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PROOF OF A CONJECTURE OF M. L. EATON ON THE CHARACTERISTIC FUNCTION OF THE WISHART DISTRIBUTION

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Let m (≥ 2) be a positive integer; I_m be the $m \times m$ identity matrix; and Σ and A be symmetric $m \times m$ matrices, where Σ is positive definite. By proving that the function $\phi_\alpha(A) = |I_m - 2iA\Sigma|^{-\alpha}$ is a characteristic function only if $\alpha \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, (m-2)/2\} \cup [(m-1)/2, \infty)$, we establish a conjecture of Eaton. A similar result is established for the rank 1 noncentral Wishart distribution and is conjectured to also be valid for any greater rank.

1. Introduction. Let m (≥ 2) be a positive integer and $\mathscr{S}_m$ denote the space of $m \times m$ symmetric matrices. Let $\Sigma, A \in \mathscr{S}_m$, where Σ is positive definite. For

$$\alpha \in \left\{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, (m-2)/2\right\} \cup \left[(m-1)/2, \infty\right), \tag{1}$$

it is well known ([1], page 329, Problem 3; [10], pages 87–88) that the function $\phi_\alpha(A) = |I_m - 2iA\Sigma|^{-\alpha}$, $A \in \mathscr{S}_m$, is a characteristic function, namely, of the (central) Wishart distribution. It has also been shown [12] that the Wishart distribution is not infinitely divisible and hence that ϕ_α is not a characteristic function for sufficiently small $\alpha > 0$. However, the problem of determining all values of α for which ϕ_α is a characteristic function has remained unsolved.

Eaton has conjectured that (1) is also necessary for ϕ_α to be a characteristic function. We will use the theory of zonal polynomials to prove Eaton's conjecture.

In the case of the noncentral Wishart distribution, the characteristic function is of the form ([10], page 444)

$$\phi_\alpha(A) = |I_m - 2iA\Sigma|^{-\alpha} e^{2i \operatorname{tr} \Omega A\Sigma(I_m - 2iA\Sigma)^{-1}}, \tag{2}$$

where 2Ω, the noncentrality parameter, is a positive semidefinite matrix. When the rank of Ω is 1, we will prove that (1) is again necessary for (2) to be a characteristic function. Moreover, we conjecture that the same result remains valid for Ω of arbitrary rank.

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2. The central case. We begin by listing some properties of the zonal polynomials $C_\kappa(\cdot)$ ([10], Chapter 7). The zonal polynomials $C_\kappa(\cdot)$ are indexed by partitions $\kappa = (k_1, \dots, k_m)$, where $k_1 \geq \cdots \geq k_m$ are nonnegative integers. Then $C_\kappa(A)$, $A \in \mathscr{S}_m$, is homogeneous of degree $k = k_1 + \cdots + k_m$. Each zonal polynomial is orthogonally invariant, that is, $C_\kappa(hAh^{-1}) = C_\kappa(A)$ for all $h \in O(m)$, the group of $m \times m$ orthogonal matrices. Moreover, the set $\{C_\kappa(\cdot)$: κ is a partition$\}$ is a basis for the vector space of all orthogonally invariant polynomials on $\mathscr{S}_m$.

If $\alpha \in \mathbf{R}$ and $k = 0, 1, 2, \dots$, let $(\alpha)_k = \alpha(\alpha + 1) \cdots (\alpha + k - 1)$ denote the shifted factorial. Define the partitional shifted factorial $(\alpha)_\kappa$ by

$$(\alpha)_\kappa = \prod_{j=1}^{m} \left(\alpha - \tfrac{1}{2}(j - 1)\right)_{k_j}. \tag{3}$$

Then we have the expansion ([10], page 259)

$$|I_m - A|^{-\alpha} = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\alpha)_\kappa C_\kappa(A)}{k!}, \tag{4}$$

valid for $\|A\| < 1$. Further if dh denotes the Haar measure on $O(m)$, normalized to have total volume 1, then for any $A, B \in \mathscr{S}_m$, we have the integral formula ([10], page 260)

$$\int_{O(m)} e^{\operatorname{tr}(Ah^{-1}Bh)}\, dh = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(A) C_\kappa(B)}{k! C_\kappa(I_m)}. \tag{5}$$

THEOREM 1. *For positive definite* $\Sigma \in \mathscr{S}_m$, *the function* $\phi_\alpha(A) = |I_m - 2iA\Sigma|^{-\alpha}$, $A \in \mathscr{S}_m$, *is a characteristic function only if* (1) *is valid.*

PROOF. Without loss of generality, we assume that $\Sigma = \frac{1}{2} I_m$. Now supposing that ϕ_α is the characteristic function of the random matrix $S \in \mathscr{S}_m$, we claim that, almost surely, S is positive semidefinite. For if ω is a $m \times 1$ unit vector, then for $t \in \mathbf{R}$, the characteristic function of $\omega' S \omega$ is

$$\begin{aligned} \mathscr{E} e^{it\omega' S\omega} &= \mathscr{E} e^{i \operatorname{tr}(t\omega\omega' S)} \\ &= |I_m - it\omega\omega'|^{-\alpha} \\ &= (1 - it)^{-\alpha}. \end{aligned} \tag{6}$$

Necessarily $\alpha \geq 0$, and then $\omega' S \omega$ follows a gamma distribution. In particular $\omega' S \omega \geq 0$ (almost surely) for all $\|\omega\| = 1$ and hence S is positive semidefinite (almost surely).

Next, it follows from (4) that for $\|A\| < 1$,

$$\phi_\alpha(A) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\alpha)_\kappa i^k C_\kappa(A)}{k!}. \tag{7}$$

Since (7) represents the Taylor expansion about zero of ϕ_α, then all moments of the random matrix S are finite.

Now we come to the heart of the proof. Since $\mathscr{E}q(S)$ is finite for any polynomial $q(\cdot)$, let us compute $\mathscr{E}C_\kappa(S)$. Because $C_\kappa(T) \geq 0$ for any positive semidefinite $T \in \mathscr{S}_m$ ([3], Section 5.3) and S is positive semidefinite (almost surely), then it follows that $\mathscr{E}C_\kappa(S) \geq 0$ for all κ. To evaluate $\mathscr{E}C_\kappa(S)$, we use a generating function method from [11]. By repeated use of Fubini's theorem and the formulas (5) and (7), we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{i^k C_\kappa(A)\mathscr{E}C_\kappa(S)}{k!C_\kappa(I_m)} &= \mathscr{E}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{i^k C_\kappa(A)C_\kappa(S)}{k!C_\kappa(I_m)}\\ &= \mathscr{E}\int_{O(m)} e^{i\,\mathrm{tr}(Ah^{-1}Sh)}\,dh\\ &= \int_{O(m)} \mathscr{E}e^{i\,\mathrm{tr}(hAh^{-1}S)}\,dh \qquad (8)\\ &= \int_{O(m)} |I_m - ihAh^{-1}|^{-\alpha}\,dh\\ &= |I_m - iA|^{-\alpha}\\ &= \sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\alpha)_\kappa i^k C_\kappa(A)}{k!}.\end{aligned}$$

Comparing coefficients of $C_\kappa(A)$ in (8), we obtain $\mathscr{E}C_\kappa(S) = (\alpha)_\kappa C_\kappa(I_m)$. Since $\mathscr{E}C_\kappa(S) \geq 0$ and $C_\kappa(I_m) > 0$, then

$$(\alpha)_\kappa \geq 0 \qquad (9)$$

for all κ. By (3), it follows that (9) holds only if (1) is satisfied. □

As a consequence of Theorem 1 we obtain the previously cited result of [12].

COROLLARY 1 (See [12]). *The characteristic functions ϕ_α are not infinitely divisible.*

3. The noncentral case. To extend Theorem 1 to the noncentral case, we will need a summation formula for the zonal polynomials. The Laguerre polynomial of argument $\Omega \in \mathscr{S}_m$ and order $\gamma = \alpha - \frac{1}{2}(m+1)$ is defined ([10], page 282) as

$$L_\kappa^\gamma(\Omega) = (\alpha)_\kappa C_\kappa(I_m)\sum_{s=0}^{k}\sum_{\sigma}\binom{\kappa}{\sigma}\frac{C_\sigma(-\Omega)}{(\alpha)_\sigma C_\sigma(I_m)}, \qquad (10)$$

where the inner sum is over all partitions $\sigma = (s_1, \ldots, s_m)$ of s and the generalized binomial coefficient $\binom{\kappa}{\sigma}$ is given by ([10], page 267)

$$\frac{C_\kappa(I_m + \Omega)}{C_\kappa(I_m)} = \sum_{s=0}^{k}\sum_{\sigma}\binom{\kappa}{\sigma}\frac{C_\sigma(\Omega)}{C_\sigma(I_m)}.$$

If ${}_0F_0(A, B)$ denotes the function in (5), then the following summation formula is valid ([10], page 283):

$$(11) \qquad |I_m - iA|^{-\alpha} {}_0F_0\big(\Omega, iA(I_m - iA)^{-1}\big) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{L_\kappa^\gamma(-\Omega) C_\kappa(iA)}{k! C_\kappa(I_m)}$$

for $\|A\| < 1$. [Usually (11) is proved for $\gamma > -1$ by the method of Laplace transforms. However, it can be shown to be valid for all α by the use of generating functions and then it is crucial to note that the quotient $(\alpha)_\kappa/(\alpha)_\sigma$ is a polynomial in α whenever $\binom{\kappa}{\sigma} \neq 0$.]

Next we derive a nonnegativity property of the generalized binomial coefficients.

LEMMA 1. *For any partition κ of k and $s = 0, \dots, k$, $\binom{\kappa}{(s)} \geq 0$.*

PROOF. If κ, σ, τ are partitions, define coefficients $g_{\sigma,\tau}^\kappa$ by

$$C_\sigma(A) C_\tau(A) = \sum_{\kappa} g_{\sigma,\tau}^\kappa C_\kappa(A).$$

Then ([10], page 289, Exercise 7.16(b))

$$\binom{\kappa}{(s)} = \binom{k}{s} \sum_{\tau} g_{(s),\tau}^\kappa.$$

The coefficients $g_{(s),\tau}^\kappa$ have been determined explicitly ([7], Section 10; [13], Section 6) and are all nonnegative. Therefore, $\binom{\kappa}{(s)} \geq 0$. □

THEOREM 2. *If Ω is of rank* 1, *then the function* (2) *is a characteristic function only if* (1) *is valid.*

PROOF. As before, we assume that $\Sigma = \frac{1}{2} I_m$. Suppose that (2) is the characteristic function of a random matrix $S \in \mathscr{S}_m$. Since

$$(I_m - it\omega\omega')^{-1} = I_m + it(1 - it)^{-1}\omega\omega'$$

for any vector ω, then proceeding as in (6) we obtain

$$\mathscr{E} e^{it\omega' S\omega} = (1 - it)^{-\alpha} \exp\big(it(1 - it)^{-1}\omega'\Omega\omega\big)$$

for any unit vector ω. Therefore $\omega' S\omega$ has a noncentral gamma distribution; $\alpha \geq 0$; S is positive semidefinite (almost surely); and, by (11), $\mathscr{E} q(S)$ is finite for all polynomials $q(\cdot)$. Further, $\mathscr{E} C_\kappa(S) \geq 0$ for all κ.

It remains to compute $\mathscr{E} C_\kappa(S)$. Proceeding as in (8) and using the expansion (11), we obtain $\mathscr{E} C_\kappa(S) = L_\kappa^\gamma(-\Omega)$; hence, by (10),

$$(12) \qquad \mathscr{E} C_\kappa(S) = (\alpha)_\kappa C_\kappa(I_m) \sum_{s=0}^{k} \sum_{\sigma} \binom{\kappa}{\sigma} \frac{C_\sigma(\Omega)}{(\alpha)_\sigma C_\sigma(I_m)}.$$

Since Ω is positive semidefinite and of rank 1, then $C_\sigma(\Omega) \geq 0$; and $C_\sigma(\Omega) > 0$ if and only if $\sigma = (s)$. Hence, (12) reduces to

$$\mathscr{E}C_\kappa(S) = (\alpha)_\kappa C_\kappa(I_m) \sum_{s=0}^{k} \binom{\kappa}{(s)} \frac{C_{(s)}(\Omega)}{(\alpha)_s C_{(s)}(I_m)}. \tag{13}$$

Since $(\alpha)_s \geq 0$ for all $\alpha \geq 0$ and, by Lemma 1, $\binom{\kappa}{(s)} \geq 0$ for all κ, s, then the sum in (13) is nonnegative. Therefore $\mathscr{E}C_\kappa(S) \geq 0$ implies $(\alpha)_\kappa \geq 0$, hence that (1) is valid. □

4. Concluding remarks. In the case of the complex Wishart distribution [4], our methods can again be applied to derive the following result.

THEOREM 3. *Let $\mathscr{S}_m(C)$ denote the space of $m \times m$ Hermitian matrices and $\Sigma, \Omega \in \mathscr{S}_m(C)$, where Σ is positive definite and rank$(\Omega) \leq 1$. Then the function*

$$\phi_\alpha(A) = |I_m - 2iA\Sigma|^{-\alpha} e^{2i \operatorname{tr} \Omega A\Sigma(I_m - 2iA\Sigma)^{-1}}, \tag{14}$$

$A \in \mathscr{S}_m(C)$, is a characteristic function only if

$$\alpha \in \{0, 1, 2, \ldots, m-2\} \cup [m-1, \infty). \tag{15}$$

In the real or complex noncentral case, when the rank of Ω is greater than 1, it seems difficult to prove that the nonnegativity of the zonal polynomial moments, $\mathscr{E}C_\kappa(S)$, for all κ implies (1) or (15), respectively. In the real case no explicit formula is available for the generalized binomial coefficients, and it does not even appear to be known whether they are nonnegative always. In the complex case it follows from the relationship between the zonal polynomials and the Schur functions ([3], page 797; [4], page 487), and the binomial theorem for the Schur functions ([9], page 30, Example 10), that the generalized binomial coefficients are given by the explicit formula

$$\binom{\kappa}{\sigma} = \left[\prod_{1 \leq i < j \leq m} \frac{s_i - s_j - i + j}{k_i - k_j - i + j}\right] \det\left(\binom{k_i + m - i}{s_j + m - j}\right). \tag{16}$$

It has been proved ([2]; [5], Section 32.2(ii)) that the determinants in (16) are nonnegative; hence so are the coefficients $\binom{\kappa}{\sigma}$.

Even if it can be proved that the generalized binomial coefficients are nonnegative, it is not clear that the sum in (12) is nonnegative for all κ only if (1) holds. Nevertheless, we make the following conjecture.

CONJECTURE 1. *For rank$(\Omega) \geq 2$, the functions* (2) *and* (14) *are characteristic functions only if α satisfies* (1) *and* (15), *respectively.*

Finally, we address the problem of the indecomposability of the characteristic functions ϕ_α. For ease of exposition we restrict our attention to the real, central Wishart distributions. In [8], Lévy proved that the characteristic function $\phi_{1/2}$ is indecomposable. This raises the problem of finding all values of α for which ϕ_α is indecomposable.

Let Φ denote the space of all characteristic functions ϕ, defined on $\mathscr{S}_m$, such that $\mathscr{E}|\mathrm{tr}(S)| < \infty$, where ϕ is the characteristic function of the random matrix S. Set $\Phi_m = \{\phi_\alpha \in \Phi\colon\ \phi_\alpha(A) = |I_m - iA|^{-\alpha},\ A \in \mathscr{S}_m\}$ and $\mathscr{A}_m = \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots, (m-2)/2\} \cup [(m-1)/2, \infty)$, the set of admissible α values.

Define $\Delta\colon \Phi \to \mathbf{R}_+$ by $\Delta(\phi) = m^{-1}\mathscr{E}\,\mathrm{tr}(S)$. Since $\mathrm{tr}(S) = C_{(1)}(S)$, then it follows from the proof of Theorem 1 that for any $\phi_\alpha \in \Phi_m$,

$$\Delta(\phi_\alpha) = m^{-1}\mathscr{E}C_{(1)}(S) = m^{-1}(\alpha)_{(1)}C_{(1)}(I_m) = \alpha.$$

Hence $\Delta(\phi_\alpha \cdot \phi_\beta) = \Delta(\phi_\alpha) + \Delta(\phi_\beta)$, $\alpha, \beta \in \mathscr{A}_m$, and $\Delta|_{\Phi_m}$ is an isomorphism between Φ_m and $\mathscr{A}_m$.

If ι denotes the identity map on $\mathbf{R}_+$, then it is not hard to prove that both $(\mathscr{A}_m, +, \iota)$ and $(\Phi_m, \cdot\,, \Delta)$, equipped with their usual topologies, are *Delphic semigroups* as defined in [6]. Further, the set of indecomposable elements of $\mathscr{A}_m$ is $\{0, \frac{1}{2}\} \cup ((m-1)/2, m/2)$. It is our belief that the decomposability properties of the semigroup $\mathscr{A}_m$ determine completely those of Φ_m, and that these properties can be established from the Delphic nature of the two semigroups. With these observations in mind, we conjecture the following result.

CONJECTURE 2. *The characteristic function ϕ_α is indecomposable if and only if*

$$\alpha \in \{0, \tfrac{1}{2}\} \cup ((m-1)/2, m/2). \qquad (17)$$

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