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EXTREMES OF MOVING AVERAGES OF RANDOM VARIABLES WITH FINITE ENDPOINT

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Consider moving average processes of the form

$$X_t = \sum_{j=0}^{\infty} c_j Z_{t-j},$$

where $\{Z_j\}$ are iid and nonnegative random variables and $c_j > 0$ are constants satisfying summability conditions at least sufficient to make the random series above converge. We suppose that the distribution of Z_j is regularly varying near 0 and discuss lower tail behavior of finite and infinite linear combinations. The behavior is quite different in the two cases. For finite linear combinations, the lower tail is again regularly varying but for infinite moving averages, the lower tail is Γ-varying, i.e., it is in the domain of attraction of a type I extreme value distribution in the sense of minima. Convergence of point processes based on the moving averages is shown to hold in both the finite and infinite order cases and suitable conclusions are drawn from such convergences. A useful analytic tool is asymptotic normality of the Esscher transform of the common distribution of the Z's. The extreme value results of this paper are in terms of minima of the moving average processes but results can be adapted to study maxima of moving averages of random variables in the domain of attraction of the type III extreme value distribution for maxima.

1. Introduction. Moving average processes and linear combinations of iid random variables are basic objects in time series analysis and in regression models. The type of processes we have in mind are of the form

$$X_t = \sum_{j=0}^{\infty} c_j Z_{t-j},$$

where $\{Z_j\}$ are iid and c_j are constants satisfying summability conditions at least sufficient to make the random series above converge. Studies of the extreme value behavior of such processes have been carried out by Rootzén (1978, 1986, 1987) and Davis and Resnick (1985, 1988). Such studies use point process methods and either assumptions about the tails of the distribution of Z_j or about the form of its density to get results about distributions of functionals of the moving averages.

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Here we suppose that $c_j \geq 0$, $Z_j \geq 0$ and that the distribution of Z_j is regularly varying near 0. We discuss lower tail behavior of finite linear combinations in Section 2 and of infinite linear combinations in Section 3. Surprisingly, the behavior is quite different in the two cases. For finite linear combinations, the lower tail is again regularly varying but in the case of infinite linear combinations, this is no longer true. In fact, a slight strengthening of assumptions allows us to show that the lower tail of the infinite moving average is Γ-varying, i.e., it is in the domain of attraction of a type I extreme value distribution in the sense of minima. This crossover phenomenon is quite interesting and shows it is impossible to obtain the correct type extreme value distribution of an infinite order moving average by truncating to a finite order moving average.

In Section 2, point process methods [cf. Resnick (1986, 1987) and Davis and Resnick (1985, 1988)] are used to derive both analytical results about lower tail behavior and results about the weak convergence properties of a sequence of point processes based on the moving average process. From such results it is straightforward to get the behavior of lower extremes of the moving average process. In Section 3 an entirely different analytic technique is needed for the study of lower tails of the infinite order moving average. The distribution of the Z's is assumed to have a density which is regularly varying at 0. The density function of the moving average is embedded in an exponential family via the Esscher transform and the transform, suitably normalized, is shown to converge to a normal density as the parameter goes to infinity. Similar techniques are used to Rootzén (1987) and Feigin and Yashchin (1989). In Section 4 these lower tail properties are used to show that a sequence of point processes based on the infinite order moving averages converges to a limiting Poisson process. From this result numerous weak convergence results about the lower extremes of $\{X_t\}$ can be read off.

The extreme value results of this paper are in terms of minima of moving average processes. The results can be adapted to study maxima of moving averages of random variables in the domain of attraction of the type III extreme value distribution for maxima. This extreme value distribution has the form

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\}, & \text{for } x \leq 0, \\ 1, & \text{for } x > 0, \end{cases}$$

where $\alpha > 0$.

2. Finite moving averages. Fix $k > 1$ and suppose $Z_1, \dots, Z_k$ are nonnegative iid random variables with common distribution $F(x)$ where F is regularly varying at 0 with index $\alpha > 0$, i.e., for all $x > 0$,

$$\lim_{t \downarrow 0} \frac{F(tx)}{F(t)} = x^\alpha. \tag{2.1}$$

We first derive the form of $G(x) := P[\sum_{i=1}^k c_i Z_i \leq x]$ as $x \to 0$ for given constants $c_i > 0$, $i = 1, \dots, k$.

In contrast to the analytic methods employed in the next section, we use point process methods to analyze the order of $G(x)$ near 0 [cf. Resnick (1986, 1987) and Davis and Resnick (1985, 1988)]. For a locally compact space E with countable base let $M_p(E)$ denote the set of Radon point measures on E and let $\mathscr{M}_p(E)$ denote the σ-field generated by the vague topology. For $x \in E$ and $A \subset E$ define

$$\varepsilon_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A, \end{cases}$$

so that a point measure in $M_p(E)$ can be represented by $\sum_i \varepsilon_{x_i}$, $x_i \in E$. We sometimes denote a Poisson process with mean measure μ by PRM(μ).

Set $a_n = F^{\leftarrow}(n^{-1/k})$ and observe that for $x_i \geq 0$, $i = 1, \dots, k$, we have

$$\begin{aligned} nP[Z_1 \leq a_n x_1, \dots, Z_k \leq a_n x_k] &= n \prod_{i=1}^{k} F(a_n x_i) \\ &= \prod_{i=1}^{k} \left(\frac{F(a_n x_i)}{n^{-1/k}} \right) \\ &\sim \prod_{i=1}^{k} \frac{F(a_n x_i)}{F(a_n)} \to \prod_{i=1}^{k} x_i^{\alpha}. \end{aligned}$$

If we define a measure ν on $\mathscr{B}([0,\infty)^k)$ by

$$\nu\{[0, x_1] \times \cdots \times [0, x_k]\} = \prod_{i=1}^{k} x_i^{\alpha},$$

then we have

$$nP\left[a_n^{-1}(Z_1, \dots, Z_k) \in \cdot\right] \to_v \nu(\cdot), \tag{2.2}$$

where the convergence is vague convergence of measures on $[0,\infty)^k$.

Let $\{\mathbf{Z}_n, n \geq 1\}$ be an iid sequence of random vectors in $\mathbb{R}_+^k$ with $\mathbf{Z}_n =_d (Z_1, \dots, Z_k)$. It follows from (2.2) and Proposition 3.21 in Resnick (1987) that as $n \to \infty$,

$$\sum_{i=1}^{n} \varepsilon_{a_n^{-1}\mathbf{Z}_i} \Rightarrow \sum_m \varepsilon_{\mathbf{j}_m} \tag{2.3}$$

in $M_p([0,\infty)^k)$ where $\Rightarrow$ denotes weak convergence and the limit is a Poisson process with mean measure ν. Define the mapping T: $[0,\infty)^k \to [0,\infty)$ by

$$T(x_1, \dots, x_k) = \sum_{i=1}^{k} c_i x_i.$$

Applying Proposition 3.18 in Resnick (1987), we get from (2.3)

$$\sum_{i=1}^{n} \varepsilon_{a_n^{-1}T\mathbf{Z}_i} \Rightarrow \sum_m \varepsilon_{T\mathbf{j}_m} \tag{2.4}$$

in $M_p([0,\infty))$ and the limit is a Poisson process with mean measure $\nu \circ T^{-1}$.

Observe that for $x > 0$,

$$\nu \circ T^{-1}([0,x]) = \nu\left\{\mathbf{y} \in [0,\infty)^k : \sum_{i=1}^{k} c_i y_i \le x\right\}$$

(2.5)
$$= \int \cdots \int_{\{(y_1,\dots,y_k):\, \sum_{i=1}^{k} c_i y_i \le x\}} \prod_{i=1}^{k} \alpha y_i^{\alpha-1}\, dy_i$$

$$= c(\alpha,k) x^{k\alpha},$$

where

$$c(\alpha,k) = \Gamma^k(\alpha+1) \Big/ \left(\Gamma(k\alpha+1)\prod_{i=1}^{k} c_i^{\alpha}\right). \tag{2.6}$$

Applying Proposition 3.21 in Resnick (1987) once again, we see that (2.4) is equivalent to

$$nP\left[a_n^{-1}\sum_{i=1}^{k} c_i Z_i \le x\right] \to \nu_k([0,x]), \tag{2.7}$$

where

$$\nu_k([0,x]) = c(\alpha,k)x^{k\alpha}.$$

By a change of variable, (2.7) is equivalent to

$$\lim_{t\downarrow 0} \frac{P\left[\sum_{i=1}^{k} c_i Z_i \le tx\right]}{F^k(t)} = c(\alpha,k)x^{k\alpha}. \tag{2.8}$$

This shows that as $t \downarrow 0$,

$$P\left[\sum_{i=1}^{k} c_i Z_i \le t\right] \sim c(\alpha,k)F^k(t),$$

whence $P[\sum_{i=1}^{k} c_i Z_i \le t]$ is regularly varying at 0 with index $k\alpha$. Compare this to equation (8.14) in Feller [(1971), page 278].

We now consider finite moving averages generated by the iid sequence $\{Z_n, -\infty < n < \infty\}$ with common distribution F satisfying (2.1). Let $c_1,\dots,c_k$ be positive constants and define the moving average process $\{X_n\}$ by

$$X_n = \sum_{j=1}^{k} c_j Z_{n-j}.$$

With a_n defined as above, (2.7) implies

$$nP\left[a_n^{-1}X_1 \in \cdot\right] \to_v \nu_k(\cdot)$$

on $[0,\infty)$, so that for all $x \ge 0$ and $j = 2,\dots,k-1$,

$$nP\left[a_n^{-1}X_1 \le x, a_n^{-1}X_j \le x\right] \le nP\left[a_n^{-1}X_1 \le x\right]P\left[a_n^{-1}c_1 Z_{j-1} \le x\right]$$
$$\to 0$$

as $n \to \infty$. Now since $\{X_n\}$ is $(k-1)$-dependent, it follows from a generalization of a theorem of Adler (1978) given in Davis and Resnick (1988) that

$$\sum_{i=1}^{\infty} \varepsilon_{(i/n, a_n^{-1}X_i)} \Rightarrow \sum_m \varepsilon_{(t_m, j_m)} \tag{2.9}$$

in $M_p([0,\infty)^2)$ where the limit is a Poisson process with mean measure $dt \times \nu_k(dx)$.

From this many extremal properties ensue [cf. Resnick (1986, 1987), Leadbetter, Lindgren and Rootzén (1983) and Davis and Resnick (1985, 1988)]. For instance,

$$\bigwedge_{i=1}^{[nt]} a_n^{-1}X_i \Rightarrow Y(t) := \bigwedge_{t_m \le t} j_m$$

in $D[0,\infty)$.

It is worthwhile to explore the relevance of this discussion to random variables in the (maximum) domain of attraction of a type III extreme value distribution

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\}, & x < 0, \\ 1, & x > 0, \end{cases}$$

for $\alpha > 0$. A distribution F is in this domain if

$$x_0 = \sup\{x: F(x) < 1\} < \infty$$

and

$$1 - F(x_0 - x^{-1}) = x^{-\alpha}L(x),$$

where $L(\cdot)$ is a slowly varying function at ∞. If Z has distribution F, we have for $x > 0$,

$$P[(x_0 - Z)^{-1} > x] = 1 - F(x_0 - x^{-1}) = x^{-\alpha}L(x).$$

Let $t = x^{-1}$ and we have as $t \downarrow 0$,

$$P[x_0 - Z \le t] = t^\alpha L(t^{-1}) =: t^\alpha L_1(t)$$

so that (2.1) holds. If $Z_1, \dots, Z_k$ are iid with common distribution F, then for $c_i > 0$, $i = 1, \dots, k$, we get from (2.8) as $t \downarrow 0$,

$$P\left[\sum_{i=1}^{k} c_i(x_0 - Z_i) \le t\right] \sim c(\alpha, k)(t^\alpha L_1(t))^k \tag{2.10}$$

so that as $x \to \infty$,

$$P\left[\left(\sum_{i=1}^{k} c_i\right)x_0 - x^{-1} \le \sum_{i=1}^{k} c_i Z_i\right] \sim c(\alpha, k)(x^{-\alpha}L(x))^k$$

and therefore the distribution of $\sum_1^k c_i Z_i$ is in the domain of attraction of $\Psi_{k\alpha}$.

Pushing this further, we may take a_n to satisfy

$$P[x_0 - Z_1 \le a_n] \sim n^{-1/k}, \qquad n \to \infty.$$

If $\{Z_n, -\infty < n < \infty\}$ are iid with distribution $F \in \mathscr{D}(\Psi_\alpha)$, then from (2.9) and (2.10),

$$\sum_{i=1}^{\infty} \varepsilon_{(i/n, a_n^{-1}\Sigma_{l=1}^k c_l(x_0 - Z_{i-l}))} \Rightarrow \sum_m \varepsilon_{(t_m, j_m)}$$

in $M_p([0,\infty)^2)$ whence, setting $X_n = \Sigma_{i=1}^k c_i Z_{n-i}$,

$$\sum_{i=1}^{\infty} \varepsilon_{(i/n, a_n^{-1}(X_i - (\Sigma_{l=1}^k c_l)x_0))} \Rightarrow \sum_m \varepsilon_{(t_m, -j_m)} \tag{2.11}$$

in $M_p([0,\infty) \times (-\infty, 0])$ and from (2.11) many maximal properties of $\{X_n\}$ are readily determined.

3. Infinite moving averages. Unlike the finite moving average case of Section 2, the distribution of the infinite order moving average is no longer regularly varying at zero. The objective of this section is in fact to show that the marginal distribution function of such a process belongs to the *minimum* domain of attraction of the extreme value distribution $1 - \exp\{-e^x\}$.

For a nonnegative random variable Z with df F, we define its Esscher transform to be a random variable $Z_{(\lambda)}$ with distribution given by

$$F_{Z_{(\lambda)}}(dx) = e^{-\lambda x}F(dx)/\phi_z(\lambda), \qquad \lambda \geq 0,$$

where

$$\phi_Z(\lambda) = Ee^{-\lambda Z}.$$

It follows directly from the definition that if Y and Z are independent and nonnegative random variables, then

$$(Y + Z)_{(\lambda)} =_d Y_{(\lambda)} + Z_{(\lambda)}$$

and

$$(cZ)_{(\lambda)} =_d cZ_{(c\lambda)}$$

for all $c > 0$.

Now suppose Z has a probability density function (pdf) $f(x) = (d/dx)F(x)$, which is regularly varying at 0 with index $\alpha - 1$, $\alpha > 0$. That is, for all $x > 0$,

$$\lim_{t \downarrow 0} \frac{f(tx)}{f(t)} = x^{\alpha - 1}, \tag{3.1}$$

which, by Karamata's theorem and Theorem 3 of Feller [(1971), page 445], implies

$$\frac{xf(x)}{F(x)} \to \alpha, \quad \text{as } x \downarrow 0, \tag{3.2}$$

and

$$\begin{aligned} \phi(\lambda) &:= \int_0^\infty e^{-\lambda x} f(x)\,dx \\ &\sim F\left(\frac{1}{\lambda}\right)\Gamma(\alpha + 1), \quad \text{as } \lambda \to \infty. \end{aligned} \tag{3.3}$$

From (3.1)–(3.3), the pdf $f_\lambda(x) = \exp\{-\lambda x\} f(x)/\phi(\lambda)$ of $Z_{(\lambda)}$ satisfies

$$\begin{aligned}\frac{1}{\lambda} f_\lambda\left(\frac{x}{\lambda}\right) &\sim \frac{e^{-x} f(x/\lambda)}{\lambda F(1/\lambda)\Gamma(\alpha+1)} \\ &= \frac{e^{-x}}{\Gamma(\alpha+1)}\left(\frac{f(x/\lambda)}{f(1/\lambda)}\right)\left(\frac{(1/\lambda) f(1/\lambda)}{F(1/\lambda)}\right) \\ &\to \frac{e^{-x} x^{\alpha-1}}{\Gamma(\alpha)}\end{aligned} \tag{3.4}$$

as $\lambda \to \infty$. Consequently,

$$\lambda Z_{(\lambda)} \Rightarrow \Gamma_\alpha$$

as $\lambda \to \infty$, where Γ_α has a gamma distribution with parameter α. Since $\phi'(\lambda)$ and $\phi''(\lambda)$ are monotone, we have by the monotone density theorem [Resnick (1987), Proposition 0.7] that the mean μ_λ and the variance σ_λ^2 of $Z_{(\lambda)}$ satisfy as $\lambda \to \infty$,

$$\begin{aligned}\lambda \mu_\lambda = E(\lambda Z_{(\lambda)}) &= -\lambda \phi'(\lambda)/\phi(\lambda) \\ &\to \alpha = E\Gamma_\alpha\end{aligned} \tag{3.5}$$

and

$$\lambda^2 \sigma_\lambda^2 := \operatorname{Var}(\lambda Z_{(\lambda)}) \to \alpha, \tag{3.6}$$

since

$$E(\lambda Z_{(\lambda)})^2 = \lambda^2 \phi''(\lambda)/\phi(\lambda) \to (\alpha+1)\alpha = E\Gamma_\alpha^2.$$

PROPOSITION 3.1. *Let $\{Z_t, t = 1, 2, \dots\}$ be an iid sequence of nonnegative random variables with finite variance and pdf f satisfying* (3.1). *Let c_j be positive constants with $\sum_{j=1}^\infty c_j < \infty$.*

(a) *The series $Y_{(\lambda)} = (\sum_{j=1}^\infty c_j Z_j)_{(\lambda)} = \sum_{j=1}^\infty c_j (Z_j)_{(c_j\lambda)}$ converges a.s. and in L^2.*

(b) *Set*

$$W_\lambda := \frac{Y_{(\lambda)} - EY_{(\lambda)}}{\sqrt{\operatorname{Var}(Y_{(\lambda)})}}.$$

Then

$$W_\lambda \Rightarrow N(0,1)$$

as $\lambda \to \infty$.

(c) *If in addition*

$$\int_0^\infty e^{-2\lambda x} f^2(x)\,dx < \infty \quad \textit{for large } \lambda \tag{3.7}$$

and c_j is a nonincreasing sequence satisfying for all $\theta \in (0,1)$,

$$\lim_{n\to\infty} \theta^n \sum_{j=\theta^{-n}}^{\infty} c_j^2/c_n^2 = 0, \tag{3.8}$$

then density convergence ensues:

$$f_{W_\lambda}(x) \to n(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{3.9}$$

uniformly in x.

REMARK 1. The assumption (3.7) together with (3.2) and (3.3) implies that

$$\begin{aligned} E\lambda^{-1} f_\lambda(Z_{(\lambda)}) &= \int_0^\infty \frac{\lambda^{-1} e^{-2\lambda x} f^2(x)}{\phi^2(\lambda)} dx \\ &\sim \frac{\lambda^{-1}(4\alpha\lambda)^{-1} f^2(1/(2\lambda))\Gamma(2\alpha+1)}{\alpha^{-2}\lambda^{-2} f^2(1/\lambda)\Gamma^2(\alpha+1)} \\ &\to \text{(const.)} \end{aligned}$$

as $\lambda \to \infty$. Hence (3.7) implies that $E(\lambda^{-1} f_\lambda(Z_{(\lambda)}))$ is bounded for large λ. This will be needed for what follows.

REMARK 2. The nondecreasing assumption on the coefficients can be dispensed with provided that condition (3.8) is expressed in terms of the ordered c_j [i.e., replace c_j by the jth largest c_k in (3.8)]. Condition (3.8) is quite mild in that it is satisfied by all but the most pathological sequences. For example, (3.8) is easily checked by $c_j \sim Kr^j$ or $c_j \sim Kj^{-1/r}$ for some $r \in (0,1)$ and $K > 0$. Weaker assumptions on $\{c_j\}$ could be imposed at the expense of a second order assumption on the regular variation of $f(x)$ near 0.

PROOF OF PROPOSITION 3.1. (a) It follows from (3.5) and the definition of $Z_{(\lambda)}$ that μ_λ is bounded for all $\lambda \geq 0$ and hence

$$E\left(\sum_{j=1}^{\infty} c_j (Z_j)_{(c_j\lambda)}\right) = \sum_{j=1}^{\infty} c_j \mu_{c_j\lambda} < \infty$$

by the summability of $\{c_j\}$. L^2 convergence is shown similarly.

(b) Define

$$g_\lambda(t) = Ee^{it\lambda(Z_{(\lambda)}-\mu_\lambda)}.$$

By (26.5) in Billingsley (1986),

$$|g_\lambda(t) - \left(1 - \tfrac{1}{2}t^2\lambda^2\sigma_\lambda^2\right)| \leq t^2 E \min\left\{|t||\lambda(Z_{(\lambda)} - \mu_\lambda)|^3, \lambda^2(Z_{(\lambda)} - \mu_\lambda)^2\right\}. \tag{3.10}$$

We first show that

$$a(t) := \sup_{\lambda\geq 0} \lambda^{-2}\sigma_\lambda^{-2} E \min\left\{|t||\lambda(Z_{(\lambda)} - \mu_\lambda)|^3, \lambda^2(Z_{(\lambda)} - \mu_\lambda)^2\right\} \to 0 \tag{3.11}$$

as $t \to 0$ so that

$$|g_\lambda(t) - \left(1 - \tfrac{1}{2}t^2\lambda^2\sigma_\lambda^2\right)| \le t^2\lambda^2\sigma_\lambda^2 a(t) \tag{3.12}$$

uniformly in $\lambda \ge 0$.

Observe that with $\tilde{Z}_{(\lambda)} = \lambda(Z_{(\lambda)} - \mu_\lambda)$,

$$0 \le \left(|t||\tilde{Z}_{(\lambda)}|^2\right) \wedge \tilde{Z}_{(\lambda)}^2 \le \tilde{Z}_{(\lambda)}^2.$$

Since $\tilde{Z}_{(\lambda)}^2 \Rightarrow (\Gamma_\alpha - \alpha)^2$, $(|t||\tilde{Z}_{(\lambda)}|^3) \wedge \tilde{Z}_{(\lambda)}^2 \Rightarrow (|t||\Gamma_\alpha - \alpha|^3) \wedge |\Gamma_\alpha - \alpha|^2$ and $E\tilde{Z}_{(\lambda)}^2 = \lambda^2\sigma_\lambda^2 \to \alpha = \mathrm{Var}(\Gamma_\alpha)$, as $\lambda \to \infty$, we get by Pratt's lemma [cf. Billingsley (1986), Exercise 16.6] that for t fixed,

$$E\left[\left(|t||\tilde{Z}_{(\lambda)}|^3\right) \wedge \tilde{Z}_{(\lambda)}^2\right] \to E\left[\left(|t||\Gamma_\alpha - \alpha|^3\right) \wedge |\Gamma_\alpha - \alpha|^2\right].$$

Moreover, since $(|t||\Gamma_\alpha - \alpha|^3) \wedge |\Gamma_\alpha - \alpha|^2 \le (\Gamma_\alpha - \alpha)^2 \in L_1$, we get by dominated convergence,

$$\lim_{t \to 0} E\left[\left(|t||\Gamma_\alpha - \alpha|^3\right) \wedge |\Gamma_\alpha - \alpha|^2\right] = 0.$$

Hence from (3.6) there exists an $M > 0$ such that

$$\sup_{\lambda \ge M} \lambda^{-2}\sigma_\lambda^{-2} E\left[\left(|t||\tilde{Z}_{(\lambda)}|^3\right) \wedge \tilde{Z}_{(\lambda)}^2\right] \le 2\alpha^{-1} E\left[\left(|t||\Gamma_\alpha - \alpha|^3\right) \wedge |\Gamma_\alpha - \alpha|^2\right]$$
$$\to 0$$

as $t \to 0$. On the other hand, for $\lambda \le M$, it follows easily from the form of the density function of $Z_{(\lambda)}$ and the boundedness of σ_λ^{-2}, $\lambda \le M$, that

$$\sup_{\lambda \le M} \lambda^{-2}\sigma_\lambda^{-2} E\left[\left(|t||\lambda(Z_{(\lambda)} - \mu_\lambda)|^3\right) \wedge \lambda^2(Z_{(\lambda)} - \mu_\lambda)^2\right] \to 0$$

as $t \to 0$. This, together with the above inequality, proves (3.11).

We now prove that W_λ is asymptotically normal. Since $(cZ)_\lambda =_d cZ_{(c\lambda)}$, we get from (3.12) that the characteristic function $g_{\lambda c_j}(t)$ of

$$\lambda\left((c_j Z_j)_{(\lambda)} - c_j \mu_{c_j\lambda}\right) = \lambda c_j\left((Z_j)_{(c_j\lambda)} - \mu_{c_j\lambda}\right)$$

satisfies

$$|g_{c_j\lambda}(t) - \left(1 - \tfrac{1}{2}t^2 c_j^2\lambda^2\sigma_{c_j\lambda}^2\right)| \le c_j^2\lambda^2\sigma_{c_j\lambda}^2 t^2 a(t). \tag{3.13}$$

We have that $c_j^2\lambda^2\sigma_{c_j\lambda}^2$ is bounded in λ and j; for $c_j\lambda$ large this follows from (3.6) and for $c_j\lambda$ small this is a consequence of

$$\lambda^2\sigma_\lambda^2 = \frac{\lambda^2\phi''(\lambda)}{\phi(\lambda)} - \left(\frac{\lambda\phi'(\lambda)}{\phi(\lambda)}\right)^2$$

being bounded for small λ due to $EZ_1 < \infty$, $EZ_1^2 < \infty$. Also we have

$$\lambda^2 S_\lambda^2 \to \infty, \tag{3.14}$$

where

$$S_\lambda^2 := \operatorname{Var}(Y_{(\lambda)}) = \sum_{j=1}^{\infty} c_j^2 \sigma_{c_j\lambda}^2,$$

since by Fatou's lemma and (3.6),

$$\liminf_{\lambda\to\infty} \lambda^2 S_\lambda^2 \geq \sum_{j=1}^{\infty} \liminf_{\lambda\to\infty} c_j^2 \lambda^2 \sigma_{c_j\lambda}^2 = \sum_{j=1}^{\infty} \alpha = \infty$$

(recall $c_j > 0$ for all j). So for λ large, we have that

$$\begin{aligned}\left| Ee^{itW_\lambda} - \prod_{j=1}^{\infty}\left(1 - \frac{t^2 c_j^2 \lambda^2 \sigma_{c_j\lambda}^2}{2\lambda^2 S_\lambda^2}\right)\right| &= \left| \prod_{j=1}^{\infty} g_{c_j\lambda}\left(\frac{t}{\lambda S_\lambda}\right) - \prod_{j=1}^{\infty}\left(1 - \frac{t^2 c_j^2 \lambda^2 \sigma_{c_j\lambda}^2}{2\lambda^2 S_\lambda^2}\right)\right| \\ &\leq t^2 \sum_{j=1}^{\infty} \frac{c_j^2 \sigma_{c_j\lambda}^2}{S_\lambda^2} a\left(\frac{t}{\lambda S_\lambda}\right) \\ &= t^2 a\left(\frac{t}{\lambda S_\lambda}\right) \to 0\end{aligned}$$

as $\lambda \to \infty$, the convergence to zero following from (3.11) and (3.14). Since

$$\prod_{j=1}^{\infty}\left(1 - \frac{t^2 c_j^2 \lambda^2 \sigma_{c_j\lambda}^2}{2\lambda^2 S_\lambda^2}\right) \to e^{-t^2/2} \quad \text{as } \lambda \to \infty,$$

it follows that

$$Ee^{itW_\lambda} \to e^{-t^2/2}, \tag{3.15}$$

which establishes the asymptotic normality of W_λ.

(b) To prove density convergence, we follow Feller [(1971), page 516]. The Fourier inversion formula gives

$$\begin{aligned}|f_{W_\lambda}(x) - n(x)| &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\prod_{j=1}^{\infty} g_{c_j\lambda}\left(\frac{t}{\lambda S_\lambda}\right) - e^{-t^2/2}\right| dt \\ &\leq \left(\int_{|t|\leq a} + \int_{a<|t|\leq \delta\lambda S_\lambda} + \int_{|t|>\delta\lambda S_\lambda}\right),\end{aligned} \tag{3.16}$$

where $a > 0$ and $\delta > 0$ are to be specified. By the dominated convergence theorem and (3.15), the first integral converges to zero as $\lambda \to \infty$. From (3.13), for each j

$$|g_{c_j\lambda}(t)| \leq \exp\left\{-\tfrac{1}{2} c_j^2 \lambda^2 \sigma_{c_j\lambda}^2 t^2 (1 - 2a(t))\right\}$$

for t small and, therefore,

$$\begin{aligned}\left|\prod_{j=1}^{\infty} g_{c_j\lambda}\left(\frac{t}{\lambda S_\lambda}\right)\right| &\leq \exp\left\{-\frac{t^2}{2}\sum_{j=1}^{\infty}\frac{c_j^2\lambda^2\sigma_{c_j\lambda}^2}{\lambda^2 S_\lambda^2}\left(1 - 2a\left(\frac{t}{\lambda S_\lambda}\right)\right)\right\} \\ &= \exp\left\{-\frac{t^2}{2}\left(1 - 2a\left(\frac{t}{\lambda S_\lambda}\right)\right)\right\}.\end{aligned}$$

Thus there exist $\beta \in (0, \frac{1}{2})$ and $\delta > 0$ such that for $|t/(\lambda S_\lambda)| \le \delta$,

$$\left| \prod_{j=1}^{\infty} g_{c_j\lambda}\left(\frac{t}{\lambda S_\lambda}\right) \right| \le \exp\{-\beta t^2\}$$

and, therefore, the second integral in (3.16) may be bounded by

$$2\int_{a<|t|\le\delta\lambda S_\lambda} e^{-\beta t^2}\,dt \le 2\int_{a<|t|} e^{-\beta t^2}\,dt,$$

which can be made arbitrarily small by taking a large.

To handle the last integral in (3.16), we note that by (3.4) and Scheffé's lemma,

$$g_\lambda(t) \to Ee^{it(\Gamma_\alpha - \alpha)}$$

uniformly in t as $\lambda \to \infty$. Therefore, given $\varepsilon > 0$ sufficiently small, there exists an $M > 0$ such that for $\lambda > M$,

$$\begin{aligned} \sup_{|t|\ge\delta} |g_\lambda(t)| &< \sup_{|t|\ge\delta} |Ee^{it(\Gamma_\alpha-\alpha)}| + \varepsilon \\ &= (1+\delta^2)^{-\alpha/2} + \varepsilon =: \eta < 1. \end{aligned} \tag{3.17}$$

Increasing M if necessary, it follows by Cauchy–Schwarz, Plancherel's identity and Remark 1 that for all $\lambda_1, \lambda_2 > M$,

$$\begin{aligned} \int_{-\infty}^{\infty} |g_{\lambda_1}(t) g_{\lambda_2}(t)|\,dt &\le \left(\int_{-\infty}^{\infty} |g_{\lambda_1}(t)|^2\,dt\right)^{1/2} \left(\int_{-\infty}^{\infty} |g_{\lambda_2}(t)|^2\,dt\right)^{1/2} \\ &= 2\pi\left(E\lambda_1^{-1} f_{\lambda_1}(Z_{(\lambda_1)})\right)^{1/2} \left(E\lambda_2^{-1} f_{\lambda_2}(Z_{(\lambda_2)})\right)^{1/2} \\ &< (\text{const.}). \end{aligned} \tag{3.18}$$

Now setting $N_\lambda = \min\{j\colon \lambda c_j \le M\}$ and using the inequalities (3.17) and (3.18), the third integral in (3.16) is bounded for λ large by

$$\begin{aligned} &\int_{|t|>\delta\lambda S_\lambda} \prod_{j=1}^{\infty} \left| g_{c_j\lambda}\left(\frac{t}{\lambda S_\lambda}\right)\right| dt + \int_{|t|>\delta\lambda S_\lambda} e^{-t^2/2}\,dt \\ &\le \int_{|t|>\delta\lambda S_\lambda} \prod_{j=1}^{N_\lambda - 1} \left| g_{c_j\lambda}\left(\frac{t}{\lambda S_\lambda}\right)\right| dt + o(1) \\ &\le \eta^{N_\lambda - 3} \int_{-\infty}^{\infty} \left| g_{c_1\lambda}\left(\frac{t}{\lambda S_\lambda}\right) g_{c_2\lambda}\left(\frac{t}{\lambda S_\lambda}\right)\right| dt + o(1) \\ &\le \eta^{N_\lambda - 3}\lambda S_\lambda (\text{const.}) + o(1) \end{aligned} \tag{3.19}$$

and it remains to check that

$$\eta^{2N_\lambda}\lambda^2 S_\lambda^2 = \eta^{2N_\lambda} \sum_{j=1}^{\infty} \lambda^2 c_j^2 \sigma_{c_j\lambda}^2 \to 0$$

as $\lambda \to \infty$. Recall that for all λ, $\lambda^2\sigma_\lambda^2$ and σ_λ^2 are bounded by a finite constant, say B. Also by definition of N_λ, $\lambda \le M/c_{N_\lambda}$ and $N_\lambda \to \infty$ as $\lambda \to \infty$, so for $\theta \in (\eta^2, 1)$, we have

$$\begin{aligned}\eta^{2N_\lambda}\lambda^2 S_\lambda^2 &\le \eta^{2N_\lambda}\left(\sum_{j=1}^{\theta^{-N_\lambda}} \lambda^2 c_j^2 \sigma_{c_j\lambda}^2 + M^2 \sum_{j=\theta^{-N_\lambda}}^{\infty} \frac{c_j^2\sigma_{c_j\lambda}^2}{c_{N_\lambda}^2}\right)\\ &\le \left(\frac{\eta^2}{\theta}\right)^{N_\lambda} B + \frac{\theta^{N_\lambda} M^2 B \sum_{\{j \ge \theta^{-N_\lambda}\}} c_j^2}{c_{N_\lambda}^2}\\ &\to 0, \quad \text{as } \lambda \to \infty,\end{aligned}$$

by (3.8). This completes the proof of the proposition. □

From (3.9) we conclude that as $\lambda \to \infty$,

$$S_\lambda e^{-\lambda(S_\lambda x + m_\lambda)} f_Y(S_\lambda x + m_\lambda)/\phi_Y(\lambda) \to n(x) \tag{3.20}$$

uniformly in x, where

$$m_\lambda = \sum_{j=1}^{\infty} c_j \mu_{c_j\lambda} = \sum_{j=1}^{\infty} c_j EZ_{(c_j\lambda)} = EY_{(\lambda)}.$$

Since $\mu_\lambda = -\phi'(\lambda)/\phi(\lambda)$ and

$$-\frac{d\mu_\lambda}{d\lambda} = \frac{\phi''(\lambda)}{\phi(\lambda)} - \left(\frac{\phi'(\lambda)}{\phi(\lambda)}\right)^2 = \mathrm{Var}(Z_{(\lambda)}) > 0,$$

we have from (3.5) that $\mu_\lambda \downarrow 0$ as $\lambda \to \infty$. Hence each summand in m_λ decreases to 0 as $\lambda \to \infty$, which by dominated convergence implies

$$m_\lambda \downarrow 0 \quad \text{as } \lambda \to \infty.$$

So as in Feigin and Yashchin (1983), if we evaluate (3.20) at $x = 0$, we get as $\lambda \to \infty$,

$$S_\lambda e^{-\lambda m_\lambda} f_Y(m_\lambda)/\phi_Y(\lambda) \to 1/\sqrt{2\pi}$$

and if $\eta = m_\lambda$, we get as $\eta \downarrow 0$,

$$S_{m^{\leftarrow}(\eta)} e^{-\eta m^{\leftarrow}(\eta)} f_Y(\eta)/\phi_Y(m^{\leftarrow}(\eta)) \to 1/\sqrt{2\pi},$$

giving an asymptotic form for the density of Y near 0. We show that this asymptotic form implies the distribution of Y is in the domain of attraction of $1 - \exp\{-e^x\}$ $x \in \mathbb{R}$.

Observe first that the function

$$g(x) = \frac{1}{m^{\leftarrow}(x)}$$

is well defined for all $x > 0$ with derivative given by

$$g'(x) = -\frac{1}{m'(m^{\leftarrow}(x))(m^{\leftarrow}(x))^2}$$
$$= (m^{\leftarrow}(x))^{-2} S_{m^{\leftarrow}(x)}^{-2}.$$

Since $\lim_{\lambda \to \infty} \lambda^2 S_\lambda^2 = \infty$, we get $\lim_{x \downarrow 0} g'(x) = 0$, whence g is self-neglecting:

$$\lim_{t \downarrow 0} g(t + xg(t))/g(t) = 1$$

locally uniformly in x [see Lemma 8.13.8 in Bingham, Goldie and Teugels (1987)]. Such a function is a suitable auxiliary function for distribution functions in the domain of attraction of $1 - \exp\{-e^x\}$, $x \in \mathbb{R}$. We show g is the auxiliary function of Y.

Since the convergence in (3.20) is uniform, we get, upon replacing x by $x/(\lambda S_\lambda)$ and remembering that $\lambda S_\lambda \to \infty$,

$$S_\lambda e^{-x} e^{-\lambda m_\lambda} f_Y(x/\lambda + m_\lambda)/\phi_Y(\lambda) \to n(0)$$

locally uniformly in x as $\lambda \to \infty$. If we make the change of variables $t = m_\lambda$ and recall $g(t) = 1/m^{\leftarrow}(t)$, then as $t \downarrow 0$,

$$\sqrt{2\pi}\, S_{m^{\leftarrow}(t)} e^{-tm^{\leftarrow}(t)} f_Y(xg(t) + t)/\phi_Y(m^{\leftarrow}(t)) \to e^x$$

locally uniformly in x so that

$$\lim_{t \downarrow 0} \frac{f_Y(xg(t) + t)}{f_Y(t)} \to e^x$$

locally uniformly. Thus f_Y is in the class Γ with auxiliary function g and hence the same is true for $F_Y(t) = \int_0^t f_Y(u)\,du$ [see Corollary 3.10.7 in Bingham, Goldie, and Teugels (1987) for the case f_Y is monotone; Vervaat (1973), page 24, for nonmonotone f_Y]. In particular,

$$g(t) = \frac{1}{m^{\leftarrow}(t)} \sim \frac{\int_0^t f_Y(y)\,dy}{f_Y(t)} = \frac{F_Y(t)}{f_Y(t)}$$

and hence

$$F_Y(t) \sim \frac{f_Y(t)}{m^{\leftarrow}(t)}.$$

The following theorem is now immediate.

THEOREM 3.2. *Under the assumptions of Proposition* 3.1 *including* (3.7) *and* (3.8), *the distribution function of the random variable* $Y = \sum_{j=1}^{\infty} c_j Z_j$ *is in the minimum domain of attraction of* $1 - \exp\{-e^x\}$ *and*

$$P[Y \le m_\lambda] \sim e^{\lambda m_\lambda} \phi_Y(\lambda)/\big(\lambda S_\lambda \sqrt{2\pi}\big), \quad \text{as } \lambda \to \infty,$$

or equivalently

$$P[Y \le x] \sim e^{x m^{\leftarrow}(x)} \phi_Y(m^{\leftarrow}(x)) / \big(m^{\leftarrow}(x) S_{m^{\leftarrow}(x)} \sqrt{2\pi}\big), \quad \text{as } x \downarrow 0.$$

Moreover, if $\{Y_n\}$ *is an iid sequence of rv's with* $Y_1 =_d Y$, *then*

$$P\left[a_n^{-1}\left(\bigwedge_{j=1}^{n} Y_j - b_n\right) \le x\right] \to 1 - \exp\{-e^x\},$$

where $b_n = F_Y^{\leftarrow}(1/n)$ *and* $a_n = g(b_n)$.

REMARK 3. If we pick r_n to satisfy

$$\exp\{r_n m_{r_n}\} \phi_Y(r_n) / \big(r_n S_{r_n} \sqrt{2\pi}\big) = n^{-1},$$

then we also have

$$b_n = m_{r_n}, \qquad a_n = r_n^{-1}.$$

4. Point process convergence. In this section, we extend the point process convergence of Section 2 to infinite order moving averages,

$$X_t = \sum_{j=0}^{\infty} c_j Z_{t-j}, \tag{4.1}$$

satisfying the conditions of Section 3. We will show that the sequence of point processes,

$$N_n := \sum_{j=1}^{\infty} \varepsilon_{(j/n,\, a_n^{-1}(X_j - b_n))},$$

where a_n and b_n are as specified in Remark 3 of Section 3, converges in $M_p([0,\infty) \times [-\infty,\infty))$ to a Poisson random measure (PRM) with mean measure $dt \times e^x\,dx$. In particular, this implies that the asymptotic behavior of the lower extremes of $\{X_t\}$ coincides with that of the associated independent sequence $\{\hat{X}_t\}$ (i.e., $\{\hat{X}_t\}$ is iid with $\hat{X}_1 =_d X_1$). Similarly, these results can be recast, as in Section 2, for the maxima of moving averages based on Z's in the domain of attraction of a type III extreme value distribution.

THEOREM 4.1. *Let* $\{X_t\}$ *be the moving average process* (4.1) *where* $\{Z_t\}$ *is an iid sequence of random variables with probability density function satisfying* (3.1), $\{c_j\}$ *is a sequence of positive constants whose ordered values* (*see Remark* 2 *of Section* 3) *satisfy* (3.8) *and* $c_j = O(j^{-q})$ *as* $j \to \infty$ *for some* $q > 2$. *Then in* $M_p([0,\infty) \times [-\infty,\infty))$,

$$N_n \Rightarrow N, \tag{4.2}$$

where N *is a* PRM($dt \times e^x\,dx$) *and* a_n *and* b_n *are as specified in Remark* 3 *of Section* 3.

PROOF. First observe that from Theorem 3.2,

$$nP[X_1 \le a_n x + b_n] \to e^x \tag{4.3}$$

for all $x \ge -\infty$ and

$$a_n \to 0, \qquad \frac{b_n}{a_n} \to \infty.$$

The first statement follows from $a_n = g(b_n) = 1/m^{\leftarrow}(b_n) \to 0$ since $m(\lambda) \to 0$ as $\lambda \to \infty$ and the second statement always holds for distributions in this domain of attraction. In addition the slow variation of auxiliary functions [see Proposition 0.12 in Resnick (1987)] implies that a_n is slowly varying, whence

$$a_n n^{\varepsilon} \to \infty \tag{4.4}$$

for all $\varepsilon > 0$. Moreover

$$\frac{b_n^2}{a_n} \to 0. \tag{4.5}$$

To see this it is enough to show, replacing r_n with λ in the definition of the normalizing constants, that

$$\lambda^{1/2} m_\lambda = \sum_{j=0}^{\infty} \lambda^{1/2} c_j \mu_{c_j\lambda} \to 0$$

as $\lambda \to \infty$. But $\lambda^{1/2}\mu_\lambda$ is a bounded function of λ, say by the constant B, which, by (3.5), vanishes at infinity. Thus

$$\begin{aligned}\lambda^{1/2} m_\lambda &= \sum_{j=1}^{J} \lambda^{1/2} c_j \mu_{c_j\lambda} + B \sum_{j=J+1}^{\infty} c_j^{1/2} \\ &\to B \sum_{j=J+1}^{\infty} c_j^{1/2} \quad \text{as } \lambda \to \infty, \\ &\to 0 \quad \text{as } J \to \infty,\end{aligned}$$

which proves (4.5).

To establish the convergence in (4.2), it suffices by Theorem 5.5.1(ii) in Leadbetter, Lindgren and Rootzén (1983) to check that the process $\{-X_t\}$ satisfies the mixing condition $D_r(\mathbf{u}_n)$ and the local dependence condition $D'(u_n)$ for all x where $u_n = -(a_n x + b_n)$. We verify $D_r(\mathbf{u}_n)$ by applying Lemma 3.1 in Rootzén (1986). With $\mu = EZ_1$, we have by Markov's inequality,

$$\begin{aligned} nP\left[a_n^{-1}\left|\sum_{j=n\nu}^{\infty} c_j Z_j\right| > \varepsilon\right] &\le \varepsilon^{-1} n a_n^{-1}\left(\sum_{j=n\nu}^{\infty} c_j\right)\mu \\ &\le (\text{const.})\, n a_n^{-1} (n\nu)^{1-q} \\ &\to 0, \end{aligned}$$

since $q > 2$, from which $D_r(\mathbf{u}_n)$ now follows directly from Lemma 3.1 in Rootzén (1986).

As for $D'(u_n)$, we must show that

$$\limsup_{n\to\infty} n \sum_{j=1}^{[n/k]} P\left[X_1 \le u_n, X_{j+1} \le u_n\right] \to 0$$

as $k \to \infty$ for all $x \ge 1$; $u_n = a_n x + b_n$. By (4.3),

$$\begin{aligned} nP\left[X_1 \le u_n X_{j+1} \le u_n\right] &\le nP\left[X_1 \le u_n, c_0 Z_{j+1} + \cdots + c_{j-1}Z_2 \le u_n\right] \\ &= nP[X_1 \le u_n] P\left[c_0 Z_{j+1} + \cdots + c_{j-1}Z_2 \le u_n\right] \qquad (4.6) \\ &= O(1) P\left[c_0 Z_{j+1} + \cdots + c_{j-1}Z_2 \le u_n\right] \end{aligned}$$

and writing $X_j' = c_j Z_1 + c_{j+1} Z_0 + \cdots$,

$$\begin{aligned} P&\left[c_0 Z_{j+1} + \cdots + c_{j-1}Z_2 \le u_n\right] \\ &= P\left[c_0 Z_{j+1} + \cdots + c_{j-1}Z_2 \le u_n, X_j' \le a_n\right] \\ &\quad + P\left[c_0 Z_{j+1} + \cdots + c_{j-1}Z_2 \le u_n, X_j' > a_n\right] \\ &\le P\left[X_{j+1} \le a_n(x+1) + b_n\right] \qquad (4.7) \\ &\quad + P\left[c_0 Z_{j+1} + \cdots + c_{j-1}Z_2 \le u_n\right] P\left[X_j' > a_n\right] \\ &\le O(n^{-1}) + F\left(\frac{u_n}{c_0}\right) \cdots F\left(\frac{u_n}{c_{j-1}}\right) P\left[X_j' > a_n\right]. \end{aligned}$$

Now define

$$c_s^* := \min_{0 \le i < s} c_i.$$

Since $a_n, u_n \to 0$, there exists a sequence of integers $w_n \uparrow \infty$ such that

$$c_{w_n}^* \ge u_n^{1/2}$$

for all n large. Moreover for a fixed integer $s > 2/\alpha$, choose a positive $\varepsilon < \alpha s - 2$, so that by the regular variation of F at zero and (4.5), we have

$$\begin{aligned} a_n^{-1} F^s(u_n/c_s^*) &\le (\text{const.}) a_n^{-1}(a_n x + b_n)^{s\alpha - \varepsilon} \quad \text{(for } n \text{ large)} \\ &\le (\text{const.}) a_n^{-1}(a_n x + b_n)^2 \\ &\sim (\text{const.}) a_n^{-1} b_n^2 \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Thus the second term in (4.7) is bounded by

$$F^j\left(u_n/c_{w_n}^*\right) \le F^j\left(u_n^{1/2}\right) \quad \text{for } j \le w_n$$

and, using Markov's inequality, by

$$F^s(u_n/c_s^*) \mu a_n^{-1} \sum_{i=j}^{\infty} c_i \le o(1) j^{1-q} \quad \text{for } j > w_n > s,$$

where the $o(1)$ term does not depend on j. Summing we obtain from (4.6), (4.7) and the two preceding inequalities

$$
\begin{aligned}
n \sum_{j=1}^{[n/k]} P\left[X_1 \le u_n, X_{j+1} \le u_n\right]
&\le O(1)\left[(n/k)O(n^{-1}) + \sum_{j=1}^{w_n} F^j\left(u_n^{1/2}\right) + o(1) \sum_{j>w_n} j^{1-q}\right] \\
&= O(1)\left[O(1)k^{-1} + o(1) + o(1)\right] \\
&= O(1)k^{-1} \quad (\text{as } n \to \infty) \\
&\to 0
\end{aligned}
$$

as $k \to \infty$ as required. This completes the proof. □

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