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GAUSSIAN MEASURE OF LARGE BALLS IN l_p [1]

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We study the behaviour of $\mu\{x \in E;\ \|x\| > t\}$ as $t \to \infty$ for a Gaussian measure μ in a Banach or quasi-Banach space in the following cases:

1. $E = l_p$, $2 < p < \infty$, and μ of diagonal form but not necessarily symmetric;
2. $E =$ Hilbert space and μ arbitrary;
3. $E = l_p^n$, $0 < p < 2$, and μ of diagonal form.

While 2 solves a problem of Hweng (1980), 1 and 3 extend some results of Dobrič, Marcus and Weber (1988).

1. Introduction. Let μ be a Gaussian (Radon) measure in a Banach space E, that is, $a(\mu)$ is Gaussian for each $a \in E'$, the dual space of E. Then there exists a uniquely determined $x_0 \in E$ for which

$$\mu_0 := \mu * \delta_{x_0} \tag{1.1}$$

is Gaussian symmetric on E [cf. Borell (1976)]. Define $\sigma = \sigma(\mu)$ by

$$\sigma := \sup\left\{\left\{\int_E \langle x, a\rangle^2 \, d\mu_0(x)\right\}^{1/2};\ \|a\| \le 1,\ a \in E'\right\}, \tag{1.2}$$

where μ_0 is as in (1.1). Then the following fundamental result holds [cf. also Landau and Shepp (1970), Fernique (1970) and Marcus and Shepp (1972)]:

THEOREM 1 [Borell (1975)]. *If μ is Gaussian on E, then*

$$\lim_{t\to\infty} t^{-2} \log \mu\{\|x\| > t\} = -1/2\sigma^2. \tag{1.3}$$

If Φ on $[0, \infty)$ is defined by

$$\Phi(t) := \sqrt{2/\pi} \int_0^t e^{-s^2/2} \, ds, \tag{1.4}$$

then Theorem 1 can be reformulated as

$$\lim_{t\to\infty} \frac{\mu\{\|x\| > t\}}{1 - \Phi(t/\sigma)} e^{-\varepsilon t^2} = 0 \tag{1.5}$$

for each $\varepsilon > 0$.

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While for arbitrary measures (1.5) is best possible, Talagrand [cf. Talagrand (1984)] verified

$$\lim_{t\to\infty} \frac{\mu\{\|x\| > t\}}{1 - \Phi(t/\sigma)} e^{-\varepsilon t} = 0, \qquad \varepsilon > 0, \tag{1.6}$$

in the case of symmetric μ. Moreover, (1.6) is optimal in the following sense: If φ: $[0, \infty) \to (0, \infty)$ is an arbitrary decreasing function with $\varphi(t) \to 0$ as $t \to \infty$, then there exists a Gaussian symmetric measure μ with $\sigma = 1$ and

$$\mu\{\|x\| > t\}/(1 - \Phi(t)) \geq e^{t\varphi(t)}, \qquad t > t_0 \tag{1.7}$$

[Talagrand (1984)].

Hence the rough behaviour of $\mu\{\|x\| > t\}$ is $1 - \Phi(t/\sigma)$, that is, $\exp(-t^2/2\sigma^2)$, while the fine behaviour, that is, the behaviour of $\mu\{\|x\| > t\}/(1 - \Phi(t/\sigma))$, may vary between 1 and $e^{t\varphi(t)}$ in the sense of (1.7) (symmetric case) or between 1 and $\psi(t)$, where $\psi(t)\exp(-\varepsilon t^2) \to 0$ as $t \to \infty$ for each $\varepsilon > 0$. Thus, given a concrete Gaussian measure μ, it is interesting to investigate the exact behaviour of $\mu\{\|x\| > t\}$ as $t \to \infty$. This was only known in a few examples, as for instance for symmetric μ in l_2 [Zolotarev (1961), Hertle (1983)] or for some nonsymmetric μ in l_2 [Hweng (1980)]. We shall determine this behaviour for arbitrary μ in l_2 which answers a question of Hweng (1980). Another example of known exact behaviour is that of symmetric measures in l_p, $2 < p < \infty$, possessing diagonal form [cf. Dobrič, Marcus and Weber (1988)]. We extend their result to the nonsymmetric case.

Finally, we investigate this question for symmetric diagonal measures on l_p^n, $p < 2$. Here we obtain the following surprising result: If $\sigma_1, \dots, \sigma_n > 0$ are arbitrary real numbers, $\sigma := \{\sum_1^n \sigma_i^r\}^{1/r}$, $1/r = 1/p - 1/2$, then

$$\lim_{t\to\infty} \mathbf{P}\left\{\sum_1^n \sigma_i^p |\theta_i|^p > t^p\right\} / (1 - \Phi(t/\sigma)) = \left(2/\sqrt{2-p}\right)^{n-1},$$

where $\theta_1, \dots, \theta_n$ are independent standard Gaussian. This also extends a result of Dobrič, Marcus and Weber (1988).

2. Notation. Here and in the sequel, $\theta_1, \theta_2, \dots$ always denotes a sequence of independent standard Gaussian random variables. Of course, we have

$$\mathbf{P}\{|\theta_1| \leq t\} = \Phi(t),$$

where Φ was defined in (1.4). Moreover, we set $\Phi(t) = 0$ for $t < 0$. Later on, the two following properties of Φ will be used several times [cf. Fernique (1976) for the first one]:

$$\sqrt{2/\pi}(1+t)^{-1} e^{-t^2/2} \leq 1 - \Phi(t) \leq (4/3)\sqrt{2/\pi}(1+t)^{-1} e^{-t^2/2} \tag{2.1}$$

for $t \geq 0$ and

$$\lim_{t\to\infty} (1 - \Phi(t)) t e^{t^2/2} = \sqrt{2/\pi}. \tag{2.2}$$

Furthermore, for each natural n we define Φ_n on $[0, \infty)$ by

$$\Phi_n(t) := \mathbf{P}\left\{ \sum_1^n \theta_i^2 \le t^2 \right\}. \tag{2.3}$$

It is well known that

$$\Phi_n(t) = \left(2^{n/2-1}\Gamma(n/2)\right)^{-1} \int_0^t s^{n-1} e^{-s^2/2}\, ds$$

and

$$\lim_{t\to\infty} (1 - \Phi_n(t))/(t^{n-2}e^{-t^2/2}) = \left(2^{n/2-1}\Gamma(n/2)\right)^{-1}.$$

3. Measures on l_p, $p > 2$. Let $\sigma_1 = \cdots = \sigma_k > \sigma_{k+1} \ge \cdots \ge 0$ be a sequence with $\sum_1^\infty \sigma_i^p < \infty$. Then Dobrič, Marcus and Weber (1988) proved

$$\lim_{t\to\infty} \mathbf{P}\left\{ \sum_1^\infty \sigma_i^p |\theta_i|^p > t^p \right\} / (1 - \Phi(t/\sigma_1)) = k$$

for $2 < p < \infty$.

We want to study the same question for

$$\mathbf{P}\left\{ \sum_1^\infty |\sigma_i \theta_i - \xi_i|^p > t^p \right\},$$

where $(\xi_i)_{i=1}^\infty$ is an arbitrary sequence in l_p. We shall need properties of two functions g and h both defined on $[0, 1]$:

$$g(u) := (1 - u^p)^{2/p} - 1 + u^2. \tag{3.1}$$

Let α, β be real numbers with $\alpha \ge \beta$ and $\alpha \ge 0$. Then $h = h_{\alpha,\beta}$ is defined by

$$h(u) := \alpha(1 - u^p)^{1/p} - \alpha + \beta u. \tag{3.2}$$

LEMMA 1. *Assume $2 < p < \infty$. Then for each $\eta > 0$,*

$$\lim_{t\to\infty} t^2 \int_\eta^1 \exp\{-t^2 g(u)/2 + t h(u)\}\, du = 0. \tag{3.3}$$

PROOF. Let us first verify that this is valid for the integral from η to $1 - \delta$ for each $\delta > 0$. Observe that $g(u) > 0$ for $u \in (0, 1)$. Hence,

$$g_0 := \inf\{g(u);\ \eta \le u \le 1 - \delta\} > 0$$

and

$$t^2 \int_\eta^{1-\delta} \exp\{-t^2 g(u)/2 + t h(u)\}\, du \le t^2 \exp\{-t^2 g_0/2 + td\},$$

where $d := \sup_{0<u<1} h(u)$. Clearly, $t^2 \exp\{-t^2 g_0/2 + td\} \to 0$ as $t \to \infty$, proving our first claim.

In order to complete the proof we first treat the case $\alpha > \beta$. Then $h(1) = \beta - \alpha < 0$ and for some $\delta > 0$, we have

$$\gamma := \sup\{h(u); 1 - \delta \le u \le 1\} < 0.$$

Then

$$t^2 \int_{1-\delta}^{1} \exp\{-t^2 g(u)/2 + th(u)\}\, du \le \delta t^2 e^{t\gamma} \to 0 \quad \text{as } t \to \infty, \tag{3.4}$$

completing the proof in this case.

Finally, we assume $\alpha = \beta \ge 0$. By a change of variables, (3.4) coincides with

$$t^2 \int_0^{\delta'} \exp\{-t^2 g(u)/2 + th(u)\} u^{p-1}(1 - u^p)^{1/p-1}\, du, \tag{3.5}$$

where $(1 - (1 - \delta)^p)^{1/p} = \delta'$.

If δ (or, equivalently, δ') is small enough, then

$$g(u) > u^2/2, \qquad h(u) < 2\beta u \quad \text{and} \quad (1 - u^p)^{1-1/p} > \tfrac{1}{2}$$

for all $u \in [0, \delta']$. Inserting this in (3.5) leads to

$$2t^2 \int_0^{\delta'} \exp\{-t^2 u^2/4 + 2\beta t u\} u^{p-1}\, du \le c(\delta', \beta) t^{2-p} \to 0$$

as $t \to \infty$ and this completes the proof. □

LEMMA 2. *Let g and h be as in* (3.1) *or* (3.2), *respectively. Then we have*

$$\lim_{\eta \downarrow 0} \limsup_{t \to \infty} t \int_0^{\eta} \exp\{-t^2 g(u)/2 + th(u)\}\, du \le e^{\beta^2/2} \int_{-\beta}^{\infty} e^{-u^2/2}\, du$$

and the reverse inequality holds for the limit inferior.

PROOF. We only prove the assertion for the limit superior. The case of the limit inferior follows after some obvious modifications in the same way. Furthermore, we suppose $\beta \ge 0$. If $\beta < 0$, some small changes are necessary.

We have $g(u) \ge u^2(1 - \varepsilon)^2$ and $h(u) \le \beta u$, $(\alpha \ge 0)$, whenever $0 \le u \le \eta$ and $\varepsilon \to 0$ as $\eta \to 0$. Consequently,

$$\begin{aligned}
&t \int_0^{\eta} \exp\{-t^2 g(u)/2 + th(u)\}\, du \\
&\quad \le t \int_0^{\eta} \exp\left\{-t^2 u^2 (1 - \varepsilon)^2/2 + t\beta u\right\} du \\
&\quad = \exp\left\{\beta^2 (1 - \varepsilon)^{-2}/2\right\} t \int_0^{\eta} \exp\left\{-\tfrac{1}{2}\left(tu(1 - \varepsilon) - \beta/(1 - \varepsilon)\right)^2\right\} du \\
&\quad \le (1 - \varepsilon)^{-1} \exp\left\{\beta^2 (1 - \varepsilon)^{-2}/2\right\} \int_{-\beta/(1-\varepsilon)}^{\infty} e^{-u^2/2}\, du
\end{aligned}$$

and the assertion follows by taking the limit $\eta \to 0$, that is, $\varepsilon \to 0$. □

Our next aim is to determine the behaviour of

$$(3.6)\qquad \mathbf{P}\left\{\sum_1^n |\theta_i - \xi_i|^p > t^p\right\}$$

as $t \to \infty$ for some finite sequence $\xi_1, \dots, \xi_n$. Observe that (3.6) does not change if we permute the ξ_i's or if we replace one (or several) ξ_i by $-\xi_i$. So we can and do assume

$$\xi_1 = \cdots = \xi_j > \xi_{j+1} \geq \cdots \geq \xi_n \geq 0.$$

For each $k = 1, \dots, n$ the function Ψ_k is defined by

$$(3.7)\qquad \Psi_k(t) := \mathbf{P}\left\{\sum_1^k |\theta_i - \xi_i|^p \leq t^p\right\}.$$

LEMMA 3. *The following is true*:

$$(3.8)\qquad \begin{aligned} 1 - \Psi_k(t) = 1 &- \{\Phi(t + \xi_k) + \Phi(t - \xi_k)\}/2 \\ &+ (2\pi)^{-1/2} \int_0^t \left(1 - \Psi_{k-1}\left((t^p - s^p)^{1/p}\right)\right) \\ &\times \left\{e^{-(s+\xi_k)^2/2} + e^{-(s-\xi_k)^2/2}\right\} ds \end{aligned}$$

for $k = 2, 3, \dots, n$.

REMARK. Defining Ψ_0 by $\Psi_0 \equiv 1$, then formula (3.8) also holds for $k = 1$.

LEMMA 4. *Let* $\xi_1, \dots, \xi_n$ *be as above and assume* $2 < p < \infty$. *Then*

$$\lim_{t\to\infty} (1 - \Psi_n(t))/(1 - \Phi(t - \xi_1)) = \begin{cases} j/2, & \xi_1 > 0, \\ n, & \xi_1 = 0. \end{cases}$$

PROOF. For $k = 1$, we have

$$(1 - \Psi_1(t))/(1 - \Phi(t - \xi_1)) = \mathbf{P}\{|\theta_1 - \xi_1| > t\}/\mathbf{P}\{|\theta_1| > t - \xi_1\},$$

which tends to $\frac{1}{2}$ or 1 according as $\xi_1 > 0$ or $\xi_1 = 0$. Thus the lemma is valid in the case $n = 1$. Let us now assume

$$(3.9)\qquad \lim_{t\to\infty} (1 - \Psi_{k-1}(t))/(1 - \Phi(t - \xi_1)) = c_{k-1},$$

where $c_{k-1} = \min\{j, k - 1\}/2$ for $\xi_1 > 0$ or $k - 1$ for $\xi_1 = 0$, respectively. Dividing (3.8) by $1 - \Phi(t - \xi_1)$, the first term on the right-hand side tends to $\frac{1}{2}$, $k \leq j$, that is, $\xi_k = \xi_1 > 0$, to zero for $\xi_k < \xi_1$, or to 1 in the case $\xi_1 = \xi_k = 0$. Consequently, it remains to prove that

$$(3.10)\qquad \begin{aligned} \lim_{t\to\infty} (2\pi)^{-1/2} &\int_0^t \frac{1 - \Psi_{k-1}\left((t^p - s^p)^{1/p}\right)}{1 - \Phi(t - \xi_1)} \\ &\times \left[e^{-(s+\xi_k)^2/2} + e^{-(s-\xi_k)^2/2}\right] ds = c_{k-1} \end{aligned}$$

in each of the cases mentioned above.

Substituting $s := tu$, (3.10) coincides with

$$(3.11)\quad (2\pi)^{-1/2} t \int_0^1 \frac{1 - \Psi_{k-1}\big(t(1-u^p)^{1/p}\big)}{1 - \Phi(t - \xi_1)} \left[e^{-(tu+\xi_k)^2/2} + e^{-(tu-\xi_k)^2/2}\right] du$$

and we claim that for each $\eta > 0$ this integral taken from η to 1 tends to zero as $t \to \infty$. By assumption,

$$1 - \Psi_{k-1}\big(t(1-u^p)^{1/p}\big) \le c\Big(1 - \Phi\big(t(1-u^p)^{1/p} - \xi_1\big)\Big)$$

for some constant $c > 0$. This and (2.1) let us conclude that it suffices to prove

$$(3.12)\quad \begin{aligned} \lim_{t\to\infty} t(1+t) \int_\eta^1 \exp\Big\{&-(t^2/2)\big[(1-u^p)^{2/p} - 1 + u^2\big] \\ &+ t\big[\xi_1(1-u^p)^{1/p} - \xi_1 + u\xi_k\big]\Big\}\, du = 0 \end{aligned}$$

for each $\eta > 0$. Defining the functions g as in (3.1) and h with $\alpha = \xi_1$ and $\beta = \xi_k$ as in (3.2), (3.12) can be written as

$$(3.13)\quad \lim_{t\to\infty} t(1+t) \int_\eta^1 \exp\{-t^2 g(u)/2 + t h(u)\}\, du$$

and this is zero in view of Lemma 2.

To complete the proof it suffices to investigate (3.10), where we integrate from zero to some $\eta > 0$.

Given $\rho > 0$, by assumption and (2.2), there exists a $t_0 > 0$ such that

$$(3.14)\quad \begin{aligned} c_{k-1}(1+\rho)^{-1} F(u,t) \le \frac{1 - \Psi_{k-1}\big(t(1-u^p)^{1/p}\big)}{1 - \Phi(t-\xi_1)} \\ \le (1+\rho) c_{k-1} F(u,t) \end{aligned}$$

for all $t > t_0$ and all $u \in [0, \eta]$. Here F is defined by

$$\begin{aligned} F(u,t) := &\frac{t - \xi_1}{t(1-u^p)^{1/p} - \xi_1} \\ &\times \exp\Big\{-(t^2/2)\big[(1-u^p)^{1/p} - 1\big] + t\xi_1\big[(1-u^p)^{1/p} - 1\big]\Big\}. \end{aligned}$$

Again we define g and h as in (3.1) and (3.2), respectively, where $\alpha = \xi_1$ and $\beta = \xi_k$. Moreover, let h^- be given by

$$h^-(u) := \xi_1\big((1-u^p)^{1/p} - 1\big) - u\xi_k, \qquad 0 \le u \le 1.$$

Then we conclude

$$(2\pi)^{-1/2}\int_0^{\eta} t(1+\rho)c_{k-1}F(u,t)\left[e^{-(tu+\xi_k)^2/2}+e^{-(tu-\xi_k)^2/2}\right]du$$

$$\tag{3.15} \le \frac{(1+\rho)c_{k-1}t(t-\xi_1)}{(2\pi)^{1/2}\left(t(1-\eta^p)^{1/p}-\xi_1\right)}e^{-\xi_k^2/2}$$

$$\times\int_0^{\eta}\exp\left(-(t^2/2)g(u)\left[e^{th(u)}+e^{th^-(u)}\right]\right)du,$$

and if we take $\lim_{\eta\downarrow 0}\limsup_{t\to\infty}$ of (3.15), Lemma 2 shows that it is less than

$$(2\pi)^{-1/2}c_{k-1}(1+\rho)e^{-\xi_k^2/2}\left[\int_{-\xi_k}^{\infty}e^{-u^2/2}\,du+\int_{\xi_k}^{\infty}e^{-u^2/2}\,du\right]e^{\xi_k^2/2}$$

$$=(1+\rho)c_{k-1}.$$

Summing up, we arrive at

$$\limsup_{t\to\infty}(2\pi)^{-1/2}t\int_0^1\frac{1-\Psi_{k-1}\left(t(1-u^p)^{1/p}\right)}{1-\Phi(t-\xi_1)}\left[e^{-(tu+\xi_k)^2/2}+e^{-(tu-\xi_k)^2/2}\right]du$$

$$\le(1+\rho)c_{k-1}$$

for any $\rho>0$.

Analogously, by (3.14), the limit inferior is larger than $(1+\rho)^{-1}c_{k-1}$ and the proof is completed by taking the limit $\rho\to 0$. □

Before we formulate Theorem 2 let us fix the notation: $\sigma_1=\cdots=\sigma_k>\sigma_{k+1}\ge\cdots\ge 0$ denotes a sequence in l_p. Given $(\xi_i)_{i=1}^{\infty}$ in l_p, we define $\xi\ge 0$ by

$$\xi:=\sup\{|\xi_i|;1\le i\le k\},$$

$$j:=\operatorname{card}\{\xi_i;|\xi_i|=\xi,1\le i\le k\}.$$

Then we have:

THEOREM 2. *If* $2<p<\infty$, *then*

$$\lim_{t\to\infty}\mathbf{P}\left\{\sum_{i=1}^{\infty}|\sigma_i\theta_i-\xi_i|^p>t^p\right\}/(1-\Phi((t-\xi)/\sigma_1))=\begin{cases}j/2, & \xi>0,\\ k, & \xi=0.\end{cases}$$

PROOF. Without losing generality, we may assume $\sigma_1=1$ and $\xi=\xi_1=\cdots=\xi_j>\xi_{j+1}\ge\cdots\ge\xi_k\ge 0$. Then we define random variables X

and Y by

$$X^p := \sum_1^k |\theta_i - \xi_i|^p, \tag{3.16}$$

$$Y^p := \sum_{k+1}^\infty |\sigma_i \theta_i - \xi_i|^p. \tag{3.17}$$

By virtue of Theorem 1,

$$\lim_{t\to\infty} t^{-2} \log \mathbf{P}\{Y > t\} = -\tfrac{1}{2}\sigma_{k+1}^2 < -\tfrac{1}{2}, \tag{3.18}$$

which implies in particular that

$$\lim_{t\to\infty} \mathbf{P}\{Y > t\}/(1 - \Phi(t - \xi)) = 0. \tag{3.19}$$

Because of

$$\mathbf{P}\{X^p + Y^p > t^p\} = \mathbf{P}\{Y > t\} + \int_0^t \left(1 - \Psi_k\left((t^p - s^p)^{1/p}\right)\right) d\mathbf{P}_Y(s)$$

[Ψ_k is defined as in (3.7)] by (3.18), it follows that

$$\lim_{t\to\infty} \frac{\mathbf{P}\{X^p + Y^p > t^p\}}{1 - \Phi(t - \xi)} = \lim_{t\to\infty} \int_0^t \frac{1 - \Psi_k\left((t^p - s^p)^{1/p}\right)}{1 - \Phi(t - \xi)} d\mathbf{P}_Y(s). \tag{3.20}$$

Lemma 4 now implies

$$\lim_{t\to\infty} \frac{1 - \Psi_k\left((t^p - s^p)^{1/p}\right)}{1 - \Phi(t - \xi)} = c_k \lim_{t\to\infty} \frac{1 - \Phi\left((t^p - s^p)^{1/p} - \xi\right)}{1 - \Phi(t - \xi)} = c_k, \tag{3.21}$$

where $c_k = j/2$, $\xi > 0$ or $c_k = k$, $\xi = 0$, respectively. Moreover, it easily follows from (2.1) that

$$\left(1 - \Phi\left((t^p - s^p)^{1/p} - \xi\right)\right)/(1 - \Phi(t - \xi)) \le \tfrac{4}{3} e^{s^2/2},$$

provided that $t \ge (s^p + \xi^p)^{1/p}$. If $s \le t \le (s^p + \xi^p)^{1/p}$, the left-hand side is less than $(1 - \phi((s^p + \xi^p)^{1/p} - \xi))^{-1} \le \frac{4}{3} \exp((s^2/2)(1 + \xi^p/s^p)^{2/p})$. By (3.18), $e^{s^2/2}$ and $\exp((s^2/2)(1 + \xi^p/s^p)^{2/p})$ are integrable with respect to $\mathbf{P}_Y$. Thus Lebesgue's theorem applies and the proof is completed by (3.20) and (3.21). Observe that by Lemma 4, $\sup_{u>0}(1 - \Psi_k(u))/(1 - \phi(u - \xi)) < \infty$. □

4. The Hilbert space case. The aim of this section is to describe the behaviour of $\mu\{\|x\| > t\}$ as $t \to \infty$ for an arbitrary Gaussian measure μ in a Hilbert space H. Equivalently, we have to study

$$\mu\{x \in H; \|x - y\| > t\}$$

for symmetric μ and arbitrary $y \in H$.

Let us first recall some well-known facts about Gaussian measures on Hilbert spaces. If μ is Gaussian symmetric on H, then there is an orthonormal system $\{x_i\}_{i=1}^{\infty} \subseteq H$ and $\sigma_i \geq 0$, $\Sigma_1^{\infty}\sigma_i^2 < \infty$, such that

$$\mu(B) = \mathbf{P}\left\{\sum_{1}^{\infty} \sigma_i \theta_i x_i \in B\right\}$$

for any measurable subset $B \subseteq H$. Moreover, the number σ defined by (1.2) coincides with $\sup_i \sigma_i$. Consequently, the investigation of $\mu\{x \in H; \|x - y\| > t\}$ is equivalent to the study of

$$\mathbf{P}\left\{\sum_{1}^{\infty} |\sigma_i \theta_i - \eta_i|^2 > t^2\right\}$$

for some $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$, $\Sigma_1^{\infty}\sigma_i^2 < \infty$ and $(\eta_i)_{i=1}^{\infty}$ with $\Sigma_1^{\infty}\eta_i^2 < \infty$. For each $\alpha \geq 0$ and natural n, we define $\Phi_{n,\alpha}$ on $[0, \infty)$ by

$$\Phi_{n,\alpha}(t) := \mathbf{P}\left\{|\theta_1 - \alpha|^2 + |\theta_2|^2 + \cdots + |\theta_n|^2 \leq t^2\right\}.$$

It is well known and easy to see that for real $\xi_1, \dots, \xi_n$,

$$\mathbf{P}\left\{|\theta_1 - \xi_1|^2 + \cdots + |\theta_n - \xi_n|^2 \leq t^2\right\} = \Phi_{n,\alpha}(t),$$

where

$$\alpha = \left\{\sum_{1}^{n} |\xi_i|^2\right\}^{1/2}.$$

We shall need the following result of Hweng (1980):

LEMMA 5. *If $\alpha > 0$, then $1 - \Phi_{n,\alpha}(t)$ behaves like*

$$(2\pi)^{-1/2}\alpha^{-1}(t/\alpha)^{(n-3)/2} \exp\left(-(t-\alpha)^2/2\right)$$

as $t \to \infty$.

REMARK. In view of (2.2), this is equivalent to

$$\lim_{t\to\infty} (1 - \Phi_{n,\alpha}(t))/\left(t^{(n-1)/2}(1 - \Phi(t-\alpha))\right) = \tfrac{1}{2}\alpha^{-(n-1)/2}.$$

Let us fix the notation again. Let $\sigma_1 = \cdots = \sigma_k > \sigma_{k+1} \geq \cdots \geq 0$ be a fixed sequence in l_2. Given $(\eta_i)_{i=1}^{\infty}$ in l_2, we define $\alpha \geq 0$ by

$$\alpha := \left\{\sum_{i=1}^{k} |\eta_i|^2\right\}^{1/2}.$$

THEOREM 3. *If $\alpha > 0$, then we have*

$$\lim_{t\to\infty} \mathbf{P}\left\{\sum_{1}^{\infty} (\sigma_i\theta_i - \eta_i)^2 > t^2\right\}/\left(t^{(k-1)/2}(1 - \Phi((t-\alpha)/\sigma_1))\right)$$

$$= \tfrac{1}{2}\alpha^{-(k-1)/2} \prod_{k+1}^{\infty} \left(1 - \sigma_i^2/\sigma_1^2\right)^{-1/2} \exp\left\{\frac{1}{2}\sum_{k+1}^{\infty} \eta_i^2/\left(\sigma_1^2 - \sigma_i^2\right)\right\}.$$

PROOF. For simplicity, we assume $\sigma_1 = 1$. But observe that we have to divide σ_i, $i \geq k+1$, η_i, α and t by σ_1 in order to obtain the general case out of this special one. As in (3.16) and (3.17), X and Y are defined by

$$X^2 := \sum_1^k |\theta_i - \eta_i|^2,$$

$$Y^2 := \sum_{k+1}^\infty |\sigma_i \theta_i - \eta_i|^2,$$

and again (1.3) leads to

$$\lim_{t\to\infty} \mathbf{P}\{Y > t\} / \left(t^{(k-1)/2}(1 - \Phi(t-\alpha))\right) = 0.$$

Since

$$\mathbf{P}\{X^2 + Y^2 > t^2\} = \mathbf{P}\{Y > t\} + \int_0^t \left(1 - \Phi_{k,\alpha}\left((t^2 - s^2)^{1/2}\right)\right) d\mathbf{P}_Y(s),$$

we only have to investigate

$$\lim_{t\to\infty} \int_0^t \left(1 - \Phi_{k,\alpha}\left((t^2 - s^2)^{1/2}\right)\right)\left((1 - \Phi(t-\alpha))t^{(k-1)/2}\right)^{-1} d\mathbf{P}_Y(s).$$

By Lemma 5 and (2.2) it follows that

$$\lim_{t\to\infty} \left(1 - \Phi_{k,\alpha}\left((t^2 - s^2)^{1/2}\right)\right)\left(t^{(k-1)/2}(1 - \Phi(t-\alpha))\right)^{-1} = \tfrac{1}{2}\alpha^{-(k-1)/2} e^{s^2/2}. \tag{4.1}$$

Moreover, it is easy to see [use (2.1)] that [recall that $\phi(u) = 0$ for $u \leq 0$]

$$\left(1 - \Phi\left((t^2 - s^2)^{1/2} - \alpha\right)\right)(1 - \Phi(t-\alpha))^{-1} < \tfrac{4}{3} e^{\alpha^2/2}(1+s)e^{s^2/2}$$

for all $t \geq s$. Consequently, for every fixed $t \geq \alpha$, the left-hand side of (4.1) is less than $c(1+s)\exp(s^2/2)$ for some $c > 0$ (use Lemma 5). An application of (1.3) implies

$$\int_0^\infty (1+s)e^{s^2/2}\, d\mathbf{P}_Y(s) < \infty,$$

hence, by Lebesgue's theorem and (4.1) we obtain

$$\lim_{t\to\infty} \mathbf{P}\{X^2 + Y^2 > t^2\} / \left(t^{(k-1)/2}(1 - \Phi(t-\alpha))\right) = \tfrac{1}{2}\alpha^{-(k-1)/2} \mathbf{E} e^{Y^2/2}.$$

But

$$\begin{aligned} \mathbf{E} e^{Y^2/2} &= \prod_{k+1}^\infty \mathbf{E} \exp\left\{(\sigma_i\theta_i - \eta_i)^2/2\right\} \\ &= \prod_{k+1}^\infty (1 - \sigma_i^2)^{-1/2} \exp\{\eta_i^2/2(1 - \sigma_i^2)\} \\ &= \prod_{k+1}^\infty (1 - \sigma_j^2)^{-1/2} \exp\left\{\frac{1}{2}\sum_{k+1}^\infty \eta_i^2/(1-\sigma_i^2)\right\}, \end{aligned}$$

which completes the proof. □

REMARK 1. If $\alpha = 0$, that is, $\eta_1 = \cdots = \eta_k = 0$, then the tail behaviour

$$\mathbf{P}\left\{\sum_1^\infty |\sigma_i\theta_i - \eta_i|^2 > t^2\right\}$$

differs from that for $\alpha > 0$. Recall that for $\alpha = 0$,

$$\lim_{t\to\infty} \mathbf{P}\left\{\sum_1^\infty |\sigma_i\theta_i - \eta_i|^2 > t^2\right\} / (1 - \Phi_k(t/\sigma_1))$$
$$= \prod_{k+1}^\infty \left(1 - \sigma_j^2/\sigma_1^2\right)^{-1/2} \exp\left\{\frac{1}{2}\sum_{k+1}^\infty \eta_i^2\left(\sigma_1^2 - \sigma_i^2\right)^{-1}\right\},$$

where Φ_k was defined in (2.3) [cf. Hweng (1980)]. Recall that the behaviour of $1 - \Phi_k(t)$ is

$$(t^{k-2}e^{-t^2/2})/(2^{k/2-1}\Gamma(k/2))$$

while for $\alpha > 0$, the behaviour of $t^{(k-1)/2}(1 - \Phi(t - \alpha))$ is equal to

$$\sqrt{2/\pi}\, t^{(k-3)/2} e^{-(t-\alpha)^2/2}.$$

REMARK 2. It is not difficult to see that after some minor modifications, our proof also works in the case $\alpha = 0$. One only has to replace $t^{(k-1)/2}(1 - \Phi(t - \alpha))$ by $1 - \Phi_k(t)$ and to use

$$\lim_{t\to\infty}\left(1 - \Phi_k\left(\left(t^2 - s^2\right)^{1/2}\right)\right)/(1 - \Phi_k(t)) = e^{s^2/2}.$$

5. Diagonal measures on l_p^n, $p < 2$. Finally, we want to study the case $p < 2$. It turns out that this case is much more complicated than $p \geq 2$ because of the following fact: If μ on l_p, $p < 2$, is defined by

$$\mu(B) := \mathbf{P}\left\{\sum_1^\infty \sigma_i\theta_i e_i \in B\right\},$$

where e_i denotes the ith unit vector in l_p, then the rough behaviour of $\mu\{\|x\| > t\}$ is $\exp\{-t^2/2\sigma^2\}$, where

$$\sigma = \left\{\sum_1^\infty \sigma_i^r\right\}^{1/r}, \qquad 1/r = 1/p - 1/2.$$

Especially, this behaviour depends on the *whole* sequence $(\sigma_i)_{i=1}^\infty$ and not only on the first k terms as for $p \geq 2$. Thus, even a very small change of any of the σ_i's changes the rough behaviour, hence it may destroy the fine behaviour completely. In particular, no approximation argument applies and we only have results in the case of finite sequences. These improve some results of Dobrič, Marcus and Weber (1988), where the asymptotic behaviour for finite sequences ($1 \leq p < 2$) was determined up to a constant (for $p = 1$ this constant was already known).

Let us fix the notation. Given $\sigma_1, \dots, \sigma_n > 0$, we define Λ_k on $[0, \infty)$ by

$$(5.1) \qquad \Lambda_k(t) := \mathbf{P}\left\{ \sum_1^k |\sigma_i \theta_i|^p \le t^p \right\}, \qquad 1 \le k \le n.$$

Furthermore, we set $a_k := \{\sum_1^k \sigma_i^r\}^{1/r}$, where $1/r := 1/p - 1/2$, that is, $r = 2p/(2-p)$. So we have

$$a_{k-1}^r + \sigma_k^r = a_k^r \quad \text{and} \quad a_n = \sigma,$$

and we define functions f_k on $[0,1]$ by

$$(5.2) \qquad f_k(u) := (1-u^p)^{2/p}/a_{k-1}^2 + u^2/\sigma_k^2 - 1/a_k^2,$$

$2 \le k \le n$. We shall see that the behaviour of $\mathbf{P}\{\sum_1^n \sigma_i^p |\theta_i|^p > t^p\}$ depends heavily on properties of the functions f_k. Let us summarize some of these properties for later use.

LEMMA 6. *Assume $0 < p < 2$ and define numbers $u_k \in (0,1)$ by*

$$(5.3) \qquad u_k := (\sigma_k / a_k)^{r/p}.$$

Then the functions f_k defined in (5.2) *possess the following properties*:

(i) *We have $f_k(u_k) = f_k'(u_k) = 0$.*

(ii) *Each f_k is strictly decreasing on $[0, u_k]$ and strictly increasing on $[u_k, 1]$.*

(iii) *The Taylor expansion of f_k at u_k is*

$$f_k(u) = (2-p)(a_k/a_{k-1})^r (u-u_k)^2/\sigma_k^2 + o\big((u-u_k)^2\big).$$

PROOF. We only give some steps of the proof of (iii). All other properties are easy to verify. Since

$$\begin{aligned} f_k''(u) &= (4-2p)(1-u^p)^{2/p-2} u^{2p-2} a_{k-1}^{-2} \\ &\quad -(2p-2)(1-u^p)^{2/p-1} u^{p-2} a_{k-1}^{-2} + 2/\sigma_k^2, \end{aligned}$$

we obtain

$$\begin{aligned} f_k''(u_k) &= (4-2p) a_{k-1}^{-2} (a_{k-1}/a_k)^{2r/p-2r} (\sigma_k/a_k)^{2r-2r/p} \\ &\quad + (2-2p) a_{k-1}^{-2} (a_{k-1}/a_k)^{2r/p-r} (\sigma_k/a_k)^{r-2r/p} + 2/\sigma_k^2 \\ &= (4-2p) a_{k-1}^{-r} \sigma_k^{r-2} - (2p-2)\sigma_k^{-2} + 2\sigma_k^{-2} \\ &= (4-2p)\sigma_k^{-2}(1 + \sigma_k^r / a_{k-1}^r) \\ &= (4-2p)\sigma_k^{-2}(a_k/a_{k-1})^r \end{aligned}$$

as asserted. □

REMARK. Later on, exactly the factor $2-p$ of the Taylor expansion will appear in the limit.

THEOREM 3. *Suppose* $0 < p < 2$. *Then we have*

$$\lim_{t\to\infty} \mathbf{P}\left\{\sum_{1}^{n} |\sigma_i\theta_i|^p > t^p\right\}/(1 - \Phi(t/\sigma)) = \left(2/\sqrt{2-p}\right)^{n-1}.$$

Here $\sigma = a_n = \{\sum_1^n \sigma_i^r\}^{1/r}$ *and* $1/r = 1/p - 1/2$.

PROOF. Again we prove this by induction. If $n = 1$, there is nothing to prove, so we assume

$$\lim_{t\to\infty} (1 - \Lambda_{k-1}(t))/(1 - \Phi(t/a_{k-1})) = \left(2/\sqrt{2-p}\right)^{k-2},$$

where Λ_k was defined in (5.1).

It is easy to see that

$$1 - \Lambda_k(t) = 1 - \Phi(t/\sigma_k) + \sqrt{2/\pi}\,\sigma_k^{-1}\int_0^t \left(1 - \Lambda_{k-1}\left((t^p - s^p)^{1/p}\right)\right)e^{-s^2/2\sigma_k^2}\,ds$$

for $k = 2, \dots, n$. Moreover, if $k \geq 2$, then $a_k > \sigma_k$, and thus

$$\lim_{t\to\infty} (1 - \Phi(t/\sigma_k))/(1 - \Phi(t/a_k)) = 0,$$

which implies

$$\lim_{t\to\infty} \frac{1 - \Lambda_k(t)}{1 - \Phi(t/a_k)} = \sqrt{2/\pi}\,\sigma_k^{-1}\lim_{t\to\infty}\int_0^t \frac{1 - \Lambda_{k-1}\left((t^p - s^p)^{1/p}\right)}{1 - \Phi(t/a_k)}e^{-s^2/2\sigma_k^2}\,ds. \tag{5.4}$$

The right-hand side of (5.4) can be written as

$$\lim_{t\to\infty} \sqrt{2/\pi}\,(t/\sigma_k)\int_0^1 \frac{1 - \Lambda_{k-1}\left(t(1 - u^p)^{1/p}\right)}{1 - \Phi(t/a_k)}e^{-t^2u^2/2\sigma_k^2}\,du, \tag{5.5}$$

and our first aim is to show that this limit is zero provided that we only integrate over $A_k(\eta) := \{u \in [0,1];\ |u - u_k| \geq \eta\}$ for some $\eta > 0$. The number u_k was defined in (5.3). To verify this, observe that

$$1 - \Lambda_{k-1}\left(t(1 - u^p)^{1/p}\right) \leq c\left(1 - \Phi\left(t(1 - u^p)^{1/p}/a_{k-1}\right)\right)$$

for some $c > 0$ and $t \geq t_0$, so in view of (2.1), it suffices to show that

$$\lim_{t\to\infty} t(1 + t/a_k)\int_{A_k(\eta)} \exp\left\{-t^2(1 - u^p)^{2/p}/2a_{k-1}^2 - t^2u^2/2\sigma_k^2 + t^2/2a_k^2\right\}du$$

$$= \lim_{t\to\infty} t(1 + t/a_k)\int_{A_k(\eta)} \exp\left\{-t^2 f_k(u)/2\right\}du = 0,$$

where f_k was defined in (5.2). But by Lemma 6,

$$\gamma_k := \inf\{f_k(u);\ u \in A_k(\eta)\} > 0,$$

which proves

$$t(1+t/a_k)\int_{A_k(\eta)}\exp\{-t^2 f_k(u)/2\}\,du \le t(1+t/a_k)e^{-t^2\gamma_k/2},$$

which clearly tends to zero as $t \to \infty$.

So it remains to study (5.5), where the integral is taken over $[u_k-\eta, u_k+\eta]$. Let us assume that η is small enough, that is, we have $u_k - \eta > 0$ and $u_k + \eta < 1$. Then by assumption,

$$\lim_{t\to\infty}\left(1-\Lambda_{k-1}\left(t(1-u^p)^{1/p}\right)\right)/\left(1-\Phi\left(t(1-u^p)^{1/p}/a_{k-1}\right)\right) = \left(2/\sqrt{2-p}\right)^{k-2}$$

and this limit is uniform with respect to $u \in [u_k-\eta, u_k+\eta]$.

In view of this uniformity, we have

$$\begin{aligned}(5.6)\qquad &\limsup_{t\to\infty}\sqrt{2/\pi}\,(t/\sigma_k)\int_{u_k-\eta}^{u_k+\eta}\frac{1-\Lambda_{k-1}\left(t(1-u^p)^{1/p}\right)}{1-\Phi(t/a_k)}e^{-t^2u^2/2\sigma_k^2}\,du\\ &\le \left(2/\sqrt{2-p}\right)^{k-2}\limsup_{t\to\infty}\sqrt{2/\pi}\,(t/\sigma_k)\\ &\quad\times\int_{u_k-\eta}^{u_k+\eta}\frac{1-\Phi\left(t(1-u^p)^{1/p}/a_{k-1}\right)}{1-\Phi(t/a_k)}e^{-t^2u^2/2\sigma_k^2}\,du\end{aligned}$$

as well as the reverse inequality for the limit inferior. Furthermore, (2.2) implies the following: Given $\rho > 0$, there exists a $t_0 > 0$ such that for all $t > t_0$ and all $u \in [u_k-\eta, u_k+\eta]$,

$$\begin{aligned}(5.7)\qquad &(1+\rho)^{-1}(a_{k-1}/a_k)(1-u^p)^{-1/p}\exp\left\{-t^2(1-u^p)^{2/p}/2a_{k-1}^2+t^2/2a_k^2\right\}\\ &\le \left(1-\Phi\left(t(1-u^p)^{1/p}/a_{k-1}\right)\right)/\left(1-\Phi(t/a_k)\right)\\ &\le (1+\rho)(a_{k-1}/a_k)(1-u^p)^{-1/p}\\ &\quad\times\exp\left\{-t^2(1-u^p)^{2/p}/2a_{k-1}^2+t^2/2a_k^2\right\}.\end{aligned}$$

Thus, the right-hand side of (5.6) can be estimated (for large t) by

$$\begin{aligned}(5.8)\qquad &(t/\sigma_k)(1+\rho)\left(2/\sqrt{2-p}\right)^{k-2}\sqrt{2/\pi}\,(a_{k-1}/a_k)\\ &\quad\times\int_{-\eta}^{\eta}\left(1-(u+u_k)^p\right)^{-1/p}e^{-t^2f_k(u+u_k)/2}\,du.\end{aligned}$$

Lemma 6 implies

$$f_k(u+u_k) \ge u^2\alpha_k(\eta),$$

where

$$\alpha_k(\eta) := (2-p)\sigma_k^{-2}(a_k/a_{k-1})^r - \gamma(\eta)$$

and $\gamma(\eta) \to 0$ as $\eta \to \infty$. Thus (5.8) is less than

$$(1+\rho)\left(2/\sqrt{2-p}\right)^{k-2}\sqrt{2/\pi}\,(t/\sigma_k)(a_{k-1}/a_k)\left(1-(u_k+\eta)^p\right)^{-1/p}$$
$$\times\int_{-\eta}^{\eta}\exp\{-t^2u^2\alpha_k(\eta)/2\}\,du,$$

which tends to

$$(5.9)\qquad \begin{aligned}&(1+\rho)(a_{k-1}/a_k)\left(2/\sqrt{2-p}\right)^{k-2}\sqrt{2/\pi}\\ &\times\sigma_k^{-1}\left(1-(u_k+\eta)^p\right)^{-1/p}\sqrt{2\pi}\,\alpha_k(\eta)^{-1/2}\end{aligned}$$

as $t \to \infty$.

Summing up, the limit superior

$$\limsup_{t\to\infty}\,(1-\Lambda_k(t))/(1-\Phi(t/a_k))$$

is less than (5.9) and this is valid for all $\rho, \eta > 0$. Now, if $\eta \to 0$, then $(1-(u_k+\eta)^p)^{1/p}$ tends to $(a_{k-1}/a_k)^{r/p}$ and $\alpha_k(\eta)$ tends to

$$(2-p)\sigma_k^{-2}(a_k/a_{k-1})^r.$$

Hence (5.9) tends to

$$\begin{aligned}&\left(2/\sqrt{2-p}\right)^{k-1}(a_{k-1}/a_k)(a_k/a_{k-1})^{r/p}(a_{k-1}/a_k)^{r/2}\\ &=\left(2/\sqrt{2-p}\right)^{k-1}\quad\text{as } \rho,\eta\to 0.\end{aligned}$$

Similar arguments prove the same estimate from below for the limit inferior. One only has to use (5.6) together with (5.7) and to replace $\alpha_k(\eta)$ by $(2-p)\sigma_k^{-2}(a_k/a_{k-1})^r + \gamma(\eta)$ and $u_k + \eta$ by $u_k - \eta$. This completes the proof. □

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REFERENCES

BORELL, C. (1975). The Brunn–Minkowski inequality in Gauss spaces. *Invent. Math.* **30** 207–216.

BORELL, C. (1976). Gaussian Radon measures on locally convex spaces. *Math. Scand.* **38** 265–284.

DOBRIČ, V., MARCUS, M. B. and WEBER, M. (1988). The distribution of large values of the supremum of a Gaussian process. *Astérisque* **157–158** 95–127.

FERNIQUE, X. (1970). Intégrabilité des vecteurs gaussiens. *C.R. Acad. Sci. Paris Ser. A–B* **270** 1698–1699.

FERNIQUE, X. (1976). Evaluation de processus gaussiens composés. *Probability in Banach Spaces. Lecture Notes in Math.* **526** 64–85.

HERTLE, A. (1983). On the asymptotic behaviour of Gaussian spherical integrals. *Probability in Banach Spaces IV. Lecture Notes in Math.* **990** 221–234.

HWENG, C.-R. (1980). Gaussian measure of large balls in a Hilbert space. *Proc. Amer. Math. Soc.* **78** 107–110 [cf. also Erratum (1985), *Proc. Amer. Math. Soc.* **94** 188].

LANDAU, H. and SHEPP, L. A. (1970). On the supremum of a Gaussian process. *Sankhyā Ser. A* **32** 369–378.

MARCUS, M. B. and SHEPP, L. A. (1972). Sample behavior of Gaussian processes. In *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* 423–441. Univ. California Press, Berkeley.

TALAGRAND, M. (1984). Sur l'integrabilité des vecteurs gaussiens. *Z. Wahrsch. Verw. Gebiete* **68** 1–8.

TALAGRAND, M. (1988). Small tails for the supremum of a Gaussian process. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **24** 307–315.

ZOLOTAREV, V. M. (1961). Concerning a certain probability problem. *Theory Probab. Appl.* **6** 201–204.

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