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PERCOLATION CRITICAL EXPONENTS UNDER THE TRIANGLE CONDITION

BY D. J. BARSKY[1] AND M. AIZENMAN[2]

University of California, Davis, and Courant Institute of Mathematical Sciences

For independent percolation models, it is shown that if the diagrammatic "triangle condition" is satisfied, then the critical exponents δ and $\hat{\beta}$ exist and take their mean-field values, generalizing the criterion introduced in 1984 by Aizenman and Newman for the mean-field value of γ in nonoriented percolation. The results apply to a broad class of nonoriented, as well as oriented, weakly homogeneous models, in which the range of the connecting bonds need not be bounded. For the nonoriented case, the condition reduces to the finiteness at the critical point of $\nabla \equiv \sum_{x,y}\tau(0,x)\tau(x,y)\tau(y,0)$ [with $\tau(u,v)$ the probability that the site u is connected to v], which was recently established by Hara and Slade for models with sufficiently spread out connections in $d > 6$ dimensions. Our analysis proceeds through the derivation of complementary differential inequalities for the percolation order parameter $M(\beta, h)$—whose value at $h = 0+$ yields the percolation density, with β parametrizing the bond, or site, occupation probabilities and with h, $h \geq 0$, a "ghost field." The conclusion is that under the triangle condition, in the vicinity of the critical point $(\beta_c, 0)$, $M(\beta, 0+) \approx (\beta - \beta_c)_+^{\hat{\beta}}$ and $M(\beta_c, h) \approx h^{1/\delta}$, with $\hat{\beta} = 1$ and $\delta = 2$.

1. Introduction. For percolation models (with and without orientation), the triangle condition is symptomatic of the reduction to mean-field (Bethe lattice) critical behavior occurring in high dimensions. For nonoriented percolation, the condition reduces to finiteness of the triangle diagram at the percolation threshold: $\nabla(p_c) < \infty$. The criterion was introduced, within that context, in Aizenman and Newman (1984), where it was proven to imply that the expected cluster size diverges with the mean-field power law: $\chi = E(|C(0)|) \approx (p - p_c)^{-\gamma}$ as $p \uparrow p_c$, with

$$\gamma = 1. \tag{1.1}$$

That result is extended here to the critical exponents δ and $\hat{\beta}$, which concern the critical behavior along other directions of approach to the critical point in the relevant two-parameter space. Specifically, it is shown that, in models for which the triangle condition is satisfied, δ and $\hat{\beta}$ are also well-defined—in the

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[2]Currently at Princeton University. Research supported in part by NSF Grants PHY-88-96163 and PHY-89-12067.

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sense that the corresponding quantities obey upper and lower power-law bounds—and the exponents assume their mean-field values:

$$\delta = 2 \quad \text{and} \quad \hat{\beta} = 1. \tag{1.2}$$

A feature common to the two sets of results is that, under the triangle condition, various differential inequalities which have been noted to convey "nonperturbative" information on the critical regime can be reversed through the insertion of factors which are nonvanishing and nonsingular in the neighborhood of the critical point. In the case of (1.1), that refers to an ordinary differential inequality for $\chi(p)$. Equation (1.2) rests on some partial differential inequalities.

We shall now make the preceding statements more explicit and mention their relations with other works. Our results apply to independent bond and site percolation models, on general lattices, which are "weakly homogeneous" and either "well-connected" or "uniformly long-range" (see Section 2). The bonds may carry orientation, and they may be of any range. For simplicity of presentation, we focus in this introduction on nearest-neighbor bond models on $\mathbb{Z}^d$ where the bonds (pairs of neighboring sites) are "occupied" with probability p independently of each other.

In percolation theory [the mathematics of which is discussed in the texts of Kesten (1982) and Grimmett (1989)], one studies the structure of the cluster $C(x)$ of the sites to which a site x is connected by means of the randomly occupied lattice bonds (or sites). Note that, for oriented percolation, the relation "x is connected to y" ($x \to y$) is not symmetric, and $C(x)$ denotes the "forward cluster of x." Of particular interest are the percolation density $P_\infty(p)$ and, more completely, the cluster size distribution, described by

$$P_n(p) = \text{prob}(|C(0)| = n), \tag{1.3}$$

with $1 \le n \le \infty$ and $|C|$ the number of sites in C. A generating function for $\{P_n\}$ is

$$M(p,h) = \sum_{1 \le n \le \infty} P_n(p)[1 - e^{-nh}], \qquad h > 0, \tag{1.4a}$$

with

$$M(p,0) \equiv M(p,0+) = P_\infty(p). \tag{1.4b}$$

The critical exponents mentioned previously are defined through the following power laws in the singular behavior of $M(p, h)$—which are expected to hold near the critical point ($p = p_c$, $h = 0$):

$$M(p,0+) \approx (p - p_c)^{\hat{\beta}} \quad \text{for } p > p_c \text{ (and } h = 0), \tag{1.5}$$

$$M(p_c,h) \approx h^{-1/\delta} \quad \text{for } (p = p_c \text{ and) } h > 0 \tag{1.6}$$

and

$$\frac{\partial M}{\partial h}(p,0+) \approx (p_c - p)^{-\gamma} \quad \text{for } p < p_c \text{ (and } h = 0). \tag{1.7}$$

The quantity $M(p,h)$ admits a geometric interpretation (see Section 2) which was quite helpful in the derivation of the following pair of inequalities [Aizenman and Barsky (1987)]:

$$M - h\frac{\partial M}{\partial h} \le pM\frac{\partial M}{\partial p} + M^2 \tag{1.8}$$

and

$$\frac{\partial M}{\partial p} \le \frac{2d}{1-p}M\frac{\partial M}{\partial h}. \tag{1.9}$$

In particular, (1.8) and (1.9) yield

$$M - h\frac{\partial M}{\partial h} \le \frac{2dp}{1-p}M^2\frac{\partial M}{\partial h} + M^2, \tag{1.10}$$

which is an ordinary differential inequality along lines of constant p.

In this paper, we prove that, for models in which the triangle condition holds, the preceding differential inequalities can be reversed throughout the regime $\mathscr{R} = \{(p,h)\colon\ p \le p_c,\ h \ge 0\}$; for example, (1.10) can be supplemented by

$$M - h\frac{\partial M}{\partial h} \ge c(p,h)M^2\frac{\partial M}{\partial h} \quad \text{for } (p,h) \in \mathscr{R}, \tag{1.11}$$

with $c(p,h)$ a continuous function, satisfying $c(p_c, 0) > 0$. [The region $\mathscr{R}$ arises here for technical reasons. However, results about $\mathscr{R}$ can be extrapolated to the supercritical regime $\{(p,h)\colon\ p > p_c,\ h \ge 0\}$ by the technique of Aizenman and Fernández (1986).]

Inequalities (1.8) and (1.9), along with some simple monotonicity properties, imply the coincidence of two natural notions of the critical point p_c. Additionally, they yield the bounds

$$\delta \ge 1/\hat{\beta} + 1 \ge 2. \tag{1.12}$$

The integration of (1.11) shows that $\delta \le 2$ under the triangle condition, and thus the two inequalities in (1.12) are saturated. More completely, the triangle condition implies that (1.2) holds in the sense that there exist constants $C_i \in (0,\infty)$ such that for (p,h) in some neighborhood of $(p_c, 0)$

$$C_1 h^{1/2} \le M(p_c, h) \le C_2 h^{1/2}, \qquad h \ge 0 \tag{1.13}$$

and

$$C_3(p - p_c) \le M(p, 0+) \le C_4(p - p_c), \qquad p \ge p_c. \tag{1.14}$$

We mention that (1.10) and (1.11) additionally show that

$$C_5 h^{-1/2} \le \frac{\partial M(p_c, h)}{\partial h} \le C_6 h^{-1/2}.$$

All that remains to make our statements concrete is to define the triangle condition. For the models considered in this section, the following quantity is referred to as the open-triangle function:

$$\nabla_R(p) = \sup\{\nabla(p;z): z \in \mathbb{Z}^d, |z| \ge R\}, \tag{1.15a}$$

where

$$\nabla(p;z) = \sum_{x,y} \tau_p(0,x)\tau_p(x,y)\tau_p(z,y) \tag{1.15b}$$

and $\tau_p(x,y)$ denotes the probability that x is connected to y (at bond density p). The triangle condition [for these models—see (2.8) for the general case] is

$$\lim_{R\to\infty} \nabla_R(p_c) = 0. \tag{1.16}$$

The "(closed-) triangle diagram" is

$$\nabla(p) \approx \nabla(p;0) = \frac{1}{|B|}\int_B |\hat{\tau}_p(k)|^2\hat{\tau}_p(k)\,dk, \tag{1.17}$$

where B is the Brillouin zone appropriate for the lattice; for example, $B = [0,2\pi]^d$ and $|B| = (2\pi)^d$ for $\mathbb{Z}^d$. In nonoriented percolation models, $\hat{\tau}_p(\cdot)$ [the Fourier transform of $\tau_p(\cdot)$] is nonnegative, and a sufficient condition for (1.16) is (Lemma 2.1)

$$\nabla(p_c) < \infty. \tag{1.18}$$

Since $\nabla(p_c) = \lim_{p\uparrow p_c} \nabla(p)$ [and $\nabla(p)$ is increasing in p], condition (1.18) is equivalent to the uniform boundedness of $\nabla(p)$ throughout the subcritical regime.

REMARK 1. An explicit implication of (1.13) and (1.14) is that

$$P_\infty(p_c) = 0, \tag{1.19}$$

that is, the percolation density is continuous at the critical point. In the nonoriented case, a considerably weaker condition for (1.19) is contained in Aizenman, Kesten and Newman (1987): that $\tau_{p_c}(x,y) \to 0$ as $|x-y| \to \infty$. Also, a lattice-animal argument of Newman (1986) shows that (1.19) holds if $\gamma < 2$, which is implied by the triangle condition.

REMARK 2. Our result (1.2) is closely related to the inequality

$$\frac{1}{\delta} \ge \frac{2-\gamma}{2} \tag{1.20}$$

of Newman (1986). Combining this inequality with (1.12) shows that when γ takes on its mean-field value ($\gamma = 1$), then so do δ and $\hat{\beta}$. However, the derivation of (1.20) leaves room for logarithmic deviations in the upper bounds for M which are not present in (1.13) and (1.14).

REMARK 3. In another result concerning the triangle criterion, Nguyen (1987) showed that the "gap exponents" (characterizing the divergence of higher moments of the cluster size) also assume their mean-field values

($\Delta_k = 2$, $k \geq 2$) when $\gamma = 1$, and thus, in particular, when the triangle condition is satisfied.

REMARK 4. In (1.12), the relation between δ and $\hat{\beta}$ is based on (1.9) and is identical to that satisfied in some ferromagnetic spin models which also obey (1.9). In that context, the implication was noted by Newman (1987a). The bound on $\hat{\beta}$ was first derived by Chayes and Chayes (1986).

The reader is referred to Aizenman and Newman (1984) for comments on the relation of the triangle criterion to the notion that the upper critical dimension for percolation is $d_{\text{uc}} = 6$, for the nonoriented case. Similar arguments suggest $d_{\text{uc}} = 5$ for d-dimensional percolation which is oriented in one direction. One expects that in models with sufficiently slow decay for the density of long bonds the triangle condition is satisfied even in low dimensions—in analogy with the fact that the effective dimensionality of a ferromagnetic spin system may be increased by long-range interaction [Fisher, Ma and Nickel (1972) and Aizenman and Fernández (1988)].

It may now be added [as a side benefit of the delay in the preparation of this manuscript, whose main results were included (with more involved derivations) in the dissertation work Barsky (1987)], that the triangle condition was recently established by Hara and Slade (1990) for nonoriented percolation models in high dimensions. Their current results cover $d > 6$ for finite-range models having sufficiently "spread-out" connections, and much larger d for the nearest-neighbor model.

2. The setup. In this section we present the general percolation models to which our results apply, establish the notation used in the next section and recall some useful inequalities.

The setup is that of partially oriented percolation (POP) as in Aizenman and Barsky (1987). We denote by $\mathbb{L}$ the set of sites and assume it to be a lattice with the following properties: $\mathbb{L}$ is equipped with a metric $\text{dist}(x, y)$ and is invariant under the action of a group $\mathscr{G}$ of translations which are isometries with the additional property $\text{dist}(Tx, x) = \text{dist}(Ty, y) \equiv |T|$, for all $x, y \in \mathbb{L}$. We write $\Lambda_L(x)$ for the box $\{y\colon \text{dist}(x, y) \leq L\}$ and assume that $V(L) = \sup\{|\Lambda_L(x)|\colon x \in \mathbb{L}\}$ is finite.

The lattice bonds are a set, denoted by $\mathbb{B}$ and closed under the action of $\mathscr{G}$, of unordered and ordered pairs of sites in $\mathbb{L}$. For each bond $b \in \mathbb{B}$, there is an (occupation status) indicator random variable n_b, $n_b \in \{0, 1\}$. When occupied ($n_b = 1$), a nonoriented bond $b = \{x, y\}$ connects both x to y and y to x, while an oriented bond $b = (x, y)$ connects only x to y. A bond configuration is a specification of the values of the independent random variables $\{n_b\}_{b \in \mathbb{B}}$. In the models discussed here, their distribution is a product probability measure, invariant under the induced action of $\mathscr{G}$, with

$$\text{prob}(n_b = 1) = \begin{cases} 1 - e^{-\beta J_{x,y}}, & b = \{x, y\} \\ 1 - e^{-\beta J_{x \to y}}, & b = (x, y), \end{cases} \tag{2.1}$$

with $J_{x,y} = J_{Tx,Ty} > 0$ and $J_{x\to y} = J_{Tx\to Ty} > 0$ for all bonds [$\{x, y\}$ nonoriented and (x, y) oriented] in $\mathbb{B}$ and each $T \in \mathscr{G}$. We sometimes write $J_{[x,y]}$ for $J_{x,y} + J_{x\to y}$. The notion of cluster is as in the introduction. Such models were first considered by Hammersley (1957).

Each site $x \in \mathbb{L}$ has a cluster size distribution, $P_n(\beta; x) = \text{prob}(|C(x)| = n)$, which can be used to define a quantity $M(\beta, h; x)$ as in (1.4) with the role of the bond density parameter played by β (instead of p of the nearest-neighbor models). Some fixed site $0 \in \mathbb{L}$ is referred to as the origin, and we write M for $M(\beta, h; 0)$. When the parameters β and h are understood from the context, we will write $M(x)$ for $M(\beta, h; x)$.

In the multiparameter space, $\{\beta, J_{x,y}, J_{x\to y}, h\}$—with h added for convenience—there is a "critical manifold" (along which $h = 0$). We study the critical behavior of the model as this "manifold" is approached by varying (β, h) with $\{J\}$ held fixed, satisfying

$$|J| \equiv \sup\left\{\sum_y (J_{x,y} + J_{x\to y}) : x \in \mathbb{L}\right\} < \infty. \tag{2.2}$$

For every such set of parameters $\{J\}$, we find a critical value of β,

$$\beta_c = \sup\{\beta : M(\beta, 0; 0) = 0\}. \tag{2.3}$$

The parameter h [the conjugate to n in (1.4a)] is turned into a ghost field by the introduction of independent random site variables $\{m_x\}_{x\in\mathbb{L}}$, with values in $\{0, 1\}$, for which $\text{prob}(m_x = 1) = 1 - e^{-h}$. Sites x having $m_x = 1$ are referred to as *green*, and their collection is denoted by G. For $h > 0$, the quantities M and $\partial M/\partial h$ admit the convenient geometric representations

$$M(\beta, h; x) = \text{prob}(x \to G) \tag{2.4}$$

and

$$\frac{\partial M}{\partial h} \equiv \frac{\partial M}{\partial h}(\beta, h; 0) = \sum_x \text{prob}(0 \to x, 0 \nrightarrow G). \tag{2.5}$$

We use here the convention that "$x \to D$," for x a site and D a (possibly random) set, means that x is connected to some site in D; $x \nrightarrow D$ denotes the complementary event.

On occasion we will want to specify that the site x is connected in the complement of a subset $A \subset \mathbb{L}$ (i.e., by a path of occupied bonds none of which have endpoints in A) to a site y or a set D. We refer to these events by "$x \to y$ off A" or "$x \to D$ off A" and denote by $C_{A^c}(x)$ the corresponding (forward) cluster of x. The similar notation "$x \to D$ off $[y, z]$" is used to denote the event that x is connected to D by a path of occupied bonds using neither $\{y, z\}$ nor (y, z).

We modify the triangle condition as follows. The open triangle diagram is now defined to be

$$\nabla_R(\beta) \equiv \sup\{\nabla(\beta; T) : T \in \mathscr{G}, |T| \geq R\}, \tag{2.6}$$

with

$$\nabla(\beta;T) = \sup\left\{ \sum_{x,y} \tau_\beta(w,x)\tau_\beta(x,y)\tau_\beta(Tw,y) \colon w \in \mathbb{L}\right\}. \tag{2.7}$$

The triangle condition is

$$\lim_{R\to\infty} \nabla_R(\beta_c) = 0. \tag{2.8}$$

For nonoriented percolation, the condition may be stated in a simpler form. The more useful portion of the equivalence statement is seen in the following lemma.

LEMMA 2.1. *For nonoriented, translation-invariant percolation models on* $\mathbb{Z}^d$, $\nabla(\beta_c) < \infty$ *implies* $\nabla_R(\beta_c) \to 0$ *as* $R \to \infty$.

PROOF. If $\nabla(\beta_c) < \infty$, then $\tau_{\beta_c}(0,x)$ is square-summable over $\mathbb{Z}^d$, and hence it has a Fourier transform $\hat{\tau}_{\beta_c}(k)$—which exists as an L_2 function on the torus $[0,2\pi]^d$. By the positivity of $\hat{\tau}(\cdot)$ [proven in Aizenman and Newman (1984)] we can write

$$\nabla(\beta_c) = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} \hat{\tau}_{\beta_c}(k)^3\,dk. \tag{2.9}$$

Hence $\nabla(\beta_c) < \infty$ implies actually that $[\hat{\tau}_{\beta_c}(k)]^3 \in L_1([0,2\pi]^d)$ and

$$\nabla(\beta_c;T) = \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} \hat{\tau}_{\beta_c}(k)^3 e^{-ik\cdot T0}\,dk. \tag{2.10}$$

By the Riemann–Lebesgue lemma, the quantity in (2.10) tends to zero as $|T| \to \infty$. □

At one point of the analysis we require the model $(\mathbb{L},\{J\})$ to be well-connected in the sense defined next, unless it has bonds of unbounded range.

DEFINITION. A POP model is well-connected with respect to a translation T if the following holds for some $L = L(T) < \infty$: Given any pair of paths (made of bonds in $\mathbb{B}$, with oriented bonds being used only in the appropriate orientation) which connect pairs of sites $w \to x$ and $Tx \to z$ with $z \in \Lambda_L(x)^c$, there exists a path connecting $w \to z$ which differs from the union of the first two paths only within $\Lambda_L(x)$. The model is said to be *well-connected* if, for each $R < \infty$, it is well-connected with respect to each of a pair of translations T_1 and T_2 with $|T_1|, |T_2|, |T_1^{-1}T_2| \geq R$.

To see, for example, that d-dimensional percolation with oriented bonds in only one direction is well-connected, consider translations along the axis of orientation.

The well-connectedness assumption is not necessary for models having enough long-range connections. For example, it can be avoided if it is known that for each $R < \infty$ there exists a translation T with $|T| \geq R$ such that $\inf\{J_{x,Tx} + J_{x \to Tx}\colon x \in \mathbb{L}\} > 0$. The next definition provides a simple generalization of this situation that is particularly appropriate to lattices which can be tiled by a finite cell (of N sites).

DEFINITION. A POP model is said to be *uniformly long-range* if, for some $N < \infty$ and every $R < \infty$, there exist N distinct translations $T_1, \dots, T_N \in \mathscr{G}$ with $|T_i| \geq R$ and $\inf\{\sum_{i=1}^N J_{x, T_i x} + J_{x \to T_i x}\colon x \in \mathbb{L}\} > 0$.

It is assumed throughout that the models have the following property, which is certainly satisfied by indecomposable models on lattices with finite tiles.

DEFINITION. A POP model is *weakly homogeneous* if there exists a continuous function $\mu = \mu(\beta) \in (0, \infty)$ such that, for h sufficiently small,

$$\mu^{-1} M(\beta, h; 0) \leq M(\beta, h; x) \leq \mu M(\beta, h; 0). \tag{2.11}$$

REMARK. Our argument requires (2.11) only for those sites x which can be reached from 0.

In proving the differential inequality (1.11) on which the results of this paper are based, we will make use of the following "diagrammatic inequalities."

PROPOSITION 2.2. *For any $A, D \subset \mathbb{L}$ and $x, y, z \in \mathbb{L}$,*

$$\operatorname{prob}(x \to D \textit{ off } A) \geq \operatorname{prob}(x \to D) - \sum_{u \in A} \operatorname{prob}(x \to u) \operatorname{prob}(u \to D), \tag{2.12}$$

$$\operatorname{prob}(x \to y,\, x \to D) \leq \sum_{v} \operatorname{prob}(x \to v) \operatorname{prob}(v \to y) \operatorname{prob}(v \to D) \tag{2.13}$$

and

$$\begin{aligned} &\operatorname{prob}(x \to y,\, x \to z,\, x \nrightarrow D) \\ &\qquad \leq \sum_{v} \operatorname{prob}(x \to v,\, x \nrightarrow D) \operatorname{prob}(v \to y) \operatorname{prob}(v \to z). \end{aligned} \tag{2.14}$$

Inequalities (2.12) and (2.13) are just a POP version of the diagrammatic bounds of Aizenman and Newman (1984), and (2.14) is easily derived from (2.13) by conditioning on the "self-determined" set of sites connected to D. Such diagrammatic bounds have nowadays a fairly direct derivation [Durrett (1985)] enabled by the advent of the inequalities of van den Berg and Kesten (1985) and van den Berg and Fiebig (1987).

3. Complementary differential inequalities. The main result of this section is the derivation of the following differential inequality for the percolation models discussed in Section 2.

PROPOSITION 3.1. *Suppose a POP model with $\beta_c < \infty$ is either finite-range and well-connected, or is uniformly long-range. Then, throughout the regime $\mathscr{R} = \{(\beta, h): \beta \le \beta_c,\ h \ge 0\}$, for each $R < \infty$,*

$$M - h\frac{\partial M}{\partial h} \ge \varepsilon_R(\beta)[1 - f(\beta)\nabla_R(\beta_c)]M^2\frac{\partial M}{\partial h} - g_R(\beta)hM\frac{\partial M}{\partial h}, \tag{3.1}$$

where ε_R ($\neq 0$), f and g_R are some model-dependent functions with $1/\varepsilon_R$, f and g_R uniformly bounded in a neighborhood of β_c.

Inequality (3.1) is complementary to (1.10). The method presented in this section allows also the reversal of the other inequalities cited in the introduction. Specifically, one may prove

$$\frac{\partial M}{\partial \beta} \ge \varepsilon'_R(\beta)[1 - f(\beta)\nabla_R(\beta_c)]M\frac{\partial M}{\partial h}, \tag{3.2}$$

with ε'_R and f' having the same properties as ε_R and f. Although these results are similar to the bound [with a different definition for $\nabla_R(\beta_c)$] of Aizenman and Newman (1984),

$$\left.\frac{\partial \chi}{\partial \beta}\right|_{h=0} \ge \varepsilon_R(\beta)[1 - \nabla_R(\beta_c)]\chi^2, \tag{3.3}$$

the additional subtractions required for $h \neq 0$ make the present discussion more involved. For completeness of the treatment of oriented percolation, (3.3) should be extended to POP models. The derivation of (3.2) and the extension of (3.3) are omitted here since they require no new ideas, and (3.1) suffices for our main result.

The proof of Proposition 3.1 starts from an identity. We then proceed in two steps. The first involves a "regularization" which amounts to the separation of points to which explicit reference is being made. In the second step, the regularized quantity is shown to be of the order of a product of simple functions, up to a correction which is small provided the triangle diagram, opened by the separation induced in step 1, is smaller than $f = 1/3\mu^4$.

As is explained in Aizenman and Barsky (1987), the quantity on the left-hand side in (3.1) admits the representation (with $\hat{h} = e^h - 1$)

$$\begin{aligned} M - \hat{h}\frac{\partial M}{\partial h} &= \text{prob}(0 \text{ is connected to more than one green site}) \\ &= \sum_x \text{prob}(0 \to x,\ x \text{ is doubly connected to } G,\ 0 \nrightarrow G \text{ off } \{x\}) \\ &\equiv \sum_x \text{prob}(\mathscr{E}(x)), \end{aligned} \tag{3.4}$$

where "x is doubly connected to G" means that x is connected to a pair of distinct sites in G by two disjoint paths of occupied bonds.

3.1. *Ultraviolet regularization.* The point x plays a triple role for the event $\mathscr{E}(x)$—once as a site to which 0 is connected, and twice as a site connected to G. In the following point-split quantity $\mathscr{E}_s$, these roles are played by three different sites:

$$\mathscr{E}_s(x,y,z) = \text{the event "}0 \to x,\ 0 \nrightarrow G \text{ and there exist path-disjoint connections of } y \text{ and } z \text{ to a pair of distinct sites in } G.\text{"} \tag{3.5}$$

The following lemma is a regularized lower bound for the quantity described in (3.4). We refer to it as ultraviolet regularization because the scale of the point-splitting is, in the critical regime, much smaller than the correlation length. Such a statement is needed here only for finite-range models, since for the other case we have the explicit lower bound presented in Lemma 3.3.

LEMMA 3.2. *In a finite-range POP model which is well-connected with respect to T_1 and T_2, there exist continuous $c_i = c_i(\beta, T_1, T_2) > 0$, such that*

$$M - \hat{h}\frac{\partial M}{\partial h} \geq c_0 \sum_x \text{prob}(\mathscr{E}_s(x, T_1x, T_2x)) - c_1 hM\frac{\partial M}{\partial h} - c_2 h^2 \frac{\partial M}{\partial h}. \tag{3.6}$$

PROOF. Let L denote the maximum of the lengths appearing in the definitions of the well-connectedness w.r.t. T_1 and T_2, and let r denote the range of the model [i.e., $r = \sup\{\text{dist}(x,y)\colon \{x,y\} \in \mathbb{B} \text{ or } (x,y) \in \mathbb{B}\}$]. We first show that

$$\begin{aligned} &\text{prob}(\mathscr{E}_s(x, T_1x, T_2x)) \\ &\quad \leq \tilde{c}_0 \sum_{x'\colon \text{dist}(x,x') \leq L} \text{prob}(\mathscr{E}(x')) + \tilde{c}_1 hM\, \text{prob}(0 \to x, 0 \nrightarrow G) \\ &\qquad + \tilde{c}_2 h^2\, \text{prob}(0 \to x, 0 \nrightarrow G), \end{aligned} \tag{3.7}$$

for some $\tilde{c}_0 = \tilde{c}_0(L, \beta)$, $\tilde{c}_1 = \tilde{c}_1(L, \beta)$, $\tilde{c}_2 = \tilde{c}_2(L) > 0$.

Defining K to be the random variable which counts the number of green sites in $\Lambda_L(x)$, we first consider those configurations for which $\mathscr{E}_s(x, T_1x, T_2x)$ occurs and $K = 0$. We wish to show that any such configuration can be locally modified to yield a configuration which makes a contribution to the last term in (3.4). The first step in the modification is to "thin" the connection $0 \to x$ by changing (from occupied to vacant) the occupation status of all bonds in $C(0) \cap \Lambda_{L+r}(x)$ except for a minimal set which is necessary to preserve the connection $0 \to x$. Call the reduced cluster of the origin $C'(0)$ and note that $C'(0) \cap \Lambda_L(x)$ consists of disjoint paths of bonds which are traversed by the connection $0 \to x$ in some definite order. For each i, $i = 1, 2$, we can use well-connectedness w.r.t. T_i to obtain a minimal set of bonds B_i in $\Lambda_L(x)$ which, when changed from vacant to occupied, connect 0 to one of the green sites in $\Lambda_L(x)^c$ which were reached from T_ix; observe that each B_i is a path of bonds which originates at some site x_i in $C'(0) \cap \Lambda_L(x)$. By Lemma 3.5 of Aizenman and Barsky (1987), there must then exist for the (thrice) modified configuration, a site x' with the properties that $0 \to x'$, $0 \nrightarrow G$ off $\{x'\}$ and x' is doubly connected to G. It is readily seen that x' is either x_1 or x_2, whichever site is encountered first in $C'(0)$.

Thus there is a map associating to each configuration in the set $\mathscr{E}_s(x, T_1x, T_2x) \cap \{K = 0\}$ a configuration in $\bigcup_{x':\ \mathrm{dist}(x,x') \le L} \mathscr{E}(x') \cap \{K = 0\}$. This map [which changes the configuration only in $\Lambda_{L+r}(x)$] is neither measure-preserving nor surjective. However, it changes the probability density by a factor nowhere greater than $\lambda(L, \beta, x)$, the ratio of the maximum probability to the minimum probability of bond configurations in $\Lambda_{L+r}(x)$, and it is at most κ – to-1, where $\kappa(L, x)$ is the number of bond configurations in $\Lambda_{L+r}(x)$. It thus follows that

$$\begin{aligned} &\mathrm{prob}(\mathscr{E}_s(x, T_1x, T_2x) \cap \{K = 0\}) \\ &\qquad \le \lambda(L, \beta, x)\kappa(L, x) \sum_{x':\ \mathrm{dist}(x,x') \le L} \mathrm{prob}(\mathscr{E}(x')). \end{aligned} \tag{3.8}$$

To obtain the two remaining terms on the right-hand side of (3.7), which will be shown in Section 4 to be of lower order, we investigate the situations where $\mathscr{E}_s(x, T_1x, T_2x)$ occurs and either $K = 1$ or $K \ge 2$. In the first case, at least one of the two sites T_1x and T_2x must be connected to a green site outside $\Lambda_{L+r}(x)$, so

$$\begin{aligned} &\mathrm{prob}(\mathscr{E}_s(x, T_1x, T_2x) \cap \{K = 1\}) \\ &\qquad \le |\Lambda_L(x)|(1 - e^{-h})[\mathrm{prob}(0 \to x, 0 \nrightarrow G, T_1x \to G) \\ &\qquad\qquad + \mathrm{prob}(0 \to x, 0 \nrightarrow G, T_2x \to G)]. \end{aligned} \tag{3.9}$$

Similarly, one has

$$\begin{aligned} &\mathrm{prob}(\mathscr{E}_s(x, T_1x, T_2x) \cap \{K \ge 2\}) \\ &\qquad \le |\Lambda_L(x)|^2(1 - e^{-h})^2 \mathrm{prob}(0 \to x, 0 \nrightarrow G). \end{aligned} \tag{3.10}$$

The van den Berg–Fiebig inequality can now be used to further simplify the right-hand side of (3.9):

$$\mathrm{prob}(0 \to x, 0 \nrightarrow G, T_1x \to G) \le M(T_ix)\mathrm{prob}(0 \to x, 0 \nrightarrow G). \tag{3.11}$$

The combination of inequalities (3.8)–(3.11) yields (3.7) with $\tilde{c}_0 = \sup\{\lambda(L, \beta, x)\kappa(L, x)\colon x \in \mathbb{L}\}$, $\tilde{c}_1 = 2\mu(\beta)V(L)$ and $\tilde{c}_2 = V(L)^2$. Summing over sites x in (3.7) and applying (2.5) yields

$$\begin{aligned} &\tilde{c}_0 \sum_x \sum_{x':\ \mathrm{dist}(x,x') \le L} \mathrm{prob}(\mathscr{E}(x')) \\ &\qquad \ge \sum_x \mathrm{prob}(\mathscr{E}_s(x, T_1x, T_2x)) - \tilde{c}_1 h M \frac{\partial M}{\partial h} - \tilde{c}_2 h^2 \frac{\partial M}{\partial h}. \end{aligned} \tag{3.12}$$

The desired inequality (3.6), with $c_0 = [\tilde{c}_0 V(L)]^{-1}$, $c_1 = c_0\tilde{c}_1$ and $c_2 = c_0\tilde{c}_2$, now follows from (3.12) and (3.4). □

The next result relates the probabilities of the events $\mathscr{E}$ and $\mathscr{E}_s$ in a way which is useful for models with sufficiently long bonds.

LEMMA 3.3. *In any POP model, for every three sites x, y and z,*

$$\operatorname{prob}(\mathscr{E}(x)) \geq \beta^2 J_{[x,y]} J_{[x,z]} \operatorname{prob}(\mathscr{E}_s(x,y,z)). \tag{3.13}$$

PROOF. The event $\mathscr{E}(x)$ has probability no smaller than the event that the two connections from x to G must first visit the particular pair of sites y and z. For compactness of notation, we write the union of the bonds $\{x,y\}$ and (x,y) as $[x,y]$, and we interpret $n_{[x,y]} = 1$ (resp., $n_{[x,y]} = 0$) to mean $n_{\{x,y\}} + n_{(x,y)} \geq 1$ (resp., $n_{\{x,y\}} = n_{(x,y)} = 0$). Thus,

$$\operatorname{prob}(\mathscr{E}(\mathrm{x})) \geq \operatorname{prob}(0 \to x,\ 0 \nrightarrow G \text{ off } [x,y] \text{ and } [x,z],\ n_{[x,y]} = 1,\ n_{[x,z]} = 1 \text{ and there exist path-disjoint connections of } y \text{ and } z \text{ to a pair of distinct sites in } G).$$

As the occupation states of the bonds from x to y and z are independent of the event that "$0 \to x$, $0 \nrightarrow G$ off $[x,y]$ and $[x,z]$ and there exist path-disjoint connections of y and z to a pair of distinct sites in G," we have

$$\begin{aligned}\operatorname{prob}(\mathscr{E}(x)) &\geq \frac{\operatorname{prob}(n_{[x,y]} = 1)}{\operatorname{prob}(n_{[x,y]} = 0)} \frac{\operatorname{prob}(n_{[x,z]} = 1)}{\operatorname{prob}(n_{[x,z]} = 0)} \\ &\quad \times \operatorname{prob}(0 \to x,\ 0 \nrightarrow G \text{ off } [x,y] \text{ and } [x,z],\ n_{[x,y]} = 0,\ n_{[x,z]} = 0 \text{ and there exist path-disjoint connections of } y \text{ and } z \text{ to a pair of distinct sites in } G) \\ &= \left(\exp\left[\beta(J_{x,y} + J_{x \to y})\right] - 1\right) \\ &\quad \times \left(\exp\left[\beta(J_{x,z} + J_{x \to z})\right] - 1\right) \operatorname{prob}(\mathscr{E}_s(x,y,z)),\end{aligned}$$

from which (3.13) trivially follows. □

3.2. *Factorization.* We now turn to the proof of (3.1) starting from the point-split expressions provided by Lemmas 3.2 and 3.3.

LEMMA 3.4. *For every pair of translations $T_1, T_2 \in \mathscr{G}$ with $|T_1|$, $|T_2|$, $|T_1^{-1}T_2| \geq R$, and for all $\beta \leq \beta_c$,*

$$\sum_x \operatorname{prob}(\mathscr{E}_s(x, T_1x, T_2x)) \geq \left(\frac{1}{\mu^2} - 3\mu^2 \nabla_R(\beta_c)\right) M^2 \frac{\partial M}{\partial h}. \tag{3.14}$$

PROOF. The main technique used here is an inclusion–exclusion argument. Various terms will be generated and then bounded. Only the leading term and the first of the corrections are treated in detail; the remaining terms are left for the reader.

Observe that one way in which the event $\mathscr{E}_s(x, T_1x, T_2x)$ may occur is if T_2x is connected to G in the complement of the $C(0)$ and $C_{C(0)^c}(T_1x)$. Partitioning this subevent of $\mathscr{E}_s(x, T_1x, T_2x)$ according to these clusters, and using independence, we have

$$\text{prob}(\mathscr{E}_s(x, T_1x, T_2x)) \tag{3.15}$$
$$\geq \sum_{A, B: x \in A} \text{prob}(A = C(0), A \cap G = \varnothing) \times \text{prob}(B = C_{A^c}(T_1x), B \cap G \neq \varnothing)\text{prob}(T_2x \to G \text{ off } A \cup B).$$

From (2.4) and (2.12), it follows that

$$\text{prob}(T_2x \to G \text{ off } A \cup B) \geq M(T_2x) - \sum_{u \in A \cup B} \text{prob}(T_2x \to u)M(u),$$

and a similar bound exists for

$$\sum_B \text{prob}(B = C_{A^c}(T_1x), B \cap G \neq \varnothing) = \text{prob}(T_1x \to G \text{ off } A).$$

Using these inequalities in (3.15), one obtains

$$\text{prob}(\mathscr{E}_s(x, T_1x, T_2x)) \geq \text{I} - \text{II} - \text{IIIA} - \text{IIIB}, \tag{3.16}$$

with

$$\text{I} = M(T_1x)M(T_2x)\text{prob}(0 \to x, 0 \nrightarrow G),$$
$$\text{II} = M(T_2x) \sum_{u, A: x, u \in A} M(u)\text{prob}(A = C(0), A \cap G = \varnothing) \times \text{prob}(T_1x \to u),$$
$$\text{IIIA} = \sum_{u, A: x, u \in A} M(u)\text{prob}(A = C(0), A \cap G = \varnothing) \times \text{prob}(T_1x \to G \text{ off } A)\text{prob}(T_2x \to u)$$

and

$$\text{IIIB} = \sum_{u, A, B: x \in A, u \in B} M(u)\text{prob}(A = C(0), A \cap G = \varnothing) \times \text{prob}(B = C_{A^c}(T_1x), B \cap G \neq \varnothing)\text{prob}(T_2x \to u).$$

The leading term can be bounded and summed [see (2.11) and (2.5)] to yield

$$\sum_x I(x, T_1x, T_2x) \geq \frac{1}{\mu^2}M^2\frac{\partial M}{\partial h}, \tag{3.17}$$

which is proportional to the important factor in the original upper bound (1.10).

The sums of terms II, IIIA and IIIB will lead to the triangle corrections that appear in the lower bound (3.14), and they have similar treatments. We will bound Σ_xII, and indicate how Σ_xIIIA and Σ_xIIIB are to be handled. For II, one notes that by the tree-diagram bound (2.14),

$$\sum_{A: x, u \in A} \text{prob}(A = C(0), A \cap G = \varnothing) \leq \sum_v \text{prob}(0 \to v, 0 \nrightarrow G)\text{prob}(v \to u)\text{prob}(v \to x),$$

and thus

$$
\begin{aligned}
&\sum_x \mathrm{II}(x, T_1x, T_2x)\\
&\quad\le \sum_{x,u,v} M(T_2x)M(u)\mathrm{prob}(0 \to v, 0 \nrightarrow G)\mathrm{prob}(v \to u)\\
&\qquad\times \mathrm{prob}(v \to x)\mathrm{prob}(T_1x \to u)\\
&\quad\le (\mu M)^2 \sum_{x,u',v} \mathrm{prob}(0 \to v, 0 \nrightarrow G)\mathrm{prob}(T_1^{-1}v \to u')\\
&\qquad\times \mathrm{prob}(v \to x)\mathrm{prob}(x \to u')\\
&\quad\le \triangledown(\beta; T_1^{-1})(\mu M)^2 \frac{\partial M}{\partial h}.
\end{aligned}
\tag{3.18}
$$

Since $\mathrm{prob}(T_1x \to G \text{ off } A) \le \mu M$, it follows that Σ_xIIIA has a similar upper bound with T_1 replaced by T_2. To bound Σ_xIIIB, one begins by using the tree-diagram bound (2.13) and concludes with

$$
\sum_x \mathrm{IIIB}(x, T_1x, T_2x) \le \triangledown(\beta; T_1^{-1}T_2)(\mu M)^2 \frac{\partial M}{\partial h}. \tag{3.19}
$$

The estimates (3.17)–(3.19) on the (sums of) terms in the decomposition (3.16) of $\mathrm{prob}(\mathscr{E}_s(x, y, z))$—along with the inequality $\triangledown(\beta; \cdot\,) \le \triangledown_R(\beta_c)$—imply the desired lower bound (3.14). □

3.3. *Proof of the main inequality.*

Proof of Proposition 3.1. In the finite-range case, for a given R we first pick a pair of translations T_1 and T_2 with $|T_1|, |T_2|, |T_1^{-1}T_2| \ge R$, for which the model is well-connected. With such a choice, Lemmas 3.2 and 3.4 yield

$$
\begin{aligned}
M - \hat{h}\frac{\partial M}{\partial h} &\ge c_0\left(\frac{1}{\mu^2} - 3\mu^2 \triangledown_R(\beta_c)\right) M^2 \frac{\partial M}{\partial h}\\
&\quad - c_1 hM\frac{\partial M}{\partial h} - c_2h^2\frac{\partial M}{\partial h},
\end{aligned}
\tag{3.20}
$$

from which the desired result (3.1) follows with $\varepsilon_R = c_0/\mu^2$, $f = 3\mu^4$ and $g_R = (c_1 + c_2)$.

In the uniformly long-range case we have to rework slightly the argument of Lemma 3.4, because now the choice of the translations T_1 and T_2 will depend (in a mild way) on the site x. First, it is elementary to show that for each R there are two collections of N translations, $\{T_{1,i}\}$ and $\{T_{2,j}\}$, such that $|T_{1,i}|, |T_{2,j}|, |T_{1,i}^{-1}T_{2,j}| \ge R$ for each $1 \le i \le n$, and

$$
K_R \equiv \inf_{x\in\mathbb{L}}\left\{\max_{1\le i,j\le N}\left\{J_{[x, T_{1,i}x]}J_{[x, T_{2,j}x]}\right\}\right\} > 0.
$$

Lemma 3.3 can now be used to point-split the event $\mathscr{E}_s(x)$—with $y = T_{1,i}x$ and $z = T_{2,j}x$, where i and j are x-dependent, chosen so that $J_{[x, T_{1,i}x]}J_{[x, T_{2,j}x]} \ge K_R$ (e.g., by maximizing the product). One obtains bounds similar to (3.18) and (3.19) with the triangle factors $\nabla(\beta, \cdot)$ appearing in those expressions replaced by $N \nabla_R(\beta_c)$. Thus,

$$M - \hat{h}\frac{\partial M}{\partial h} \geq c_R(\beta)\left(\frac{1}{\mu^2} - 3\mu^2 N \nabla_R(\beta_c)\right) M^2 \frac{\partial M}{\partial h}, \tag{3.21}$$

and so (3.1) holds with $\varepsilon_R = K_R \beta^2/\mu^2$, $f = 3N\mu^4$ and $g_R \equiv 0$. □

4. Implications of the triangle condition. We now turn to the implications of the differential inequality derived in the previous section. Following is the main result.

PROPOSITION 4.1. *In any weakly homogeneous POP model (which is either finite-range and well-connected or else uniformly long-range), if the triangle condition (2.8) is satisfied, then there exist constants $C_i \in (0, \infty)$ such that, for small $h \geq 0$,*

$$C_1 h^{1/2} \leq M(\beta_c, h) \leq C_2 h^{1/2} \tag{4.1}$$

and, in the vicinity of β_c,

$$C_3(\beta - \beta_c)_+ \leq P_\infty(\beta) \leq C_4(\beta - \beta_c)_+, \tag{4.2}$$

where $P_\infty(\beta) \equiv M(\beta, 0+)$ and $x_+ = \max\{x, 0\}$.

REMARK. The new results in the conclusion of Proposition 4.1 are the upper bounds [the lower bounds in (4.2) and (4.1) were proven in Chayes and Chayes (1986) and Aizenman and Barsky (1987), correspondingly].

PROOF OF PROPOSITION 4.1 (The upper bounds). We begin by dividing both sides of inequality (3.1) by M to obtain

$$1 \geq \varepsilon_R(\beta)[1 - f(\beta)\nabla_R(\beta_c)]\left[1 - O\left(\frac{h}{M}\right)\right] M \frac{\partial M}{\partial h}. \tag{4.3}$$

By the concavity of M in h, $\lim_{h \downarrow 0}(M/h) = \partial M/\partial h (= \chi(\beta)$ for $\beta < \beta_c)$, which diverges [continuously by Aizenman and Newman (1984)] as $\beta \uparrow \beta_c$. Thus, in the neighborhood of the critical point $(\beta_c, 0)$, the h/M correction can be neglected. Taking R sufficiently large [and using the assumption that the triangle condition (2.8) is satisfied], we obtain the simple inequality

$$\frac{\partial}{\partial h} M^2 \leq \text{const.}^2, \tag{4.4}$$

which holds in the vicinity of the critical point: $(\beta_0, \beta_c) \times (0, h_0)$ for some $h_0 > 0$ and $\beta_0 < \beta_c$.

The integration of (4.4) yields

$$M(\beta, h) - M(\beta, 0+) \leq \text{const.}\, h^{1/2}. \tag{4.5}$$

for $0 < h \leq h_0$ and $\beta \in (\beta_0, \beta_c]$. For all $\beta < \beta_c$, $M(\beta, 0+) = 0$ and hence

$M(\beta, h) \le$ const. $h^{1/2}$. By the continuity (in β) of $M(\beta, h)$, for $h > 0$ [see, e.g., Aizenman and Barsky (1987)], the latter implies the upper bound in (4.1).

Seeking an upper bound on $M(\beta, 0+)$, for $\beta > \beta_c$, one may at first be discouraged by the fact that the "corrected inequalities" derived here are rather useless in the region $\{\beta > \beta_c,\ h \ge 0\}$. A similar situation was encountered in Aizenman and Fernández (1986), where the problem was resolved by means of certain "extrapolation principles" [derived there from a Burgers inequality like (1.9)]. These methods were applied to percolation models in Aizenman and Barsky (1987), where Proposition 6.2(ii) states that if

$$M(\beta_c + t, h) \le ch^{\alpha}|\ln h|^{\omega}(1 + O(h)) \tag{4.6}$$

along a ray $t = ah$, $h \ge 0$, with $c > 0$, $0 < \alpha < 1$ and $\omega \ge 0$, then

$$\begin{aligned} &M(\beta_c + t, 0+) \\ &\quad \le \left(|J|c^{1/\alpha}\right)^{\alpha/(1-\alpha)} t^{\alpha/(1-\alpha)} |\ln(|J|Mt)|^{\omega/(1-\alpha)}(1 + O(t)), \end{aligned} \tag{4.7}$$

for $t \ge 0$. [Furthermore, (4.7) is valid also along any other ray $t = a'h$.] Thus, the upper bound of (4.1) implies directly the one in (4.2). For completeness, we mention that there is another extrapolation principle for percolation [Newman (1987b)] which alternatively could be used here. □

COROLLARY 4.2. *Under the hypotheses of Proposition* 4.1, *the limits*

$$\delta = \lim_{h \downarrow 0} \frac{\ln h}{\ln M(\beta_c, h)} \tag{4.8}$$

and

$$\hat{\beta} = \lim_{\beta \downarrow \beta_c} \frac{\ln M(\beta, 0+)}{\ln(\beta - \beta_c)} \tag{4.9}$$

exist, and the critical exponents they define take on their "mean-field" values

$$\hat{\beta} = 1 \quad \text{and} \quad \delta = 2. \tag{4.10}$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA 95616

PRINCETON UNIVERSITY
JADWIN HALL
P.O. BOX 708
PRINCETON, NEW JERSEY 08544