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PRODUCT MARTINGALES AND STOPPING LINES FOR BRANCHING BROWNIAN MOTION

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For a branching Brownian motion, a probability space of trees is defined. By analogy with stopping times on $\mathbb{R}$, stopping lines are defined to get a general branching property. We exhibit an intrinsic class of martingales which are products indexed by the elements of a stopping line. We prove that all these martingales have the same limit which we identify. Two particular cases arise: the line of particles living at time t and the first crossings of a straight line whose equation is $y = at - x$ in the plane (y, t).

1. Introduction. Let us consider a branching Brownian motion where the particles reproduce according to a Galton–Watson process with law $p = (p_n, n \in \mathbb{N})$. The generating function of p is denoted by f and we assume that the mean m of p is finite. Each particle lives an $\exp(\alpha)$-distributed time ($\alpha > 0$) and during its life performs a Brownian motion on $\mathbb{R}$. Offspring particles move independently of each other and start off at the position where the parent particle died.

The probability space is a space of branching Brownian trees which are a special case of marked trees introduced in Neveu (1986) and Chauvin (1986). They are defined in Section 2 whose purpose is to get a general branching property which is a strong Markov property expressed with stopping lines.

The independence properties of this process provide a class of product martingales (Section 3). Multiplicative functions have been introduced by Watanabe and others [in Ikeda, Nagasawa and Watanabe (1968, 1969)] and product martingales for the Galton–Watson process have been used by Joffe and Spitzer (1967). More recently, Neveu (1987) has exhibited two product martingales which were the motivation of our construction: First,

$$M_t = \prod_{L_t} \psi(X_t - at),$$

where L_t is the population living at time t, X_t denotes the position, $a \in \mathbb{R}_+$ and ψ is a solution of the Kolmogorov equation [the above martingale appears also in Lalley and Sellke (1987)]; second,

$$M_x = \psi(z - x)^{\mu_x},$$

where $x, z \in \mathbb{R}$ and μ_x is the number of particles reaching the line whose equation in the plane (y, t) is $y = at - x$. We prove here that the previous two martingales are in fact the same one, stopped in two different ways by line L_t

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and straight line $y = at - x$, respectively. It allows us to prove that they have the same limit when $t \to +\infty$ and $x \to +\infty$, respectively.

2. The tree model and the branching property. A *tree* ω is by definition a subset of

$$U = \{\varnothing\} \cup \bigcup_{n=1}^{\infty} (\mathbb{N}^*)^n$$

such that

$$\varnothing \in \omega,$$
$$\forall\, u, v \in U, \qquad (uv \in \omega) \Rightarrow (u \in \omega),$$
$$\forall\, u \in \omega, \forall\, j \in \mathbb{N}^*, \qquad (uj \in \omega) \quad \Leftrightarrow \quad (1 \le j \le \nu_u(\omega)),$$

where $\nu_u(\omega)$ is a nonnegative integer. We denote by Ω the space of the trees and any $u \in U$ belonging to a tree ω is called a node of ω (or a particle when the branching process is a spatial one). For $u \in U$,

$$\Omega_u = \{\omega \in \Omega,\, u \in \omega\}$$

is the set of trees having u as a node. Hence, ν_u defines a map from Ω_u to $\mathbb{N}$. The notation $v < u$ means that v is an ancestor of u and $|u|$ denotes the length of u.

Consider now as the space of marks the space Γ of continuous real functions γ with lifetime σ and such that $\gamma(0) = 0$. We call $\mathscr{H}(t)$, $t \in \mathbb{R}_+$, the natural filtration on Γ. For every $t \in \mathbb{R}_+$, $\mathscr{H}(t) \subset \mathscr{H}(\sigma)$ and $\gamma(\sigma)$ is $\mathscr{H}(\sigma)$-measurable. By definition, a *marked tree* is

$$\overline{\omega} = (\omega, (\gamma_u,\, u \in \omega)),$$

where $\omega \in \Omega$ and $\gamma_u \in \Gamma$. Notice that other choices for the space of marks give rise to other models of branching processes [see, e.g., Neveu (1986), Chauvin (1986), (1988)].

We call π the canonical projection from the space $\overline{\Omega}$ of marked trees onto Ω. For any $u \in U$,

$$\overline{\Omega}_u = \pi^{-1}(\Omega_u)$$

is the set of marked trees having u as a node. The map induced by ν_u on $\overline{\Omega}_u$ by the canonical projection π is still called ν_u. Marks γ_u are maps from $\overline{\Omega}_u$ to Γ. Denote by $\sigma_u = \sigma \circ \gamma_u$ the lifetime of u. Hence σ_u are maps from $\overline{\Omega}_u$ to $\mathbb{R}_+$.

The birthtime S_u of particle u is defined on $\overline{\Omega}_u$ by

$$S_u = S_v + \sigma_v \qquad (v \text{ is } u\text{'s parent}),$$
$$S_\varnothing = 0.$$

To define the position of a particle we have to consider

$$\tilde{\Omega} = \mathbb{R} \times \overline{\Omega}.$$

We call $\tilde{\pi}$ the projection from $\tilde{\Omega}$ to $\overline{\Omega}$ and we still denote by ν_u, σ_u, γ_u and S_u the maps induced on $\tilde{\Omega}_u = \tilde{\pi}^{-1}(\overline{\Omega}_u)$ by ν_u, σ_u, γ_u and S_u. For any $u \in U$, we denote by $\mathscr{H}_u(t) = \gamma_u^{-1}(\mathscr{H}(t))$ the filtration on $\tilde{\Omega}_u$, generated by γ_u. Thus σ_u is a stopping time for $\mathscr{H}_u(\cdot)$. The initial position Y_u of particle u is defined on $\tilde{\Omega}_u$ by

$$Y_u = Y_v + \gamma_v(\sigma_v),$$
$$Y_\varnothing(x, \bar{\omega}) = x.$$

If $\mathscr{G}_\varnothing$ is the σ-algebra generated by $Y_\varnothing$ and $\mathscr{G}_u$, $u \neq \varnothing$, is the sub σ-algebra on $\tilde{\Omega}_u$ defined inductively by

$$\mathscr{G}_u = (\mathscr{G}_v \vee \mathscr{H}_v(\sigma_v)) \cap \tilde{\Omega}_u \qquad (v \text{ is } u\text{'s parent}),$$

then S_u and Y_u are $\mathscr{G}_u$-measurable for every $u \in U$. For $s \in \mathbb{R}_+$, there is a filtration on $\tilde{\Omega}_u$ defined by

$$\mathscr{A}_u(s) = \mathscr{G}_u \vee \mathscr{H}_u(s).$$

A particle whose age is s will be said to be s-old. The position of a particle u which is s-old is now defined on $\tilde{\Omega}_u \cap \{s \leq \sigma_u\}$ by

$$X_u(s) = Y_u + \gamma_u(s)$$

and $X_u(s)$ is $\mathscr{A}_u(s)$-measurable. In the following, $X_u(\sigma_u)$ stands for $X_u(\sigma_u^-)$.

The space $\tilde{\Omega}$ is endowed with the σ-algebra $\tilde{\mathscr{F}}$ generated by $\{\tilde{\Omega}_u, X_u\}$, $u \in U$, for which the previous defined maps are measurable.

By definition, a *stopping line* τ is a family of positive random variables τ_u: $\tilde{\Omega}_u \to [0, \sigma_u]$, indexed by $u \in U$, such that: (i) τ_u is a stopping time for $\mathscr{A}_u(\cdot)$; (ii) $L_\tau(\tilde{\omega}) = \{u, u \in \tilde{\pi}(\tilde{\omega}), 0 \leq \tau_u(\tilde{\omega}) < \sigma_u(\tilde{\omega})\}$ has the line property, that is, no strict ancestor of a node in L belongs to L (this idea is also found in the notion of prefix code).

For a stopping line τ, define D_τ as the set of strict descendants of the line, say

$$D_\tau = \{u, u \in U: \exists\, v \in U, v < u, v \neq u, v \in L_\tau\}.$$

Notice that any family of r.v. τ_u: $\tilde{\Omega}_u \to [0, \sigma_u]$, indexed by $u \in U$, satisfying condition (i) can be modified to become a stopping line in the following way. Let

$$(2.1) \qquad \tau'_u = \begin{cases} \sigma_u & \text{if } \exists\, v \in U, v < u, v \neq u, \tau_v < \sigma_v, \\ \tau_u & \text{otherwise.} \end{cases}$$

The σ-algebra associated with a stopping line is defined on $\tilde{\Omega}$ by

$$\mathscr{F}_\tau = \bigvee_{u \in U} \{u \notin D_\tau\} \cap \mathscr{A}_u(\tau_u)$$

so that $\mathscr{F}_\tau$ contains on the one hand the whole life of v, $v \in U$, for v ancestor of some u in L_τ or v in a branch not yet at L_τ, and on the other hand $\mathscr{F}_\tau$ contains the life of u until τ_u for u in L_τ.

We define a partial order relation among stopping lines, setting $\tau \leq \tau'$ if and only if $D_\tau \supseteq D_\tau$ and $(u \in L_\tau \cap L_{\tau'} \Rightarrow \tau_u \leq \tau'_u)$.

Finally, for $u \in U$ and $s \in \mathbb{R}_+$, we define shift operators $T_{u,s}$: $\tilde{\Omega}_u \cap \{\sigma_u > s\} \to \tilde{\Omega}$ by

$$\begin{aligned} Y_\varnothing \circ T_{u,s}(\tilde{\omega}) &= X_u(s), \\ \tilde{\pi}(T_{u,s}(\tilde{\omega})) &= \{v, v \in U, uv \in \tilde{\pi}(\tilde{\omega})\}, \\ \gamma_v \circ T_{u,s}(\tilde{\omega}) &= \gamma_{uv}(\tilde{\omega}), \qquad \tilde{\omega} \in \tilde{\Omega}_{uv}, \qquad v \neq \varnothing, \\ \sigma_\varnothing \circ T_{u,s}(\tilde{\omega}) &= \sigma_u(\tilde{\omega}) - s, \\ \gamma_\varnothing(t) \circ T_{u,s}(\tilde{\omega}) &= \gamma_u(s+t)(\tilde{\omega}) - \gamma_u(s)(\tilde{\omega}) \quad \text{on} \quad \{s + t \leq \sigma_u\}. \end{aligned}$$

In other words, $T_{u,s}(\tilde{\omega})$ is the tree beginning at node u, when u is s-old, with initial position $X_u(s)$.

Examples.

Example 1. For $n \in \mathbb{N}$, $u \in U$, let us define on $\tilde{\Omega}_u$

$$\tau_u^{(n)} = \begin{cases} \sigma_u & \text{if } |u| \neq n, \\ 0 & \text{if } |u| = n, \end{cases}$$

so that $\tau^{(n)}$ is a stopping line and

$$L_n(\tilde{\omega}) = L_{\tau^{(n)}}(\tilde{\omega}) = \{u, u \in \tilde{\pi}(\tilde{\omega}), |u| = n\}$$

is the nth generation of a tree $\tilde{\omega}$. The associated σ-algebra is denoted by $\mathscr{F}_n$.

Example 2. For $t \in \mathbb{R}_+$, $u \in U$, let us define on $\tilde{\Omega}_u$,

$$\tau_u^{(t)} = \begin{cases} \sigma_u & \text{if } t < S_u \text{ or } S_u + \sigma_u \leq t, \\ t - S_u & \text{if } S_u \leq t < S_u + \sigma_u, \end{cases}$$

so that $\tau^{(t)}$ is a stopping line (the line property is ensured because the birthtime of a particle is the deathtime of its parent) and

$$L_t(\tilde{\omega}) = L_{\tau^{(t)}}(\tilde{\omega}) = \{u, u \in \tilde{\pi}(\tilde{\omega}), S_u(\tilde{\omega}) \leq t < S_u(\tilde{\omega}) + \sigma_u(\tilde{\omega})\}$$

is the population living at time t. The σ-algebra associated with this line is denoted by $\mathscr{F}_t$ and $(\mathscr{F}_t, t \in \mathbb{R}_+)$ is a filtration on $\tilde{\Omega}$.

Example 3. For $x, a \in \mathbb{R}_+$, $u \in U$, let us define on $\tilde{\Omega}_u$ the first age when particle u meets the line whose equation in the plane (y, t) is $y = at - x$, by

$$\begin{aligned} \tau_u'^x &= \inf\{s, 0 \leq s < \sigma_u, X_u(s) = a(S_u + s) - x\} \\ &= \sigma_u \quad \text{if } \forall\, s, \qquad 0 \leq s < \sigma_u, \qquad X_u(s) \neq a(S_u + s) - x. \end{aligned}$$

Thus $\tau_u'^x$ is a stopping time for $\mathscr{A}_u(\cdot)$ and we get a stopping line τ^x by putting as in (2.1),

$$\tau_u^x = \begin{cases} \sigma_u & \text{if } \exists\, v \in U, v < u, v \neq u, \tau_v'^x < \sigma_v, \\ \tau_u'^x & \text{otherwise,} \end{cases}$$

so that the line property is satisfied. We denote by $\mathscr{F}_x$ the associated σ-algebra; $(\mathscr{F}_x, x \in \mathbb{R}_+)$ is a filtration on $\tilde{\Omega}$ and L_x is written instead of L_{τ^x}. For u in L_x, times $S_u + \tau_u^x$ are the first crossings of Neveu (1987).

There are many analogies between stopping lines and stopping times [some of them can be found in Chauvin (1988) or in Jagers (1989)] which allow the establishment of a strong Markov property for spatial trees as follows.

For convenience, we write ν, σ, γ, X and Y instead of $\nu_\varnothing$, $\sigma_\varnothing$, $\gamma_\varnothing$, $X_\varnothing$ and $Y_\varnothing$, respectively.

PROPOSITION 2.1. *For every reproduction law $p = (p_n, n \in \mathbb{N})$, for every probability law λ on $\mathbb{R}$, there is a probability P_λ on a space $(\tilde{\Omega}, \tilde{\mathscr{F}})$ such that*:

(i) *ν is p-distributed, γ is a standard Brownian motion killed at time σ which is $\exp(\alpha)$-distributed, Y is λ-distributed, ν, γ and Y are independent*;

(ii) *the following branching property is valid. Let τ be a stopping line. Conditioned on $\mathscr{F}_\tau$, shifted trees T_{u,τ_u}, $u \in L_\tau$, are independent, $P_{X_u(\tau_u)}$-distributed (P_x stands for P_{ε_x}). In other words, for every nonnegative $(\tilde{\Omega}, \tilde{\mathscr{F}})$-measurable function f_u indexed by $u \in U$, for every $x \in \mathbb{R}$,*

$$E_\lambda^{\mathscr{F}_\tau}\Big(\prod_{u \in L_\tau} f_u \circ T_{u,\tau_u}\Big) = \prod_{u \in L_\tau} E_{X_u(\tau_u)}(f_u). \tag{2.2}$$

The proof is analogous to Neveu's [in Neveu (1986)] for Galton–Watson trees.

The following two corollaries should not be considered as new results: The martingale Z_t in Corollary 2.2 has been much used by Biggins (1978), Uchiyama (1982) and others, and the branching process μ_x in Corollary 2.3 has been introduced by Neveu (1987). We state them here to show how these martingale properties are nothing but the branching property.

COROLLARY 2.2 (Example 2). *For $\lambda \in \mathbb{R}_+$, for $\phi: \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$E_0(\nu e^{\lambda X(\sigma) - \sigma\phi(\lambda)}) = 1, \tag{2.3}$$

$$Z_t(\lambda) = \sum_{u \in L_t} e^{\lambda X_u(t - S_u) - t\phi(\lambda)}$$

is a positive $\mathscr{F}_t$-martingale whose expectation equals 1 [*in our model $\phi(\lambda)$ equals $\lambda^2/2 + \alpha(m-1)$*].

PROOF. For $s, t \in \mathbb{R}_+$, $s \le t$, apply the branching property to L_s. □

COROLLARY 2.3 (Example 3). *Let $x \in \mathbb{R}_+$ and denote by*

$$\mu_x = \mathrm{Card}(L_x) = \sum_{u \in U} 1_{\{0 \le \tau_u^x < \sigma_u\}} \tag{2.4}$$

the number of first crossings of line L_x. Then $(\mu_x, x \in \mathbb{R}_+)$ is a branching process.

PROOF. For $x, y \in \mathbb{R}_+$, $x \leq y$, $s \in [0,1]$, express $E_0^{\mathscr{F}_x}(s^{\mu_y})$ and apply the branching property to L_x. □

The last useful tool is the following lemma where a decomposition of stopping lines with pasting is made as for stopping times. This is essentially the theorem of Courrege and Priouret [in Dellacherie and Meyer (1975), Section 94-103].

LEMMA 2.4. *Let $\tau = (\tau_u,\ u \in U)$ and $\rho = (\rho_u,\ u \in U)$ be two stopping lines such that $\tau \leq \rho$. Then, for any $v \in U$, there is a family $(\mu_w^v,\ w \in U)$, μ_w^v: $\tilde{\Omega}_v \times \tilde{\Omega} \to [0, \sigma_{vw}]$ such that*:

(i) *$\mu_w^v(\cdot,\cdot)$ is $\mathscr{A}_v(\tau_v) \otimes \tilde{\mathscr{F}}$-measurable*;
(ii) *for every $\tilde{\omega} \in \tilde{\Omega}_v$, $\mu_w^v(\tilde{\omega}, \cdot)$ is a stopping time for $\mathscr{A}_w(\cdot)$*;
(iii) *for every $\tilde{\omega} \in \tilde{\Omega}_v$,*

$$1_{vw \in L_\rho} 1_{v \in L_\tau} \mu_w^v(\tilde{\omega}, T_{v,\tau_v}(\tilde{\omega})) = \rho_{vw}(\tilde{\omega}) 1_{vw \in L_\rho} 1_{v \in L_\tau}, \qquad w \neq \varnothing$$

$$1_{v \in L_\rho \cap L_\tau} \mu_\varnothing^v(\tilde{\omega}, T_{v,\tau_v}(\tilde{\omega})) = (\rho_v(\tilde{\omega}) - \tau_v(\tilde{\omega})) 1_{v \in L_\rho \cap L_\tau}.$$

3. Product martingale associated with a family of stopping lines. Let us consider a supercritical branching process for which $m > 1$. For simplicity, let us assume that $p_0 = 0$. Let $a \in \mathbb{R}_+$ and let $\psi \in C^2(\mathbb{R})$ be a solution of the K-P-P differential equation [Kolmogorov, Petrovski and Piscounov (1937)]

$$\begin{aligned} &\tfrac{1}{2}\psi'' - a\psi' + \alpha(f(\psi) - \psi) = 0 \\ &\psi(-\infty) = 1, \qquad \psi(+\infty) = 0. \end{aligned} \tag{3.1}$$

It is known by the theory of differential equations [see, e.g., Bramson (1983)] that equation (3.1) has a solution with values in $[0,1]$ if and only if $a^2 \geq 2\alpha(m-1)$. In this case the solution is unique up to a translation.

THEOREM 3.1. *Let $a \geq \sqrt{2\alpha(m-1)}$ and let $\psi \in C^2(\mathbb{R})$ be a solution of equation (3.1), where f is the generating function of $p = (p_n, n \geq 1)$ with mean $m > 1$. Let $(\tau^x,\ x \in \mathbb{R}_+)$, $\tau^x = (\tau_u^x,\ u \in U)$, be a family of stopping lines indexed by $\mathbb{R}_+$, such that*

$$\forall\, x, y \in \mathbb{R}_+, \qquad x \leq y \Rightarrow \tau^x \leq \tau^y \tag{3.2}$$

so that $(\mathscr{F}_{\tau^x},\ x \in \mathbb{R}_+)$ defines a filtration on space $(\tilde{\Omega}, \tilde{\mathscr{F}})$.

For any stopping line τ, for $n \in \mathbb{N}$, let

$$A_\tau^{(n)} = \{u,\ u \in U,\ |u| = n,\ u \notin D_\tau,\ u \notin L_\tau\}$$

be the set of particles of the nth generation which have not yet reached the line L_τ. Assume that for every $x \in \mathbb{R}_+$, $y \in \mathbb{R}$, the following barrier condition holds:

$$A_{\tau^x}^{(n)} \to_{n \to +\infty} \varnothing \qquad P_y\text{-}a.s. \tag{3.3}$$

For $x \in \mathbb{R}_+$, let

$$M_{\tau^x} = \prod_{u \in L_{\tau^x}} \psi(X_u(\tau_u^x) - a(S_u + \tau_u^x)). \tag{3.4}$$

Then $M_{\tau^\cdot}$ is a nonnegative $\mathscr{F}_{\tau^\cdot}$-martingale.

PROOF. Let $x, y \in \mathbb{R}_+$, $x \le y$. By (3.2), $\tau^x \le \tau^y$ and every particle in L_{τ^y} is a descendant of some particle in L_{τ^x}. Hence

$$M_{\tau^y} = \prod_{v \in L_{\tau^x}} \prod_{\substack{w \in T_{v,\tau_v^x} \\ vw \in L_{\tau^y}}} \psi(X_{vw}(\tau_{vw}^y) - a(S_{vw} + \tau_{vw}^y)).$$

As in the proof of Corollary 2.3, we need to express τ^y with T_{v,τ_v^x}. The decomposition was explicit in the case of Example 3 but here we need the decomposition of Lemma 2.4:

$$\tau_{vw}^y(\tilde{\omega}) = \mu_w^v(\tilde{\omega}, T_{v,\tau_v^x}(\tilde{\omega})), \qquad w \ne \varnothing,$$
$$\tau_v^y(\tilde{\omega}) = \tau_v^x(\tilde{\omega}) + \mu_\varnothing^v(\tilde{\omega}, T_{v,\tau_v^x}(\tilde{\omega}))$$

for $v \in L_{\tau^x}$ and $vw \in L_{\tau^y}$. Let us apply branching property (2.2). We get

$$E_\lambda^{\mathscr{F}_{\tau^x}}(M_{\tau^y}) = \prod_{v \in L_{\tau^x}} \int \prod_{\mu_w^v(\tilde{\omega}, \tilde{\omega}') < \sigma_v(\tilde{\omega}')} \psi(-a(S_v + \tau_v^x) + X_w(\mu_w^v(\tilde{\omega}, \tilde{\omega}'))$$
$$-a(S_w(\tilde{\omega}') + \mu_w^v(\tilde{\omega}, \tilde{\omega}'))) \, dP_{X_v(\tau_v^x)}(\tilde{\omega}')$$

which can be written (because $\mu^v = \{\mu_w^v(\tilde{\omega}, \cdot\,), w \in U\}$ is a stopping line)

$$E_\lambda^{\mathscr{F}_{\tau^x}}(M_{\tau^y}) = \prod_{v \in L_{\tau^x}} \int M_{\mu^v} \, dP_{X_v(\tau_v^x) - a(S_v + \tau_v^X)}.$$

By the barrier condition (3.3) let us notice that, conditioned on $\mathscr{F}_{\tau^x}$, for any $v \in L_{\tau^x}$, $\tilde{\omega} \in \tilde{\Omega}_v$,

$$A_{\mu^v(\tilde{\omega}, \cdot)}^{(n)} \to_{n \to +\infty} \varnothing \qquad P_{X_v(\tau_v^x)}\text{-a.s.}$$

so that it is sufficient to prove:

LEMMA 3.2. *Let $\tau = (\tau_u, u \in U)$ be a stopping line such that*

$$A_\tau^{(n)} \to_{n \to +\infty} \varnothing \qquad P_x\text{-}a.s. \ (\textit{for some } x \in \mathbb{R}).$$

Then, for every $z \in \mathbb{R}$,

$$E_{x+z}(M_\tau) = \psi(x + z).$$

PROOF OF LEMMA 3.2. For $n \in \mathbb{N}$, let us introduce the following approximation of M_τ:

$$M_\tau^{(n)} = \prod_{\substack{u \in L_\tau \\ |u| \le n}} \psi(X_u(\tau_u) - a(S_u + \tau_u)) \prod_{u \in A_\tau^{(n)}} \psi^{\nu_u}(X_u(\sigma_u) - a(S_u + \sigma_u)),$$

where ν_u denotes the number of children of particle u. The assumption of the lemma gives the L^1-convergence of $M_\tau^{(n)}$ to M_τ (ψ is $[0,1]$-valued). It is then sufficient to show that $M_\tau^{(n)}$ is a nonnegative $\mathscr{F}_{n+1}$-martingale ($\mathscr{F}_n$ has been introduced in Example 1). Without loss of generality it suffices to consider $z = 0$ so that the purpose is to prove

$$\forall\, n \in \mathbb{N}, \qquad E_x^{\mathscr{F}_{n+1}}\big(M_\tau^{(n+1)}\big) = M_\tau^{(n)}, \tag{3.5}$$

$$E_x\big(M_\tau^{(0)}\big) = \psi(x). \tag{3.6}$$

Here X denotes a Brownian motion on $\mathbb{R}$ and we get by a standard computation from Itô's formula, for every $t \in \mathbb{R}_+$,

$$E_x\big(1_{\sigma > t}\psi(X(t) - at) + 1_{\sigma \le t}\psi^\nu(X(\sigma) - a\sigma)\big) = \psi(x), \tag{3.7}$$

which gives (3.6).

For the martingale property (3.5), let us write

$$\begin{aligned} E_x^{\mathscr{F}_{n+1}}\big(M_\tau^{(n+1)}\big) = {} & \prod_{\substack{u \in L_\tau \\ |u| \le n}} \psi(X_u(\tau_u) - a(S_u + \tau_u)) \\ & \times E_x^{\mathscr{F}_{n+1}}\Bigg(\prod_{\substack{u \in L_\tau \\ |u| = n+1}} \psi(X_u(\tau_u) - a(S_u + \tau_u)) \\ & \qquad \times \prod_{u \in A_\tau^{(n+1)}} \psi^{\nu_u}(X_u(\sigma_u) - a(S_u + \sigma_u))\Bigg). \end{aligned}$$

Let us consider particles u such that $|u| = n+1$ and $u \notin D_\tau$. If such a particle is in L_τ, it occurs in the second product; if not, it occurs in the third one. Hence the expectation in the R.H.S. equals

$$\begin{aligned} E_x^{\mathscr{F}_{n+1}}\Bigg(\prod_{\substack{|u| = n+1 \\ u \notin D_\tau}} \big(& 1_{\tau_u < \sigma_u}\psi(X_u(\tau_u) - a(S_u + \tau_u)) \\ & + 1_{\tau_u \ge \sigma_u}\psi^{\nu_u}(X_u(\sigma_u) - a(S_u + \sigma_u))\big)\Bigg) \end{aligned}$$

and we use now the branching property applied to the line $\tau^{(n+1)}$ (introduced in Example 1). Let us moreover notice that $\{u \notin D_\tau\}$ is in $\mathscr{F}_{n+1}$, S_u is $\mathscr{F}_{n+1}$-measurable and by the decomposition lemma applied to stopping lines $\tau^{(n+1)} \wedge \tau$ and τ, there is some μ^U such that

$$\tau_u(\tilde\omega) = \mu_\varnothing^u(\tilde\omega, T_{u,0}(\tilde\omega)).$$

Thus the above expectation becomes

$$\begin{aligned} \prod_{\substack{|u| = n+1 \\ u \notin D_\tau}} E_{Y_u}\big(& 1_{\mu_\varnothing^u(\tilde\omega,\cdot) < \sigma}\psi(-aS_u + X(\mu_\varnothing^u(\tilde\omega,\cdot)) - a\mu_\varnothing^u(\tilde\omega,\cdot)) \\ & + 1_{\mu_\varnothing^u(\tilde\omega,\cdot) \ge \sigma}\psi^\nu(-aS_u + X(\sigma) - a\sigma)\big), \end{aligned}$$

which equals, by (3.7),

$$\prod_{\substack{|u|=n+1\\ u\notin D_\tau}} \psi(Y_u - aS_u).$$

Putting together the children of the same particle it becomes

$$\prod_{\substack{|v|=n,\, v\notin D_\tau\\ v\notin L_\tau}} \psi^{\nu_v}(X_v(\sigma_v) - a(S_v + \sigma_v)), \tag{3.8}$$

and (3.5) is realized. □

Our purpose is now to identify the limit when $x \to +\infty$ of the martingale M_{τ^x} for a family of stopping lines which tends to infinity as in Definition 3.3.

DEFINITION 3.3. An increasing family of stopping lines $(\tau^x,\, x \in \mathbb{R}_+)$ is said to tend to infinity when x tends to infinity if

$$\forall\, \tilde{\omega} \in \tilde{\Omega}, \qquad \inf\{|u|,\, u \in L_{\tau^x}\} \to_{x\to+\infty} +\infty \tag{3.9}$$

(in other words, for x sufficiently large, L_{τ^x} is posterior to any generation line L_n).

THEOREM 3.4. *Let $(\tau^x,\, x \in \mathbb{R}_+)$ be an increasing family of stopping lines such that* (3.3) *and* (3.9) *hold. Then*

$$\lim_{x\to+\infty} M_{\tau^x} = W_\psi = e^{-Z_\psi} \tag{3.10}$$

does not depend on the family $(\tau^x,\, x \in \mathbb{R}_+)$ of stopping lines. The law of Z_ψ is given by its Laplace transform

$$E_0\big(e^{-e^{-bz}Z_\psi}\big) = \psi(z), \tag{3.11}$$

where $b = a - \sqrt{a^2 - 2a(m-1)}$.

PROOF. Suppose that τ^x and $\tilde{\tau}^x$ are two increasing families of stopping lines indexed by $\mathbb{R}_+$ satisfying (3.3) and (3.9). For any x, there exists a perhaps random $y(x)$ such that

$$\tau^x \le \tilde{\tau}^{y(x)}$$

[indeed the barrier condition implies that $N(x) = \sup\{|u|,\, u \in L_{\tau^x}\}$ is a.s. finite and (3.9) provides $y(x)$]. Moreover $y(x)$ is a $\mathcal{F}_{\tilde{\tau}^\cdot}$ stopping time and by Theorem 3.1,

$$M_{\tau^x} = E^{\mathcal{F}_{\tau^x}}\big(M_{\tilde{\tau}^{y(x)}}\big).$$

In the same way there exists a $y'(x)$ such that

$$\tilde{\tau}^x \le \tau^{y'(x)},$$

$y'(x)$ is a $\mathscr{F}_{\tau}$. stopping time and

$$M_{\bar{\tau}^x} = E^{\mathscr{F}_{\bar{\tau}^x}}(M_{\tau^{y'(x)}}).$$

Then M_{τ^x} and $M_{\bar{\tau}^x}$ have the same limit when x tends to infinity.

For the second part of the theorem, notice that if

$$X_t^* = \sup_{u \in L_t} X_u(t - S_u)$$

denotes the position of the rightmost particle at time t, it is known [Bramson (1983), Neveu (1987)] that for $a \geq \sqrt{2\alpha(m-1)}$,

$$X_t^* - at \to_{t \to +\infty} -\infty.$$

Together with (3.9) this implies

$$\inf\{X_u(\tau_u^x) - a(S_u + \tau_u^x),\, u \in L_{\tau^x}\} \to_{x \to +\infty} -\infty. \tag{3.12}$$

Moreover we know [Bramson (1983)] the asymptotic behaviour of the function ψ when $x \to -\infty$:

$$-\log \psi(x) \sim ce^{bx} \quad \text{if } a > \sqrt{2\alpha(m-1)},$$

$$-\log \psi(x) \sim (c' - cx)e^{bx} \quad \text{if } a = \sqrt{2\alpha(m-1)},$$

where $c' \in \mathbb{R}$, $c \in \mathbb{R}_+$, $b = a - \sqrt{a^2 - 2\alpha(m-1)}$. Thus

$$\frac{\log \psi(z+x)}{\log \psi(x)} \to_{x \to -\infty} e^{bz}$$

and with (3.12) this leads to (3.11). □

COROLLARY 3.5. *With the notation of Examples* 2 *and* 3, *let*

$$M_t = \prod_{u \in L_t} \psi(X_u(t - S_u) - at)$$

and

$$M_x = \prod_{u \in L_x} \psi(X_u(\tau_u^x) - a(S_u + \tau_u^x)) = (\psi(-x))^{\mu_x}.$$

Then, M_t *and* M_x *are, respectively*, $\mathscr{F}_t$ *and* $\mathscr{F}_x$ *martingales and they have the same limit when* $t \to +\infty$ *and* $x \to +\infty$, *respectively.*

We may conclude that with any family of stopping lines satisfying some reasonable conditions (satisfied by the line of particles living at time t, the generation line, the stopping line associated with an oblique straight line) one can associate a product martingale and all these martingales have the same limit.

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