathscr{B}} f| \, dP$ of conditional expectation $E_{\mathscr{B}} f$, given $\mathscr{B}$, of every bounded function f. The convergence of these quantities $\int |E_{\mathscr{B}} f| \, dP$ was used, instead of the strong convergence of σ-algebras (see [5] IV.3.2, page 124), by the author in his previous article [3] in an application to asymptotic theory in statistics. In Section 2, we shall show the equivalence of the (strong) convergence of σ-algebras $\mathscr{B}_n$ to that of the quantities $\int |E_{\mathscr{B}_n} f| \, dP$ for every bounded f, and to the convergence of these quantities for merely every f of the form $f = u - I_A$ with $0 < u < 1$ and a measurable set A. In Section 3, we concern ourselves with the relationship between the inclusion ordering of σ-algebras $\mathscr{B}$ and the magnitude ordering of the quantities $\int |E_{\mathscr{B}} f| \, dP$; as a consequence of this discussion upper and lower limits of σ-algebras naturally come to be introduced. The existence of these limits is shown and another condition for the strong convergence of $\mathscr{B}_n$ is given; namely: $\limsup \mathscr{B}_n = \liminf \mathscr{B}_n$.

2. Conditions equivalent to strong convergence. Let $(X, \mathscr{A}, P)$ be a probability space consisting of a set X, a σ-algebra $\mathscr{A}$ of subsets of X and a probability measure P on $\mathscr{A}$. Throughout this paper all sub-σ-algebras of $\mathscr{A}$ are assumed to be complete, that is, to contain all sets in $\mathscr{A}$ of P-measure zero; they will be referred to simply as "σ-algebras". These σ-algebras are denoted by script letters $\mathscr{B}, \mathscr{C}, \cdots$, etc. The conditional expectation $E_{\mathscr{B}} f$, given a σ-algebra $\mathscr{B}$, is given, by definition, by the Radon–Nikodym derivative, on $\mathscr{B}$, of the measure $\int_B f \, dP$, $B \in \mathscr{B}$, with respect to P. In particular, if f is the indicator function I_A of a set $A \, (\in \mathscr{A})$, then $E_{\mathscr{B}} f$ is called the *conditional probability of* A, *given* $\mathscr{B}$, and denoted by $P(A : \mathscr{B})$. In the sequel, an integral sign without any affix will be understood to denote integration over the whole space X.

We shall begin by establishing some lemmas which will be needed later.

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LEMMA 2.1. *Let $\mathscr{B}$ and $\mathscr{C}$ be two σ-algeblas and u a real between 0 and 1. For any C in $\mathscr{C}$ we have*

$$(2.1) \qquad 2 \inf_{B \in \mathscr{B}} \{(1-u)P(C-B) + uP(B-C)\} = \int |u - P(C:\mathscr{C})|\, dP - \int |u - P(C:\mathscr{B})|\, dP .$$

PROOF. Since $(1-u)P(C-B)+uP(B-C) = (1-u)P(C)+\int_B (u-P(C:\mathscr{B}))\, dP$ for any $B \in \mathscr{B}$, the left-hand side of (2.1) equals

$$(2.2) \qquad 2(1-u)P(C) + 2\int_{B^-} (u - P(C:\mathscr{B}))\, dP$$

with $B^- = \{x : u < P(C:\mathscr{B})\}$. Denote by H and K the values of the integrals of a function $u - P(C:\mathscr{B})$ over domains B^- and $X - B^-$, respectively, with respect to P. It is easily verified that $H + K = u - P(C)$ and $K - H = \int |u - P(C:\mathscr{B})|\, dP$, and hence

$$(2.3) \qquad 2H = u - P(C) - \int |u - P(C:\mathscr{B})|\, dP .$$

Replacing the integral part of (2.2) by (2.3), the left-hand side of (2.1) equals

$$u + (1-2u)P(C) - \int |u - P(C:\mathscr{B})|\, dP .$$

Since $P(C:\mathscr{C}) = 1$, a.e., on C and $= 0$, a.e., on $X - C$, we have $\int |u - P(C:\mathscr{C})|\, dP = u + (1-2u)P(C)$. Thus (2.1) has been obtained for every $C \in \mathscr{C}$.

LEMMA 2.2. *Let $\{\mathscr{B}_n : n = 0, 1, 2, \cdots\}$ be an arbitrary sequence of σ-algebras and u an arbitrary real between 0 and 1. If B is a member of $\mathscr{B}_0$ satisfying*

$$(2.4) \qquad \lim_{n\to\infty} \int |u - P(B:\mathscr{B}_n)|\, dP = \int |u - P(B:\mathscr{B}_0)|\, dP ,$$

then we have

$$(2.5) \qquad \lim_{n\to\infty} \int_B E_{\mathscr{B}_n} f\, dP = \int_B E_{\mathscr{B}_0} f\, dP$$

for any bounded $\mathscr{A}$-measurable function f.

PROOF. Let B_n be an arbitrary member of $\mathscr{B}_n$, $n = 1, 2, \cdots$. If f is bounded by M, then

$$|\int_{B_n} E_{\mathscr{B}_n} f\, dP - \int_B E_{\mathscr{B}_0} f\, dP| = |\int_{B_n} f\, dP - \int_B f\, dP| \leqq MP(B_n \triangle B) ,$$

and

$$|\int_B E_{\mathscr{B}_n} f\, dP - \int_{B_n} E_{\mathscr{B}_n} f\, dP| \leqq \int_{B\triangle B_n} E_{\mathscr{B}_n} f\, dP \leqq MP(B_n \triangle B) ,$$

where $\triangle$ denotes the symmetric difference operation. Therefore

$$|\int_B E_{\mathscr{B}_n} f\, dP - \int_B E_{\mathscr{B}_0} f\, dP| \leqq 2MP(B \triangle B_n) \leqq 2M_1\{(1-u)P(B - B_n) + uP(B_n - B)\} ,$$

where $M_1 = M \max\{u^{-1}, (1-u)^{-1}\}$. Since B_n is arbitrary in $\mathscr{B}_n$, it follows from Lemma 2.1 that

$$|\int_B E_{\mathscr{B}_n} f\, dP - \int_B E_{\mathscr{B}_0} f\, dP| \leqq M_1\{\int |u - P(B:\mathscr{B}_0)|\, dP - \int |u - P(B:\mathscr{B}_n)|\, dP\} .$$

The last inequality shows that (2.4) implies (2.5).

LEMMA 2.3. *Let* $\{\mathscr{B}_n : n = 0, 1, 2, \cdots\}$ *be a sequence of* σ*-algebras. Let* u *be a given real between* 0 *and* 1. *If* (2.4) *holds for every* B *in* $\mathscr{B}_0$, *then for any bounded* $\mathscr{A}$*-measurable* f

$$\binom{\limsup}{\liminf} \int |E_{\mathscr{B}_n} f|\, dP = \int |E_{\mathscr{B}_0} f|\, dP + 2 \binom{\limsup}{\liminf} \int_{B_n \triangle B_0} |E_{\mathscr{B}_n} f|\, dP ,$$

where $B_n = \{x : E_{\mathscr{B}_n} f < 0\}$ *for* $n = 0, 1, 2, \cdots$.

PROOF. Since

$$\begin{aligned} \int |E_{\mathscr{B}_n} f|\, dP &= \int E_{\mathscr{B}_n} f\, dP - 2 \int_{B_n} E_{\mathscr{B}_n} f\, dP \\ &= \int f\, dP - 2 \int_{B_n} E_{\mathscr{B}_n} f\, dP , \qquad n = 0, 1, 2, \cdots , \end{aligned}$$

we have

$$\int |E_{\mathscr{B}_n} f|\, dP = \int |E_{\mathscr{B}_0} f|\, dP + 2K_n , \tag{2.6}$$

where

$$K_n = \int_{B_0} E_{\mathscr{B}_0} f\, dP - \int_{B_n} E_{\mathscr{B}_n} f\, dP ,$$

or

$$K_n = \int_{B_0} (E_{\mathscr{B}_0} f - E_{\mathscr{B}_n} f)\, dP + \int_{B_n \triangle B_0} |E_{\mathscr{B}_n} f|\, dP . \tag{2.7}$$

Since (2.5) with $B = B_0$ follows from our assumption and Lemma 2.2, (2.7) yields

$$\binom{\limsup}{\liminf} K_n = \binom{\limsup}{\liminf} \int_{B_n \triangle B_0} |E_{\mathscr{B}_n} f|\, dP . \tag{2.8}$$

By (2.6) and (2.8) we have our desired result.

THEOREM 2.1. *The following three statements are equivalent to each other*:

(i) $P(A : \mathscr{B}_n)$ *converges to* $P(A : \mathscr{B}_0)$ *in probability for any* A *in* $\mathscr{A}$.

(ii) $\lim_{n\to\infty} \int |E_{\mathscr{B}_n} f|\, dP = \int |E_{\mathscr{B}_0} f|\, dP$ *for any bounded* $\mathscr{A}$*-measurable function* f.

(iii) *For any real* u *between* 0 *and* 1 *and any* A *in* $\mathscr{A}$, *the integral* $\int |u - P(A : \mathscr{B}_n)|\, dP$ *converges to* $\int |u - P(A : \mathscr{B}_0)|\, dP$ *as* $n \to \infty$.

PROOF. Since the convergence of $P(A : \mathscr{B}_n)$ to $P(A : \mathscr{B}_0)$ in probability for any A in $\mathscr{A}$ yields the L^1-convergence of $E_{\mathscr{B}_n} f$ to $E_{\mathscr{B}_0} f$ for any bounded $\mathscr{A}$-measurable f (see Neveu [5]), the statement (ii) above is implied by statement (i), due to the inequality: $|\int |E_{\mathscr{B}_n} f|\, dP - \int |E_{\mathscr{B}_0} f|\, dP| \leqq \int |E_{\mathscr{B}_n} f - E_{\mathscr{B}_0} f|\, dP$. Statement (iii) follows directly from statement (ii) by taking $f = u - I_A$. Assume now that statement (iii) holds. From Lemma 2.3 it follows that

$$\begin{aligned} \lim_{n\to\infty} \int |v - P(A : \mathscr{B}_n)|\, dP &= \int |v - P(A : \mathscr{B}_0)|\, dP \\ &\quad + 2 \lim_{n\to\infty} \int_{B_n \triangle B_0} |v - P(A : \mathscr{B}_n)|\, dP \end{aligned}$$

for every v in (0, 1), where $B_n = B_n(v) = \{x : P(A : \mathscr{B}_n) > v\}$ for $n = 0, 1, 2, \cdots$. Comparing this equation with statement (iii), we get

$$\lim_{n\to\infty} \int_{B_n(v) \triangle B_0(v)} |v - P(A : \mathscr{B}_n)|\, dP = 0 . \tag{2.9}$$

Let ε be a positive number and take a set $S_n(v, \varepsilon) = \{x : P(A : \mathscr{B}_0) > v$ and $P(A : \mathscr{B}_n) < v - \varepsilon\} \cup \{x : P(A : \mathscr{B}_0) \leqq v$ and $P(A : \mathscr{B}_n) > v + \varepsilon\}$ for every v in

(0, 1) and $\varepsilon > 0$. Obviously we have $S_n(v, \varepsilon) \subset B_n(v) \triangle B_0(v)$, and so $\varepsilon P(S_n(v, \varepsilon)) < \int_{S_n(v,\varepsilon)} |v - P(A : \mathscr{B}_n)| \, dP \leqq \int_{B_n(v) \triangle B_0(v)} |v - P(A : \mathscr{B}_n)| \, dP$. Therefore it follows from (2.9) that

$$\lim_{n\to\infty} P(S_n(v, \varepsilon)) = 0 . \tag{2.10}$$

Let k be an arbitrary positive integer. Since

$$\left\{x : |P(A : \mathscr{B}_n) - P(A : \mathscr{B}_0)| > \frac{1}{k}\right\} \subset \bigcup_{i=1}^{2k-1} S_n\left(\frac{i}{2k}, \frac{1}{2k}\right),$$

it follows from (2.10) that

$$\lim_{n\to\infty} P\left(\left\{x : |P(A : \mathscr{B}_n) - P(A : \mathscr{B}_0)| > \frac{1}{k}\right\}\right) \leqq \lim_{n\to\infty} \sum_{i=1}^{2k-1} P\left(S_n\left(\frac{i}{2k}, \frac{1}{2k}\right)\right) = 0 ;$$

that is to say, $P(A : \mathscr{B}_n)$ converges to $P(A : \mathscr{B}_0)$ in probability.

REMARK 2.1. According to Neveu [5], $\{\mathscr{B}_n\}$ is said to converge to $\mathscr{B}_0$ strongly if statement (i) of Theorem 2.1 holds. Hence, the other statements of Theorem 2.1 may be regarded as equivalent definitions of strong convergence. The second of these, statement (ii), which is based on the quantity $\int |E_{\mathscr{B}} f| \, dP$ associated with a σ-algebra $\mathscr{B}$, is suitable for generalization of the concept of limit, as is seen in the next section. The third, statement (iii), is more intuitive than the others. When $\mathscr{A}$ is regarded as a metric space equipped with a distance $P(A \triangle B)$, every σ-algebra is a closed subset of $\mathscr{A}$ and $d(A, \mathscr{B}) = \inf_{B \in \mathscr{B}} P(A \triangle B)$ denotes the distance of a point A from a closed set $\mathscr{B}$. In this geometrical setup, a new concept of convergence of σ-algebras is introduced as follows: a sequence $\{\mathscr{B}_n\}$ converges to $\mathscr{B}_0$ "in d-sense", if $d(A, \mathscr{B}_n)$ converges to $d(A, \mathscr{B}_0)$, as $n \to \infty$, for every A in $\mathscr{A}$. Since $d(A, \mathscr{B}) = \frac{1}{2} - \int |\frac{1}{2} - P(A : \mathscr{B})| \, dP$, according to Lemma 2.1, the convergence of $\{\mathscr{B}_n\}$ to $\mathscr{B}_0$ in d-sense is equivalent to $\lim_{n\to\infty} \int |\frac{1}{2} - P(A : \mathscr{B}_n)| \, dP = \int |\frac{1}{2} - P(A : \mathscr{B}_0)| \, dP$. Therefore it follows from Theorem 2.1 that convergence in d-sense is weaker than strong convergence. The pseudometric recently introduced by Boylan [1] is closely related to convergence in d-sense.

3. Existence of the upper and lower limits. In this section we are mainly concerned with the monotonicity of the expression $\int |E_{\mathscr{B}} f| \, dP$ pertaining to a σ-algebra $\mathscr{B}$. This property is given in the following

THEOREM 3.1. *$\int |E_{\mathscr{C}} f| \, dP \leqq \int |E_{\mathscr{B}} f| \, dP$ holds for every bounded $\mathscr{A}$-measurable f if and only if $\mathscr{C} \subset \mathscr{B}$.*

PROOF. If $\mathscr{C} \subset \mathscr{B}$, then by Jensen's inequality we have

$$\int |E_{\mathscr{B}} f| \, dP = \int E_{\mathscr{C}} |E_{\mathscr{B}} f| \, dP \geqq \int |E_{\mathscr{C}} E_{\mathscr{B}} f| \, dP = \int |E_{\mathscr{C}} f| \, dP .$$

Conversely, suppose that $\int |E_{\mathscr{C}} f| \, dP \leqq \int |E_{\mathscr{B}} f| \, dP$, and take a set C in $\mathscr{C}$. Putting $f = \frac{1}{2} - I_C$, we have $\frac{1}{2} = \int |\frac{1}{2} - P(C : \mathscr{C})| \, dP \leqq \int |\frac{1}{2} - P(C : \mathscr{B})| \, dP \leqq \frac{1}{2}$.

Hence $P(C : \mathscr{B}) = 0$ or 1, a.e., on X. Since $\mathscr{B}$ is complete, C belongs to $\mathscr{B}$, which completes the proof.

DEFINITION 3.1. A σ-algebra $\mathscr{B}_0$ is called the *lower limit* of $\{\mathscr{B}_n\}$, and denoted by P-lim inf $\mathscr{B}_n$, if

(a) $\mathscr{B} = \mathscr{B}_0$ satisfies

$$\liminf_{n\to\infty} \int |E_{\mathscr{B}_n} f|\, dP \geqq \int |E_{\mathscr{B}} f|\, dP \tag{3.1}$$

for every bounded $\mathscr{A}$-measurable f, and

(b) any σ-algebra $\mathscr{B}$ satisfying (3.1) is contained in $\mathscr{B}_0$.

DEFINITION 3.2. A σ-algebra $\mathscr{B}_0$ is called the *upper limit* of $\{\mathscr{B}_n\}$, and denoted by P-lim sup $\mathscr{B}_n$, if

(a) $\mathscr{B} = \mathscr{B}_0$ satisfies

$$\limsup_{n\to\infty} \int |E_{\mathscr{B}_n} f|\, dP \leqq \int |E_{\mathscr{B}} f|\, dP \tag{3.2}$$

for every bounded $\mathscr{A}$-measurable f, and

(b) any σ-algebra $\mathscr{B}$ satisfying (3.2) contains $\mathscr{B}_0$ as its subset.

THEOREM 3.2. *For any sequence $\{\mathscr{B}_n\}$ of-algebras, the class of sets*

$$\mathscr{B}_0 = \{A \in \mathscr{A} : \lim_{n\to\infty} \inf_{B \in \mathscr{B}_n} P(A \triangle B) = 0\}$$

is the lower limit of $\{\mathscr{B}_n\}$.

PROOF. From the inequalities

$$P((A_1 \cap A_2) \triangle (B_1 \cap B_2)) \leqq P(A_1 \triangle B_1) + P(A_2 \triangle B_2)$$

and

$$P((A_1 \cup A_2) \triangle (B_1 \cup B_2)) \leqq P(A_1 \triangle B_1) + P(A_2 \triangle B_2)\,,$$

for any sets A_1, A_2, B_1 and B_2 in $\mathscr{A}$, it follows directly that $A_1 \in \mathscr{B}_0$ and $A_2 \in \mathscr{B}_0$ imply $A_1 \cap A_2 \in \mathscr{B}_0$ and $A_1 \cup A_2 \in \mathscr{B}_0$. Suppose that B_n, $n = 1, 2, \cdots$, are in $\mathscr{B}_0$, and take

$$B = \bigcup_{n=1}^{\infty} B_n \qquad \text{and} \qquad B_{(m)} = \bigcup_{n=1}^{m} B_n\,.$$

Let ε be a given positive number. There is a positive integer m such that $P(B \triangle B_{(m)}) < \varepsilon/2$. Since $B_{(m)}$ is in $\mathscr{B}_0$ as just shown, there are a positive integer N and a set C_n in $\mathscr{B}_n$ for each $n \geqq N$ such that $P(C_n \triangle B_{(m)}) < \varepsilon/2$. Hence $\inf_{B_n \in \mathscr{B}_n} P(B \triangle B_n) \leqq P(B \triangle C_n) \leqq P(B \triangle B_{(m)}) + P(B_{(m)} \triangle C_n) < \varepsilon$ for each $n \geqq N$. Since ε is arbitrary, we have $B \in \mathscr{B}_0$. Thus, $\mathscr{B}_0$ is closed under the formation of countable unions. Since $\mathscr{B}_0$ is also closed under the formation of complements, $\mathscr{B}_0$ is a σ-algebra. Next we shall show that $\mathscr{B} = \mathscr{B}_0$ satisfies (3.1). According to Lemma 2.1, any set B in $\mathscr{B}_0$ satisfies $\lim_{n\to\infty} \int |\frac{1}{2} - P(B : \mathscr{B}_n)|\, dP = \int |\frac{1}{2} - P(B : \mathscr{B}_0)|\, dP$. Therefore, by Lemma 2.3, (3.1) holds with $\mathscr{B} = \mathscr{B}_0$. Lastly, we shall show that a σ-algebra $\mathscr{B}$ satisfying (3.1) is a subset of $\mathscr{B}_0$. Suppose that $\mathscr{B}$ satisfies (3.1) and B is a member of $\mathscr{B}$. Taking $f = \frac{1}{2} - I_B$ in (3.1), we have $\frac{1}{2} \geqq \limsup_{n\to\infty} \int |\frac{1}{2} - P(B : \mathscr{B}_n)|\, dP \geqq \liminf_{n\to\infty} \int |\frac{1}{2} - P(B : \mathscr{B}_n)|\, dP \geqq \int |\frac{1}{2} - P(B : \mathscr{B})|\, dP = \int |\frac{1}{2} - I_B|\, dP = \frac{1}{2}$. Thus we have $\lim_{n\to\infty} \int |\frac{1}{2} - P(B : \mathscr{B}_n)|\, dP = \frac{1}{2}$, that is, by Lemma 2.1, $\lim_{n\to\infty} \inf_{B_n \in \mathscr{B}_n} P(B \triangle B_n) = 0$. Thus we have $B \in \mathscr{B}_0$, which completes the proof.

Now we shall proceed to establish the existence of the upper limit of $\mathscr{B}_n$.

LEMMA 3.1. *If a σ-algebra $\mathscr{B}$ satisfies* (3.2), *then we have*

$$\limsup_{n\to\infty} \int |E_{\mathscr{B}_n} E_{\mathscr{B}} f| \, dP = \limsup_{n\to\infty} \int |E_{\mathscr{B}_n} f| \, dP .$$

PROOF. Replacing f in (3.2) by $E_{\mathscr{B}} f - f$, we have

$$\begin{aligned} \limsup_{n\to\infty} \int |E_{\mathscr{B}_n}(E_{\mathscr{B}} f - f)| \, dP &\leqq \int |E_{\mathscr{B}}(E_{\mathscr{B}} f - f)| \, dP \\ &= \int |E_{\mathscr{B}} E_{\mathscr{B}} f - E_{\mathscr{B}} f| \, dP \\ &= \int |E_{\mathscr{B}} f - E_{\mathscr{B}} f| \, dP = 0 . \end{aligned}$$

Therefore,

$$\begin{aligned} &|\limsup_{n\to\infty} \int |E_{\mathscr{B}_n} E_{\mathscr{B}} f| \, dP - \limsup_{n\to\infty} \int |E_{\mathscr{B}_n} f| \, dP| \\ &\qquad \leqq \limsup_{n\to\infty} \int |E_{\mathscr{B}_n}(E_{\mathscr{B}} f - f)| \, dP = 0 , \end{aligned}$$

which completes the proof.

LEMMA 3.2. *Denote by* $\mathbf{U}$ *the family of all σ-algebras $\mathscr{B}$ satisfying* (3.2). *Then*

(i) *if $\mathscr{B}$ and $\mathscr{C}$ are members of* $\mathbf{U}$, *so is* $\mathscr{B} \cap \mathscr{C}$,
(ii) *if $\mathscr{B} \in \mathbf{U}$ and $\mathscr{B} \subset \mathscr{C}$, then $\mathscr{C} \in \mathbf{U}$.*

PROOF. The statement (ii) is clear from Theorem 3.1. To prove (i), let $\mathscr{B}$ and $\mathscr{C}$ be members of $\mathbf{U}$. We shall verify by mathematical induction that

$$\limsup_{n\to\infty} \int |E_{\mathscr{B}_n} f| \, dP \leqq \int |(E_{\mathscr{B}} E_{\mathscr{C}})^m f| \, dP \tag{3.3}$$

for $m = 1, 2, \cdots$. Replacing f in (3.2) by $E_{\mathscr{C}} f$, and applying Lemma 3.1, we can easily obtain (3.3) for $m = 1$. Suppose that (3.3) holds for some m. Replacing f in (3.3) by $E_{\mathscr{B}} E_{\mathscr{C}} f$ and applying Lemma 3.1 twice, we get $\int |(E_{\mathscr{B}} E_{\mathscr{C}})^{m+1} f| \, dP \geqq \limsup_{n\to\infty} \int |E_{\mathscr{B}_n} E_{\mathscr{B}} E_{\mathscr{C}} f| \, dP = \limsup_{n\to\infty} \int |E_{\mathscr{B}_n} E_{\mathscr{C}} f| \, dP = \limsup_{n\to\infty} \int |E_{\mathscr{B}_n} f| \, dP$. Thus (3.3) has been proved for all m. Now we shall prove that $\mathscr{B} \cap \mathscr{C} \in \mathbf{U}$. Since a bounded $\mathscr{A}$-measurable function belongs to $L^2(X, \mathscr{A}, P)$, the operator $E_{\mathscr{B}}$ is regarded as the projection of $L^2(X, \mathscr{A}, P)$ into itself. It is well known that $(E_{\mathscr{B}} E_{\mathscr{C}})^m f$ converges to $E_{\mathscr{B} \cap \mathscr{C}} f$ in the L^2-sense as $m \to \infty$ (see [2; Theorem 3]). Therefore by the Schwartz inequality,

$$\begin{aligned} \lim_{m\to\infty} &|\int |(E_{\mathscr{B}} E_{\mathscr{C}})^m f| \, dP - \int |E_{\mathscr{B} \cap \mathscr{C}} f| \, dP| \\ &\leqq \lim_{m\to\infty} \int |(E_{\mathscr{B}} E_{\mathscr{C}})^m f - E_{\mathscr{B} \cap \mathscr{C}} f| \, dP \\ &\leqq \lim_{m\to\infty} (\int |(E_{\mathscr{B}} E_{\mathscr{C}})^m f - E_{\mathscr{B} \cap \mathscr{C}} f|^2 \, dP)^{\frac{1}{2}} \cdot 1 = 0 . \end{aligned}$$

Hence, from (3.3) it follows that $\limsup_{n\to\infty} \int |E_{\mathscr{B}_n} f| dP \leqq \lim_{m\to\infty} \int |(E_{\mathscr{B}} E_{\mathscr{C}})^m f| dP = \int |E_{\mathscr{B} \cap \mathscr{C}} f| \, dP$, which shows that $\mathscr{B} \cap \mathscr{C} \in \mathbf{U}$.

THEOREM 3.3. *The upper limit of $\{\mathscr{B}_n\}$ exists, whatever the sequence $\{\mathscr{B}_n\}$ may be.*

PROOF. Let $\{\mathscr{B}_\gamma : \gamma \in \Gamma\}$ be a monotone nonascending generalized sequence of elements of $\mathbf{U}$. Since the generalized limit of $E_{\mathscr{B}_\gamma} f$ in probability exists and coincides with $E_{\mathscr{B}^*}f$, $\mathscr{B}^* = \bigcap_{\gamma \in \Gamma} \mathscr{B}_\gamma$, for every bounded $\mathscr{A}$-measurable f (see [5] page 124), $\mathscr{B}^* = \bigcap_{\gamma \in \Gamma} \mathscr{B}_\gamma$ belongs to **U**. Therefore, by Zorn's lemma, there is a minimal member $\mathscr{B}_0$ in **U**. If it were not the smallest member in U, then we could choose a σ-algebra $\mathscr{C}$ in **U** such that $\mathscr{B}_0 \cap \mathscr{C} \neq \mathscr{B}_0$. By Lemma 3.2, however, $\mathscr{B}_0 \cap \mathscr{C} \in$ **U**, which contradicts the assumption that $\mathscr{B}_0$ is minimal in **U**. Thus $\mathscr{B}_0$ must be the smallest in **U**.

THEOREM 3.4. *For any sequence $\{\mathscr{B}_n\}$ of σ-algebras, we have*

$$P\text{-}\lim\inf_{n\to\infty} \mathscr{B}_n \subset P\text{-}\lim\sup_{n\to\infty} \mathscr{B}_n ,$$

where the sign "$\subset$" is replaced by the equality sign if and only if $\{\mathscr{B}_n\}$ is strongly convergent. In this case P-lim inf $\mathscr{B}_n$ is the strong limit of $\{\mathscr{B}_n\}$.

The proof is direct from the definitions of the upper and lower limits of $\{\mathscr{B}_n\}$.

We shall now give an example of a case in which

$$P\text{-}\lim\sup \mathscr{B}_n \neq P\text{-}\lim\inf \mathscr{B}_n .$$

EXAMPLE 3.1. Take an interval $X = [0. 1)$ and the family $\mathscr{A}$ of all Borel sets in X. Let P be a probability measure on $\mathscr{A}$ defined by $P(A) = \frac{1}{2}(|A| + I_A(0))$ for $A \in \mathscr{A}$, where $|\cdot|$ stands for Lebesgue measure. Let $0.x_1x_2 \cdots$ be the binary expansion of an $x \in [0, 1)$; in particular, $0 = 0.000\cdots$. Take a set $B_n = \{x : x_n = 0\}$ and consider the complete σ-algebra $\mathscr{B}_n$ generated by $\{X, \emptyset, B_n, X - B_n\}$. Note that there is no ambiguity in defining $\mathscr{B}_n$, because B_n is determined except for a set of Lebesgue measure zero. Define probability distribution functions F_n by $F_n(x) = 0$ for $x < 0$; $= 2 \int_{B_n \cap [0,x]} dx$ for $0 \leqq x \leqq 1$; $= 1$ for $x > 1$, for $n = 1, 2, \cdots$. From the fact that the family $C[0, 1]$ of continuous functions on $[0, 1]$ is dense in $L^1[0, 1]$ in the L^1-sense, for any given bounded measurable function f on $[0, 1]$ and any given $\varepsilon > 0$ there is an element g in $C[0, 1]$ such that $\int_0^1 |f - g|\, dx < \varepsilon$ and consequently $\int |f - g|\, dF_n < 2\varepsilon$ for every n. Therefore we have $|\int f\, dF_n - \int f\, dF| \leqq \int |f - g|\, dF_n + |\int g\, dF_n - \int g\, dF| + \int |f - g|\, dF < 3\varepsilon + |\int g\, dF_n - \int g\, dF|$, where F is the probability distribution function of the uniform distribution on $[0, 1]$. Since $\lim_{n\to\infty} F_n(x) = F(x)$, by the Helly–Bray theorem (see [4]) we have $\lim_{n\to\infty} \int g\, dF_n = \int g\, dF$, and hence $\lim_{n\to\infty} |\int f\, dF_n - \int f\, dF| < 3\varepsilon$. Since ε is arbitrary, we have $\lim_{n\to\infty} \int_{B_n} f(x)\, dx = \frac{1}{2} \lim_{n\to\infty} \int f\, dF_n = \frac{1}{2} \int_0^1 f(x)\, dx$. Similarly, $\lim_{n\to\infty} \int_{X-B_n} f(x)\, dx = \frac{1}{2} \int_0^1 f(x)\, dx$.

By a simple calculation, we get $E_{\mathscr{B}_n} f = \frac{1}{2}\{\int_{B_n} f(x)\, dx + f(0)\}/(\frac{1}{2} + \frac{1}{4})$ for $x \in B_n$ and $= \frac{1}{2} \int_{X-B_n} f(x)\, dx/\frac{1}{4}$ for $x \notin B_n$. Therefore we have

$$\begin{aligned} \lim_{n\to\infty} \int |E_{\mathscr{B}_n} f|\, dP &= \tfrac{1}{2}|\lim_{n\to\infty} \textstyle\int_{B_n} f\, dx + f(0)| + \tfrac{1}{2}|\lim_{n\to\infty} \int_{X-\mathscr{B}_n} f\, dx| \\ &= \tfrac{1}{2}|f(0) + \tfrac{1}{2} \textstyle\int_0^1 f(x)\, dx| + \tfrac{1}{4}|\int_0^1 f(x)\, dx| . \end{aligned}$$

Take $\frac{1}{2} - I_E$ for f in the above expression of $\lim \int |E_{\mathscr{B}_n} f|\, dP$, where E is a measurable subset of $[0, 1]$. Since $\frac{1}{4}\{|\frac{3}{2} - 2I_E(0) - |E|| + |\frac{1}{2} - |E||\} = \frac{1}{2}$ if and only if $E = X$ or $\emptyset$(P-a.e.), by Theorem 3.2 and Lemma 2.1 we have P-lim inf $\mathscr{B}_n = \{X, \emptyset\}$, and hence $\int |E_{P\text{-}\lim\inf \mathscr{B}_n} f|\, dP = \frac{1}{2}|\int_0^1 f\, dx + f(0)|$. On the other hand, the completion $\mathscr{B}_0$ of $\{X, \emptyset, (0, 1), \{0\}\}$ is the upper limit of $\mathscr{B}_n$, because $\int |E_{\mathscr{B}_0} f| \, dP = \frac{1}{2}\{|\int_0^1 f \, dx| + |f(0)|\} \geqq \lim_{n\to\infty} \int |E_{\mathscr{B}_n} f| \, dP$ and there is no proper sub-σ-algebra of $\mathscr{B}_0$ satisfying this inequality.

REMARK 3.1. The set theoretical definition of the upper and lower limits should be $\bigcap_{k=1}^{\infty} \bigvee_{n=k}^{\infty} \mathscr{B}_n$ and $\bigvee_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \mathscr{B}_n$, respectively, where $\bigvee$ denotes the formation of the smallest σ-algebra containing all σ-algebras following the sign. If $\{\mathscr{B}_n\}$ is monotone, these limits coincide with ours. However, in general, this is not true. As is easily shown, we have $\bigvee_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \mathscr{B}_n \subset P\text{-}\liminf \mathscr{B}_n \subset P\text{-}\limsup \mathscr{B}_n \subset \bigcap_{k=1}^{\infty} \bigvee_{n=k}^{\infty} \mathscr{B}_n$.

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