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ALMOST SUBADDITIVE EXTENSIONS OF KINGMAN'S ERGODIC THEOREM

By Klaus Schürger

Australian National University

Based on two notions of almost subadditivity which were introduced by Derriennic and Schürger, two a.s. limit theorems are proved which both generalize Kingman's subadditive ergodic theorem. These results, being valid under weak moment conditions, are obtained by short proofs. One of these proofs is completely elementary and does not even make use of Birkhoff's ergodic theorem which, instead, is obtained as a by-product. Finally, an improvement of Liggett's a.s. limit theorem is given.

1. Introduction. Kingman's [4] a.s. subadditive limit theorem is a complete generalization of Birkhoff's ergodic theorem. In the present paper it is shown that similar a.s. limit theorems can be obtained if Kingman's subadditivity hypothesis is replaced by a weaker condition. In Derriennic [1] and Schürger [10] two suitable notions of "almost" subadditivity have been introduced. In the sequel let $X = (X_J)$ and $U = (U_J)$, $U_J \geq 0$, $J \in \mathscr{J}$, be two families of real random variables defined on a common probability space $(\Omega, \mathscr{A}, P)$, the index set being $\mathscr{J} = \{[i, j[\colon\ 0 \leq i < j,\ i, j \text{ integers}\}$ where $[i, j[= \{r \in R\colon\ i \leq r < j\}$. We put $\overline{X}_{[i,j[} = (j-i)^{-1} X_{[i,j[}$, $[i, j[\in \mathscr{J}$. We will say that X is DS-*subadditive* w.r.t. U provided

(DS) $$X_{[i,k[} \leq X_{[i,j[} + X_{[j,k[} + U_{[j,k[}, \qquad 0 \leq i < j < k.$$

In particular, X is called *additive* if

$$X_{[i,k[} = X_{[i,j[} + X_{[j,k[}, \qquad 0 \leq i < j < k.$$

We will say that X is AS-*subadditive* w.r.t. U provided

(AS) $$X_{J_1 \cup \cdots \cup J_k} \leq \sum_{i=1}^{k} \left(X_{J_i} + U_{J_i} \right)$$

for all disjoint sets $J_1, \ldots, J_k$ in $\mathscr{J}$ such that $J_1 \cup \cdots \cup J_k \in \mathscr{J}$.

Note that (AS) is weaker than (DS). Condition (DS) was introduced by Derriennic who was the first to prove strong almost subadditive limit theorems (see, e.g., [1], Theorem 4). Condition (AS) is a special case of an almost subadditivity condition which was introduced by Schürger [10] in the multiparameter case. In Section 2 we derive an AS-limit theorem (Theorem 1) and a DS-limit theorem (Theorem 2) which hold under weak moment conditions. In particular, Theorem 2 shows that in Derriennic's a.s. limit theorem for processes X which are DS-subadditive w.r.t. U ([1], Theorem 4), the moment

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condition on U can be considerably relaxed (in fact, $E[U_{[0,n[}]$ may be allowed to grow almost linearly).

Theorems 1 and 2 are obtained by short proofs. Moreover, the proof of Theorem 1 is completely elementary and does not even use Birkhoff's ergodic theorem which, instead, is obtained as a by-product. Motivated by Shields [11], Steele [12] recently obtained a short and very elegant proof of Kingman's a.s. subadditive limit theorem, which is based on a certain recursive ("algorithmic") procedure of constructing random decompositions of sets in $\mathscr{J}$. Applying Steele's algorithm in a more general situation, we derive a certain inequality [see (2.3)] which will be used in various ways to prove the desired AS-limit theorem. The proof of our DS-limit theorem is based on the key observation that each DS-subadditive process can be represented as a sum of an additive process and a certain DS-subadditive process (Z_J) such that $(Z_{[0,n[})$ is *decreasing*. Finally, we obtain an improvement of Liggett's [6] a.s. limit theorem under a condition corresponding to (DS) (see Theorem 3).

2. Almost subadditive limit theorems. We will say that X and U are *jointly stationary* if the finite-dimensional distributions of the random vectors (X_{J+k}, U_{J+k}), $J \in \mathscr{J}$, do not depend on $k \geq 0$ (here, $J + k = \{j + k: j \in J\}$). Let R^∞ be the space of all families (x_J), $J \in \mathscr{J}$, of real numbers (R^∞ being endowed with the usual σ-algebra of Borel sets). The indicator function of a set A will be denoted by 1_A.

LEMMA. *Let X be AS-subadditive w.r.t. U and assume that*

$$X_{[k,k+1[} + U_{[k,k+1[} \leq 0, \qquad k \geq 0. \tag{2.1}$$

Let $1 \leq m_1 < m_2 < \cdots$ be a fixed sequence of integers. Define, for a given nonpositive measurable mapping ϕ: $R^\infty \to R$, the events $D_j(K)$, $j \geq 0$, $K \geq 1$, by

$$D_j(K) = \left\{ \min_{1 \leq k \leq K} \left(\overline{X}_{[j,j+m_k[} + \overline{U}_{[j,j+m_k[} \right) > \phi(X) + \alpha \right\}, \tag{2.2}$$

where $\alpha > 0$ is fixed. Then we have

$$X_{[0,m_n[} \leq \alpha m_n + \phi(X) \left(m_n - m_K - \sum_{j=0}^{m_n - 1} 1_{D_j(K)} \right), \qquad K \geq 1, n \geq 1. \tag{2.3}$$

If, additionally, X and U are jointly stationary and if ϕ has the property that

$$\phi((X_{J+k})) = \phi((X_J)) \quad a.s., k \geq 1, \tag{2.4}$$

then

$$\left(1_{D_j(K)}\right)_{j \geq 0} \quad \textit{is stationary}, K \geq 1.$$

PROOF. Let $K \geq 1$, $n \geq 1$ and $\omega \in \Omega$ be fixed. Using Steele's [12] recursive procedure, we decompose $[0, m_n[$ into sets in $\mathscr{J}$ (depending on ω) as follows. Let $0 \leq j \leq m_n - 1$ be the smallest integer not belonging to any of the random sets constructed so far.

CASE 1. $\omega \in D_j(K)$. Take $[j, j+1[$ as the next set.

If, however, $\omega \notin D_j(K)$, then there exists some $1 \le k \le K$ such that

$$X_{[j, j+m_k[}(\omega) + U_{[j, j+m_k[}(\omega) \le m_k(\phi(X(\omega)) + \alpha). \tag{2.5}$$

Choose any such k.

CASE 2. $j + m_k \le m_n$. Take $[j, j+m_k[$ as the next set.

CASE 3. $j + m_k > m_n$. Take $[j, j+1[$ as the next set.

Let $n_i = n_i(\omega)$ denote the number of sets constructed according to Case i ($i = 1, 2, 3$). If $k_1, \dots, k_{n_2}$ are the numbers of integers contained in the sets constructed according to Case 2, we clearly have

$$n_1 + n_3 + k_1 + \cdots + k_{n_2} = m_n, \qquad n_1 \le \sum_{j=0}^{m_n - 1} 1_{D_j(K)} \quad \text{and} \quad n_3 \le m_K - 1,$$

which implies

$$k_1 + \cdots + k_{n_2} \ge m_n - m_K - \sum_{j=0}^{m_n-1} 1_{D_j(K)}. \tag{2.6}$$

From (AS) we get, taking into account (2.1), (2.5), (2.6) and the nonpositivity of ϕ,

$$X_{[0, m_n[} \le (\phi(X) + \alpha) \sum_{i=1}^{n_2} k_i \le \alpha m_n + \phi(X)\left(m_n - m_K - \sum_{j=0}^{m_n-1} 1_{D_j(K)}\right).$$

The last assertion of the lemma is a consequence of (2.4), (2.2) and the joint stationarity of X and U. □

Using the above lemma, we can prove the following AS-limit theorem (we put $a^+ = \max\{0, a\}$, $a \in R$).

THEOREM 1. (i) *Let X and U be jointly stationary and let X be AS-subadditive w.r.t. U. Assume that*

$$X^+_{[0,1[} \in L^1_+, \qquad U \subset L^1_+. \tag{2.7}$$

Let $1 \le m_1 < m_2 < \cdots$ be a sequence of integers such that

$$\liminf_n \overline{X}_{[0, j+m_n[} \ge \liminf_n \overline{X}_{[0, m_n[} \quad a.s., \ j \ge 1, \tag{2.8}$$

and

$$\lim_n \overline{U}_{[0, m_n[} = 0 \quad a.s. \tag{2.9}$$

Then

$$\lim_n \overline{X}_{[0, m_n[} = \xi_0 \quad \textit{exists } a.s., \tag{2.10}$$

where

$$(2.11)\qquad -\infty \le \xi_0 < \infty \quad a.s.$$

(ii) *If additionally,*

$$(2.12)\qquad \lim_n E\left[\overline{X}_{[0,m_n[}\right] = \gamma \quad \textit{exists and is finite}$$

and

$$(2.13)\qquad \lim_n E\left[\overline{U}_{[0,m_n[}\right] = 0,$$

then $(\overline{X}_{[0,m_n[})$ *also converges in* L^1.

REMARKS. (i) Note that (2.8) is satisfied if $m_n = n$, $n \ge 1$, or if, for almost all $\omega \in \Omega$, there are constants $c(\omega) \in R$ such that $X_{[0,n+1[}(\omega) - X_{[0,n[}(\omega) \ge c(\omega)$, $n \ge 1$.

(ii) Let $m_n = n$, $n \ge 1$. It has been noted by Kingman ([5], Theorem 1.8) that (2.10) and (2.11) hold if X satisfies the hypotheses of the first part of Theorem 1 in the case where U is the identically vanishing process.

(iii) Let $m_n = n$, $n \ge 1$. Theorem 1 shows that Theorem 3.2 of [10], restricted to the one-parameter case, holds under weaker moment conditions.

(iv) It has been shown by Derriennic and Hachem [2] that a stationary integrable process $X = (X_J)$, $J \in \mathscr{J}$, satisfying the subadditivity condition

(DHS)

$$E\left[\left(X_{[0,n+k[} - X_{[0,n[} - X_{[n,n+k[}\right)^+\right] \le c(n+k), \qquad n \ge 1, k \ge 1,$$

where $(c(n)) \subset R_+$, has the property that $(\overline{X}_{[0,n[})$ converges in L^1 provided X and $(c(n))$ satisfy certain additional hypotheses. In [2] the question is posed to find an analogue of (DHS) on which an a.s. limit theorem for $(\overline{X}_{[0,n[})$ can be based. Let us say that a process $X = (X_J) \subset L^1$ is *BS-subadditive* w.r.t. $U = (U_J) \subset L^1_+$, $J \in \mathscr{J}$, provided

$$(\mathrm{BS})\qquad X_{J_1 \cup \cdots \cup J_k} \le \sum_{i=1}^{k} X_{J_i} + U_{J_1 \cup \cdots \cup J_k}$$

for all disjoint sets $J_1, \ldots, J_k$ in $\mathscr{J}$ such that $J_1 \cup \cdots \cup J_k \in \mathscr{J}$. If X is BS-subadditive w.r.t. U, clearly (DHS) holds provided we put $c(n) = E[U_{[0,n[}]$, $n \ge 1$. It is clear that X is BS-subadditive w.r.t. U iff $(X_J - U_J)$ is AS-subadditive w.r.t. U. Hence Theorem 1(i) implies that $(\overline{X}_{[0,n[})$ converges a.s. to some random variable ξ_0 such that $-\infty \le \xi_0 < \infty$ a.s., provided X and U are jointly stationary, X is BS-subadditive w.r.t. U, and $\overline{U}_{[0,n[} \to 0$ a.s. as $n \to \infty$.

PROOF OF THEOREM 1. (i) Let us first consider the case when X and U additionally satisfy the inequalities

$$(2.14)\qquad X_{[k,k+1[} + U_{[k,k+1[} \le 0, \qquad k \ge 0.$$

Putting

$$\liminf_n \overline{X}_{[j,j+m_n[} = \xi_j, \qquad j \ge 0,$$

we have

$$\xi_j = \xi_0 \quad \text{a.s.}, \ j \ge 1. \tag{2.15}$$

In fact, since, for $j \ge 1$, $n \ge 1$,

$$X_{[0, j+m_n[} \le X_{[0,j[} + U_{[0,j[} + X_{[j, j+m_n[} + U_{[j, j+m_n[},$$

the joint stationarity of X and U combined with (2.8) and (2.9) implies $\xi_j \ge \xi_0$ a.s., $j \ge 1$. Since ξ_j and ξ_0 have the same (possibly defective) distribution, (2.15) follows. Now, let $A_j(K, a)$ be the event obtained from the event $D_j(K)$ in (2.2) by putting $\phi(X) = \max\{-a, \xi_0\}$ for $a > 0$. Dividing both sides in (2.3) by m_n and letting $n \to \infty$, we get, for $\alpha > 0$, $a > 0$ and $K \ge 1$,

$$\limsup_n \overline{X}_{[0, m_n[} \le \alpha + \max\{-a, \xi_0\} + aS(\alpha, a, K), \tag{2.16}$$

putting

$$S(\alpha, a, K) = \limsup_n \frac{1}{m_n} \sum_{j=0}^{m_n - 1} 1_{A_j(K,a)}. \tag{2.17}$$

Suppose that

$$\lim_K S(\alpha, a, K) = 0 \quad \text{a.s.}, \ \alpha > 0, \ a > 0. \tag{2.18}$$

Then, by (2.16), first letting $K \to \infty$ and then $\alpha \to 0$ and $a \to \infty$, we see that $(\overline{X}_{[0, m_n[})$ converges a.s. To prove (2.18) keep $\alpha > 0$ and $a > 0$ fixed. Since

$$S(\alpha, a, 1) \ge S(\alpha, a, 2) \ge \cdots, \qquad \lim_K S(\alpha, a, K) = S(\alpha, a) \quad \text{exists a.s.}$$

Thus (2.18) is equivalent to

$$E[S(\alpha, a)] = 0. \tag{2.19}$$

To prove (2.19) first note that, for $\varepsilon > 0$ and $K \ge 1$,

$$E[S(\alpha, a, K)] \le \varepsilon + P\left\{ \sup_k \frac{1}{k} \sum_{j=0}^{k-1} 1_{A_j(K,a)} > \varepsilon \right\}. \tag{2.20}$$

To obtain an upper bound for the probability in (2.20), keep also $K \ge 1$ fixed and apply (2.3) to the additive process Y given by

$$Y_J = - \sum_{j \in J \cap Z_+} 1_{A_j(K,a)}, \qquad J \in \mathscr{J},$$

and the sequence $m_n = n$, $n \ge 1$. Note that, by (2.15), Y is stationary. Let $\varepsilon > 0$. Replacing in (2.2) X, U, $K \ge 1$, $\alpha > 0$ and ϕ, respectively, by Y, the identically vanishing process, $L \ge 1$, $\beta > 0$ and the function identically equal to $-\beta - \varepsilon$, we get, by taking expectations on both sides of (2.3),

$$\begin{aligned} -nP(A_0(K,a)) &\le \beta n - (\beta + \varepsilon) \\ &\quad \times \left(n - L - nP\left\{ \max_{1 \le k \le L} \frac{1}{k} \sum_{j=0}^{k-1} 1_{A_j(K,a)} < \varepsilon \right\} \right) \end{aligned} \tag{2.21}$$

for $L \geq 1$, $n \geq 1$. Dividing both sides in (2.21) by n, we obtain, letting $n \to \infty$, $\beta \to 0$ and $L \to \infty$,

$$P\left\{\sup_k \frac{1}{k}\sum_{j=0}^{k-1} 1_{A_j(K,a)} > \varepsilon\right\} \leq \frac{1}{\varepsilon}P(A_0(K,a)).$$

Since, by (2.9), $P(A_0(K,a)) \to 0$ as $K \to \infty$, (2.19) follows from (2.20) and the dominated convergence theorem. If, however, the inequalities (2.14) are not all satisfied, we consider the decomposition

$$X_J = V_J - W_J, \qquad J \in \mathscr{J} \tag{2.22}$$

given by

$$V_J = X_J - \sum_{j\in J\cap Z_+} \left(X^+_{[j,j+1[} + U_{[j,j+1[}\right), \qquad J \in \mathscr{J}, \tag{2.23}$$

$$W_J = -\sum_{j\in J\cap Z_+} \left(X^+_{[j,j+1[} + U_{[j,j+1[}\right), \qquad J \in \mathscr{J}, \tag{2.24}$$

and apply the a.s. convergence result obtained above to V and W: It follows from Fatou's lemma that the a.s. limit of $(\overline{W}_{[0,m_n[})$ is integrable by (2.7) and the joint stationarity of X and U. Thus (2.10) and (2.11) also hold in the general case.

(ii) First note that the additive process $(\overline{W}_J)$ (W_J given by (2.24)) is uniformly integrable. Hence $(\overline{W}_{[0,n[})$ converges in L^1. It therefore suffices to show that $(\overline{V}_{[0,m_n[})$ converges in L^1. It follows from (2.12) that

$$\lim_n E\left[\overline{V}_{[0,m_n[}\right] = \tilde{\gamma} \quad \text{exists and is finite}.$$

Let $\lim_n \overline{V}_{[0,m_n[} = \tilde{\xi}_0$ a.s. Fatou's lemma implies $E[\tilde{\xi}_0] \geq \tilde{\gamma}$. In order to show that $E[\tilde{\xi}_0] \leq \tilde{\gamma}$, fix $N \geq 1$ and write $m_n = k_n m_N + r_n$ where $k_n \geq 0$ and $0 \leq r_n < m_N$. Since, for $n \geq 1$,

$$V_{[0,m_n[} \leq \sum_{j=0}^{k_n-1}\left(V_{[jm_N,(j+1)m_N[} + U_{[jm_N,(j+1)m_N[}\right) + V_{[k_n m_N, m_n[} + U_{[k_n m_N, m_n[}$$

(putting $V_\varnothing \equiv U_\varnothing \equiv 0$) and (by the Borel–Cantelli lemma)

$$\lim_n \frac{1}{m_n} V_{[k_n m_N, m_n[} = \lim_n \frac{1}{m_n} U_{[k_n m_N, m_n[} = 0 \quad \text{a.s.}, N \geq 1,$$

we get

$$E\left[\tilde{\xi}_0\right] \leq E\left[\overline{V}_{[0,m_N[} + \overline{U}_{[0,m_N[}\right], \qquad N \geq 1.$$

Therefore (2.13) implies $E[\tilde{\xi}_0] = \tilde{\gamma}$ which, in turn, shows that

$$E\left[|\overline{V}_{[0,m_n[} - \tilde{\xi}_0|\right] = 2E\left[\left(\overline{V}_{[0,m_n[} - \tilde{\xi}_0\right)^+\right] - E\left[\overline{V}_{[0,m_n[} - \tilde{\xi}_0\right]$$

tends to zero as $n \to \infty$ (note that the random variables $(\overline{V}_{[0,m_n[} - \tilde{\xi}_0)^+$, $n \geq 1$, are uniformly integrable). □

EXAMPLES. (i) Let Π denote a Poisson point process (intensity 1) on R^2, and denote by $\Pi(A)$, $A \subset R^2$, the set of points of Π contained in A. Let $\Pi(R \times [0,1]) = \{\eta_1, \eta_2, \dots\}$, the points η_i being indexed in any systematic order. Let $\rho_1, \rho_2, \dots$ be an i.i.d. sequence of nonnegative random variables which is independent of Π. We introduce an undirected random graph Γ without loops and multiple edges as follows. The vertex set of Γ is $\Pi(R \times [0,1])$. Two vertices η_i and η_j $(i \neq j)$ are adjacent in Γ iff $\|\eta_i - \eta_j\| \le \rho_i + \rho_j$ ($\|\cdot\|$ denotes the Euclidean norm). Let Γ_J, $J \in \mathscr{J}$, denote the following subgraph of Γ: The vertex set of Γ_J is $\Pi(J \times [0,1])$ and two vertices of Γ_J are adjacent in Γ_J iff they are adjacent in Γ. Define the random variable X_J, $J \in \mathscr{J}$, to be the number of vertices belonging to those connected components of Γ_J which contain an *even* number of vertices. Finally, we define U_J, $J \in \mathscr{J}$, to be the number of vertices x of Γ_J for which there exists a vertex $y \in \Pi((R \setminus J) \times [0,1])$ such that x and y are the end vertices of a certain path in Γ. It is easy to see that $X = (X_J)$ is AS-subadditive w.r.t. $U = (U_J)$. In the case where ρ_1 has a density which is strictly positive on some interval $[0, c]$ $(c > 0)$, it is easy to verify that there do not exist constants $a \in R$ and $b \ge 0$ such that $(X_J + aU_J)$ or $(-X_J + aU_J)$ is DS-subadditive w.r.t. (bU_J). Let us show that X and U satisfy the hypotheses of Theorem 1(i) in the case where $m_n = n$, $n \ge 1$, and $0 \le \rho_1 \le r_0$ a.s. for some constant $r_0 > 0$. Clearly, for all $J \in \mathscr{J}$,

$$X_J \le |\Pi(J \times [0,1])| \in L^1_+, \qquad U_J \le |\Pi(J \times [0,1])| \in L^1_+$$

($|A|$ denoting the cardinal number of a set A). In order to verify Condition (2.9) (for $m_n = n$, $n \ge 1$), put

$$\tau = \min\{m \ge 0 \colon \Pi([3mr_0, 3(m+1)r_0[\times [0,1]) = \varnothing\}$$

and, for $n \ge 1$,

$$\tau_n = \min\{m \ge 0 \colon \Pi([n - 3(m+1)r_0, n - 3mr_0[\times [0,1]) = \varnothing\}.$$

Clearly,

$$P\{\tau > m\} = P\{\tau_n > m\} = (1 - \exp(-3r_0))^{m+1}, \qquad m \ge 0, n \ge 1.$$

Using a Borel–Cantelli argument we obtain that a.s.

$$\tau_n \le c(r_0)\log n \quad \text{for all sufficiently large } n$$

where $c(r_0) = -2(\log(1 - \exp(-3r_0)))^{-1}$. This implies that a.s. for all sufficiently large n,

$$\begin{aligned} U_{[0,n[} &\le |\Pi([0, 3\tau r_0[\times [0,1])| + |\Pi([n - 3\tau_n r_0, n[\times [0,1])| \\ &\le |\Pi([0, 3\tau r_0[\times [0,1])| + |\Pi([n - 3r_0 c(r_0)\log n, n[\times [0,1])|. \end{aligned}$$

Hence (U_J) satisfies (2.9). By Theorem 1(i), $(\overline{X}_{[0,n[})$ converges a.s. Since $X_{[0,n[} \le |\Pi([0,n[\times [0,1])|$, $n \ge 1$, the sequence $(\overline{X}_{[0,n[})$ is uniformly integrable and therefore also converges in L^1. Similar remarks apply to the process $Y = (Y_J)$, where the random variable Y_J, $J \in \mathscr{J}$, is defined to be the number of vertices belonging to those connected components of Γ_J which contain an *odd* number of vertices.

(ii) Derriennic (see [1], page 676) has given an example of processes $X = (X_J) \subset L^1_+$ and $U = (U_J)$, $U_J \geq 0$, $J \in \mathscr{J}$, such that X and U are jointly stationary, X is DS-subadditive w.r.t. U and

$$P\{U_J > t\} \leq 2^{-t}, \qquad t \geq 0, J \in \mathscr{J}. \tag{2.25}$$

It follows from (2.25) that

$$\sup_n E\left[U_{[0,n[}\right] < \infty.$$

Hence Derriennic's almost subadditive limit theorem ([1], Theorem 4) applies and gives that

$$\lim_n \overline{X}_{[0,n[} = \xi_0 \in L^1_+ \quad \text{exists a.s.,} \tag{2.26}$$

which is the a.s. part of the Shannon–McMillan–Breiman theorem of information theory. On the other hand, (2.26) also follows from Theorem 1. In fact, by (2.25),

$$\sum_{n=1}^{\infty} P\left\{\overline{U}_{[0,n[} > \varepsilon\right\} < \infty, \qquad \varepsilon > 0,$$

and hence $\overline{U}_{[0,n[} \to 0$ a.s. as $n \to \infty$.

We can now prove the following DS-limit theorem.

THEOREM 2. *Let $X \subset L^1$ and $U \subset L^1_+$ be jointly stationary and let X be DS-subadditive w.r.t. U. Assume that*

$$\inf_n E\left[\overline{X}_{[0,n[}\right] > -\infty \tag{2.27}$$

and

$$\lim_n E\left[\overline{U}_{[0,n[}\right] = 0. \tag{2.28}$$

(i) *Suppose that, for some integer $p \geq 2$,*

$$\lim_n \overline{U}_{[0,p^n k[} = 0 \quad a.s., \ k \geq 1. \tag{Λ_p}$$

Then

$$\lim_n \overline{X}_{[0,n[} = \xi_0 \quad \textit{exists a.s. and in } L^1. \tag{2.29}$$

(ii) *Assume that, for some $\delta > 0$,*

$$E\left[U_{[0,n[}\right] = O\left(\frac{n}{\log n (\log\log n)^{1+\delta}}\right) \quad \textit{as } n \to \infty. \tag{2.30}$$

Then (2.29) *holds.*

REMARK. Using a rather difficult proof, Derriennic obtained a limit theorem for processes X which are DS-subadditive w.r.t. U ([1], Theorem 4). In particular, he assumed that

$$\sup_n E\left[U_{[0,n[}\right] < \infty. \tag{2.31}$$

Part (ii) of Theorem 2 shows that the conclusion (2.29) in Derriennic's limit theorem remains unaffected if condition (2.31) is replaced by (2.30). On the other hand, part (ii) of Theorem 2 shows that the monotonicity assumptions in Theorem 3.3 of [7] are unnecessary if the processes $(X_{m,n})$ and $(Y_{m,n})$ are assumed to be jointly stationary. In [7] and [8] a.s. limit theorems were obtained under certain stationarity assumptions weaker than joint stationarity.

PROOF OF THEOREM 2. (i) In order to prove that $(\overline{X}_{[0,n[})$ converges a.s. we use the decomposition (2.22). Since Theorem 1 applies to the process W given by (2.24), it suffices to show that $(\overline{V}_{[0,n[})$ converges a.s. [V given by (2.23)]. Clearly, V is DS-subadditive w.r.t. U and

$$V_{[k,k+1[} + U_{[k,k+1[} \leq 0, \qquad k \geq 0.$$

Since this implies that $(V_{[0,n[})$ is decreasing, it follows from Theorem 3.2 of [7] that it suffices to show that

$$\lim_n \overline{V}_{[0,p^n k[} = \eta_k \quad \text{exists a.s.}, \; k \geq 1, \tag{2.32}$$

and

$$\eta_k = \eta_1 \quad \text{a.s.}, \; k \geq 2. \tag{2.33}$$

By Lemma 3.1 of [7] we have

$$E\Big[\liminf_n \Big(\overline{V}_{[0,n[} + \overline{U}_{[0,n[}\Big)\Big] \geq \gamma, \tag{2.34}$$

where γ equals the (finite) limit

$$\lim_n E\big[\overline{V}_{[0,n[}\big] = \gamma \tag{2.35}$$

(the existence of the limit in (2.35) is a consequence of Lemma 3.1 of [7]). On the other hand, it is easy to show [see the proof of Theorem 1(ii)] that

$$E\Big[\limsup_n \overline{V}_{[0,n[}\Big] \leq E\big[\overline{V}_{[0,N[}\big] + E\big[\overline{U}_{[0,N[}\big], \qquad N \geq 1,$$

which, by (2.35), (2.28) and (2.34), implies

$$E\Big[\limsup_n \overline{V}_{[0,n[}\Big] \leq \gamma \leq E\Big[\liminf_n \Big(\overline{V}_{[0,n[} + \overline{U}_{[0,n[}\Big)\Big]. \tag{2.36}$$

Clearly, (2.32) and (2.33) follow from (Λ_p) and (2.36). In order to show that $(\overline{X}_{[0,n[})$ converges in L^1, one can use the same argument as in the proof of Theorem 1(ii).

(ii) Note that (2.30) implies

$$\sum_{n=1}^{\infty} E\big[\overline{U}_{[0,p^n k[}\big] < \infty, \qquad k \geq 1, p \geq 2.$$

Hence (Λ_p) holds for all $p \geq 2$ and (ii) follows from (i). This completes the proof of Theorem 2. □

REMARKS. (i) The proof of (2.34) (given in [7]) is relatively short but uses a less elementary compactness argument. This leads to the question whether (2.34) can be obtained by an elementary covering argument similar to that in the proof of (2.3).

(ii) Using a construction of Durrett [3], Liggett [6] obtained a new version of Kingman's [2] subadditive limit theorem in which the subadditivity and stationarity assumptions are relaxed without weakening the conclusions. Since Lemma 3.1 of [7] is also based on Durrett's construction, it is straightforward to verify that, in Theorem 2, (DS) and the requirement that X and U are jointly stationary can be replaced by the following weaker hypotheses [the proof is based on the decomposition (2.22) where, in the definitions of V_J and W_J, $X^+_{[j,j+1[}$ is replaced by $X_{[j,j+1[}$, $j \geq 0$]:

$$X_{[0,n+k[} \leq X_{[0,n[} + X_{[n,n+k[} + U_{[n,n+k[}, \qquad n \geq 1;\ k \geq 1; \tag{2.37}$$

$$\text{the processes } \{X_{[nk,(n+1)k[},\ n \geq 0\} \text{ and } \{U_{[nk,(n+1)k[},\ n \geq 0\} \text{ are stationary for each } k \geq 1; \tag{2.38}$$

and

$$\text{the joint distributions of } \left\{X_{[m,n+m[} + U_{[m,n+m[} - \sum_{j=m}^{n+m-1} (X_{[j,j+1[} + U_{[j,j+1[}),\ n \geq 1\right\} \text{ do not depend on } m \geq 0. \tag{2.39}$$

[Let us note that the proof of Lemma 3.1 of [7], applied to V and U, does not really use conditions (3.4) and (3.6) of that lemma but, instead, only uses the condition

$$E\left[V_{[m,n+m[} + U_{[m,n+m[}\right] = E\left[V_{[0,n[} + U_{[0,n[}\right], \qquad m \geq 0,\ n \geq 1,$$

which, in turn, follows from (2.38) and (2.39).] It is, however, not clear whether there exist interesting processes X and U which are not jointly stationary and fail to satisfy (DS) but do satisfy conditions (2.37)–(2.39). If, however, condition (Λ_p) of Theorem 2(i) is replaced by a suitable stronger condition on U, then it is possible to obtain an improvement of Liggett's [6] a.s. convergence result. In fact, the following result (not mentioned in [7]) is a consequence of Lemma 3.1 of [7].

THEOREM 3. *Let $X \subset L^1$ and $U \subset L^1_+$ be processes satisfying the following assumptions*:

(2.40) *for each $k \geq 1$, the distribution of $X_{[m,m+k[}$ as well as that of $U_{[m,m+k[}$ does not depend on $m \geq 0$;*

(2.41) *the joint distributions of $\{X_{[m,m+k[} + U_{[m,m+k[},\ k \geq 1\}$ are the same as those of $\{X_{[0,k[} + U_{[0,k[},\ k \geq 1\}$ for each $m \geq 1$;*

(2.42) *for each* $k \geq 1$, *the processes* $\{X_{[nk,(n+1)k[}, \ n \geq 0\}$ *and* $\{U_{[nk,(n+1)k[}, \ n \geq 0\}$ *are stationary*;

$$X_{[0,n+k[} \leq X_{[0,n[} + X_{[n,n+k[} + U_{[n,n+k[}, \qquad n \geq 1;\ k \geq 1; \tag{2.43}$$

$$\inf_n E\left[\overline{X}_{[0,n[}\right] > \infty; \tag{2.44}$$

$$\lim_n E\left[\overline{U}_{[0,n[}\right] = 0; \tag{2.45}$$

and

$$\lim_n \overline{U}_{[0,n[} = 0 \quad a.s. \tag{2.46}$$

Then

$$\left(\overline{X}_{[0,n[}\right) \textit{ converges a.s. and in } L^1.$$

Proof. It follows from Lemma 3.1 of [7] that

$$\lim_n E\left[\overline{X}_{[0,n[}\right] = \gamma$$

exists and is finite.

Proceeding as in the proof of Theorem 2(i) we obtain

$$E\left[\limsup_n \overline{X}_{[0,n[}\right] \leq \gamma \leq E\left[\liminf_n \left(\overline{X}_{[0,n[} + \overline{U}_{[0,n[}\right)\right],$$

which, by (2.46), implies the desired conclusion [note that, by (2.43) and (2.42), the sequence $((\overline{X}_{[0,n[})^+)$ is uniformly integrable]. □

Finally, we would like to mention that Durrett's [3] construction has been used in [9] to obtain subadditive limit theorems for stochastic processes with a multidimensional parameter.

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DEPARTMENT OF ECONOMICS
UNIVERSITY OF BONN
ADENAUERALLEE 24-42
D-5300 BONN 1
GERMANY