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COUPLED REACTION-DIFFUSION EQUATIONS[1]

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We consider single equations and systems of reaction-diffusion equations depending on small parameter. These equations are generalizations of the Kolmogorov–Petrovskii–Piskunov equation. Using the large deviation principle, we describe the asymptotic behavior of solutions.

1. Introduction. Statement of the problem. Consider the reaction-diffusion equation (RDE)

$$(1)\qquad \frac{\partial u(t,x)}{\partial t} = \frac{D}{2}\Delta u + f(u), \qquad u(0,x) = g(x),\ x \in R^r,\ t > 0.$$

Let $f \in C^1$, $f(u) > 0$ for $0 < u < 1$, $f(0) = f(1) = 0$, $f(u) < 0$ for $u \notin [0,1]$ and $(df/du)|_{u=0} = f'(0) = \sup_{u>0} u^{-1} f(u)$. The class of such functions f we call $\mathscr{F}_1$. The initial function g we suppose to be bounded, nonnegative and having support $G_0 = [\{x \in R^r\colon\ g(x) \neq 0\}]$ such that $G_0 = [(G_0)]$, $G_0 \neq \varnothing$. Here $[A]$ means the closure of a set $A \subset R^r$ and (A) is the interior of $A \subset R^r$. Let the initial function $g(x)$ be continuous on (G_0) and outside G_0.

It is well known that the solution $u(t,x)$ of (1) tends to 1 when $t \to \infty$, and the domain $G_t \subset R^r$, where $u(t,x)$ close to 1 is growing in a sense with the speed $\sqrt{2Df'(0)}$ (see [10], [1]).

If the diffusion coefficients, initial function and nonlinear term are nonhomogeneous in space but changing slowly so that they are functions of εx, where $x \in R^r$ and $\varepsilon > 0$ is a small parameter, a rescaling of the space and the time is useful (see discussion in [6] and [7]). In the new variables, the problem will be the following:

$$(2)\qquad \frac{\partial u(t,x)}{\partial t} = \frac{\varepsilon}{2} L u^\varepsilon + \frac{1}{\varepsilon} f(x, u^\varepsilon), \qquad u^\varepsilon(0,x) = g(x),\ x \in R^r,\ t > 0,$$

where $L = \frac{1}{2}\sum_1^r a^{ij}(x)(\partial^2/\partial x^i\,\partial x^j)$, $a^{ij} \in C^1$, $\sum_1^r a^{ij}(x)\lambda_i\lambda_j \geq a|\lambda|^2$ for any $x, \lambda \in R^r$, for some $a > 0$. Limit behavior of the solution of (2) when $\varepsilon \downarrow 0$ is studied in [6] and [7] using a large deviation principle for the family of Markov processes X_t^ε, corresponding to the operators εL, $\varepsilon \downarrow 0$.

In particular, if $f(x, \cdot) \in \mathscr{F}_1$ for every $x \in R^r$ and $\partial f(x,0)/\partial u = c = \text{const}$, it is proved in these papers that $\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x) = 1$ for $x \in G_t$ and

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$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x) = 0$ for $x \notin [G_t]$, where $G_t = \{x \in R^r, \rho(x, G_0) < t\sqrt{2c}\}$, $t > 0$. Here $G_0 = \operatorname{supp} g$, and ρ is the Riemannian metric corresponding to the operator L: $ds^2 = \sum_{i,j=1}^r a_{ij}(x)\,dx^i\,dx^j$, $(a_{ij}(x)) = (a^{ij}(x))^{-1}$. This means that the domain G_t can be described by the Huygens principle with homogeneous isotropic velocity field $v(x,e) = \sqrt{2c}$, $x, e \in R^r$, $|e| = 1$, in the Riemannian metric ρ. The boundary of the set G_t separates the set where $u^\varepsilon(t,x)$ tends to one and the set where $u^\varepsilon(t,x) \to 0$ for $\varepsilon \downarrow 0$. One can say that the boundary ∂G_t is the position of the wave front at time t.

If we have two equations of the type (1) with $D = D_1$, $f = f_1(u)$, $D = D_2$, $f = f_2(u)$ and $f_1, f_2 \in \mathcal{F}_1$, in general, different velocities of the wave fronts are established in the large time interval: $v_1 = \sqrt{2D_1 f_1'(0)}$, $v_2 = \sqrt{2D_2 f_2'(0)}$.

Let us consider the coupling of such equations,

$$\begin{aligned} \frac{\partial u_1(t,x)}{\partial t} &= \frac{1}{2} D_1 \Delta u_1 + f_1(u_1) + \varepsilon d_1(u_2 - u_1), \\ \frac{\partial u_2(t,x)}{\partial t} &= \frac{1}{2} D_2 \Delta u_2 + f_2(u_2) + \varepsilon d_2(u_1 - u_2). \end{aligned} \tag{3}$$

Here d_1, d_2 are positive constants, $\varepsilon \geq 0$ is a parameter characterizing the strength of the coupling. The physical sense of the last terms in (3) is as follows. For $d_1 = d_2 = 0$, the equations [(3)] describe diffusion and multiplication (or killing) of the particles of the first and the second types. The particles of the different types have no interaction. The new terms describe transmutation from first to second type and vice versa. The constant εd_1 is the intensity of the transition from the first to the second type, and εd_2 is the same characteristic for transition from the second to the first type.

If we consider (3) on a fixed time interval independent of ε and take $\varepsilon \downarrow 0$, the functions $u_k(t,x)$, $k = 1, 2$, tend to solutions of the equations (3) for $\varepsilon = 0$. For $\varepsilon = 0$, the equations [(3)] are independent and have, in general, different velocities of the wave fronts. But in the large time interval, growing together with ε^{-1} when $\varepsilon \downarrow 0$, one can expect that, due to interaction, some velocity of the front common for both components will be established. If the rate of transmutations is small ($\varepsilon \ll 1$), establishing the common velocity takes a great amount of time. The position of the wave front at time t also tends to infinity when $t \to \infty$ (at least in the case of initial functions with compact support). To detect the front we should rescale not only the time but the space also.

As we will see later, the proper scaling is $t \to t/\varepsilon$, $x \to x/\varepsilon$. Put $u_k^\varepsilon(t,x) = u_k(t\varepsilon^{-1}, x\varepsilon^{-1})$, $k = 1, 2$, where $u_k(t,x)$ is the solution of the equations (3). Then we have the following equations for u_1^ε and u_2^ε:

$$\begin{aligned} \frac{\partial u_1^\varepsilon(t,x)}{\partial t} &= \frac{\varepsilon D_1}{2} \Delta u_1^\varepsilon + \frac{1}{\varepsilon} f_1(u_1^\varepsilon) + d_1(u_2^\varepsilon - u_1^\varepsilon), \\ \frac{\partial u_2^\varepsilon(t,x)}{\partial t} &= \frac{\varepsilon D_2}{2} \Delta u_2^\varepsilon + \frac{1}{\varepsilon} f_2(u_2^\varepsilon) + d_2(u_1^\varepsilon - u_2^\varepsilon). \end{aligned} \tag{4}$$

Let $(u_1^\varepsilon(t,x),\ u_2^\varepsilon(t,x))$ be the solution of the system (4) with initial conditions $u_1^\varepsilon(0,x) = g_1(x)$, $u_2^\varepsilon(0,x) = g_2(x)$. We assume that $f_1, f_2 \in \mathscr{F}_1$ and initial functions g_1, g_2 satisfy the conditions listed above. Denote by G_0 the support of the function $g_1(x) + g_2(x)$. We define an increasing family of sets $G_t \subset R^r$, $t > 0$, such that $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 1$ for $x \in G_t$, $t > 0$, $k = 1, 2$, and $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 0$ for $x \in R^r \setminus [G_t]$, $t > 0$, $k = 1, 2$. The sets G_t can be described in terms of a variational problem connected with the action functional for the family of the Markov processes corresponding to the system (4). It turns out that even in the space-homogeneous and isotropic case of (4), the wave front for the coupled system can propagate faster than in each separated component, and the speed of propagation is independent of d_1, d_2, provided these constants are positive.

The generalization of (4) for the nonhomogeneous nonisotropic in-space medium has the form

$$\frac{\partial u_k^\varepsilon(t,x)}{\partial t}$$

$$= \varepsilon L_k u_k^\varepsilon(t,x) + \frac{1}{\varepsilon} f_k(x, u_k^\varepsilon) + \sum_{j=1}^{n} d_{kj}\left(u_j^\varepsilon - u_k^\varepsilon\right), \quad x \in R^r, \quad t > 0, \tag{5}$$

$$u_k^\varepsilon(0,x) = g_k(x) \geq 0, \qquad k = 1, \ldots, n.$$

In (5), $L_k = \frac{1}{2}\sum_{i,j=1}^r a_k^{ij}(x)(\partial^2/\partial x^i\, \partial x^j)$ are elliptic operators with smooth enough coefficients, $d_{kj} > 0$, $f_k(x, \cdot\,) \in \mathscr{F}_1$ for any $x \in R^r$ and $k = 1, \ldots, n$. The initial functions $g_k(x)$ have the same properties as in the case of the single equation.

We describe behavior of the $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x)$ for the solutions of (5). In the case of systems, in general, the motion of the wave front will not be continuous. As in the case of a single equation (see [6] and [7]), a wave front can have jumps. Evolution of the wave front, in general, will not be Markovian: motion of the front after time t, provided the position of the front at the time t is known, depends on the behavior of the front before t. If $\partial f_k(x,u)/\partial u|_{u=0} = c$ independent of x and k, the motion of the front will be continuous, Markovian and can be described by the Huygens principle. The corresponding velocity field is homogeneous and isotropic not in a Riemannian metric, but in a Finsler metric. The Finsler metric is defined by its unit spheres near each $x \in R^r$. In the case under consideration, the unit sphere at the point $x \in R^r$ is the convex envelope of the family of ellipsoids (Riemannian unit sphere): $S_k = \{z\colon \sum_{i,j=1}^r a_{k,ij}(x) z^i z^j \leq 1\}$, $k = 1, \ldots, n$, $(a_{k,ij}(x)) = (a_k^{ij}(x))^{-1}$.

In the next section we consider (2) and describe the behavior of the wave fronts in a rather general situation. The main machinery here is the Feynman–Kac formula and the large deviations principle. We show in Section 2 that a generalization of the results of [4] and [6], proved by analytic methods in [3] can be treated by light modification of the method of [4] and [6].

In Section 3 we apply the similar approach to (5) and prove the general result on wave front propagation for the coupled RDE. In some cases we can give explicit formulas for the position of the wave front. We consider such examples in Section 4. Section 5 is devoted to various generalizations of (2) and (5).

In conclusion, note that a class of RDE-systems similar in a sense to (1) is studied in [5], [7] and [2]. Using the approach of this paper one can consider the coupling of such systems.

2. Wave front propagation for generalized KPP-equation. If the coefficients and the nonlinear term of a RDE just as the initial function are slowly changing in space (meaning that they are functions of εx, $x \in R^r$, where ε is a small parameter), then after rescaling of the space and time we come to the following Cauchy problem:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\varepsilon}{2}\sum_{i,j=1}^{r} a^{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x^i \partial x^j} + \frac{1}{\varepsilon}f(x,u^\varepsilon), \tag{6}$$
$$u^\varepsilon(0,x) = g(x) \geq 0.$$

This is the same problem we have after time rescaling if an equation with small diffusion is considered ([6]). In this section, we study limit behavior of the solution of (6) when $\varepsilon \downarrow 0$ in the case $f(x,\cdot) \in \mathscr{F}_1$ for any $x \in R^r$. Let $c(x,u) = f(x,u)u^{-1}$. Since $f \in \mathscr{F}_1$, $c(x,1) = 0$, $c(x,u) > 0$ for $u < 1$ and $c(x,u) < 0$ for $u > 1$, $c(x) = c(x,0) = \max_{u\geq 0} c(x,u)$. We assume that the function $c(x,u)$ is continuous in x, u, and Lipschitz continuous in u. Assumptions on the coefficients $a^{ij}(x)$ and the initial function g are formulated in Section 1.

Let X_t^ε be the Markov process, corresponding to the operator

$$\varepsilon L = \frac{\varepsilon}{2}\sum_{i,j=1}^{r} a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j}$$

in the space R^r (see, for example, [7]). Using the Feynman–Kac formula we can write down the following equation for $u^\varepsilon(t,x)$:

$$u^\varepsilon(t,x) = E_x g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_s^\varepsilon, u^\varepsilon(t-s, X_s^\varepsilon))\,ds\right\}. \tag{7}$$

From (7) one can see ([6]) that

$$0 \leq u^\varepsilon(t,x) \leq 1 \vee \sup_{x\in R^r} g(x), \qquad \limsup_{\varepsilon\downarrow 0} u^\varepsilon(t,x) \leq 1 \quad \text{for } t > 0.$$

We will use also the following simple properties of (6).

1. If $u_1(t,x)$ and $u_2(t,x)$ are the solutions of (6) for $g = g_1(x)$ and $g = g_2(x)$, and $g_1(x) \geq g_2(x)$, $x \in R^r$, then $u_1(t,x) \geq u_2(t,x)$.
2. If $u_1(t,x)$ and $u_2(t,x)$ are the solutions of (6) for $f = f_1(x,u)$ and for $f = f_2(x,u)$ with the same initial function g, and $f_1(x,u) \geq f_2(x,u)$ for $x \in R^r$, $0 \leq u \leq \sup_{x\in R^r} g(x)$, then $u_1(t,x) \geq u_2(t,x)$ for all $t \geq 0$, $x \in R^r$.

The properties 1 and 2 are implication of the maximum principle for linear parabolic equations.

Taking into account that $c(x, 0) = c(x) = \max_{u \geq 0} c(x, u)$, we derive from (7)

$$u^{\varepsilon}(t, x) \leq E_x g(X_t^{\varepsilon}) \exp\left\{\frac{1}{\varepsilon} \int_0^t c(X_s^{\varepsilon})\, ds\right\}. \tag{8}$$

The asymptotics of the right side of (8) are defined by the large deviations of the process X_t^{ε} from zero. Recall that the action functional in the space C_{ot} of continuous functions φ: $[0, t] \to R^r$ for the family $\{X_t^{\varepsilon}, \varepsilon \downarrow 0\}$ has the form $\varepsilon^{-1} S_{ot}(\varphi)$, where

$$S_{ot}(\varphi) = \begin{cases} \frac{1}{2}\int_0^t \sum_{i,j=1}^{r} a_{ij}(\varphi_s)\dot{\varphi}_s^i \dot{\varphi}_s^j\, ds, & \varphi \in C_{ot},\ \varphi \text{ is absolutely continuous,} \\ +\infty, & \text{for the rest of } C_{ot}. \end{cases}$$

Here $(a_{ij}(x)) = (a^{ij}(x))^{-1}$. Taking into account the continuity of the functional $\int_0^T c(\varphi_s)\, ds$, we have the following Laplace-type asymptotic formula (see [11] and [8]):

$$\lim \varepsilon \ln E_x g(X_t^{\varepsilon}) \exp\left\{\frac{1}{\varepsilon} \int_0^t c(X_s^{\varepsilon})\, ds\right\} = V(t, x), \tag{9}$$

where $V(t, x) = \sup\{R_{ot}(\varphi) \colon \varphi \in C_{ot},\ \varphi_0 = x,\ \varphi_t \in G_0\}$,

$$R_{ot}(\varphi) = \int_0^t C(\varphi_s)\, ds - S_{ot}(\varphi), \qquad G_0 = \operatorname{supp} g.$$

One can see from (9) that $\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(t, x) = 0$ if $V(t, x) < 0$.

We say that condition (N) is fulfilled, if for every point (t, x) such that $V(t, x) \leq 0$,

$$V(t, x) = \sup\{R_{ot}(\varphi) \colon \varphi \in C_{ot},\ \varphi_0 = x,\ \varphi_t \in G_0,\ V(t - s, \varphi_s) < 0 \\ \text{for } 0 < s < T\}.$$

If condition (N) is fulfilled, one can prove ([4], [6], [7]) that $\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(t, x) = 1$ at the points (t, x), where $V(t, x) > 0$. Thus the set $\{x \in R^r \colon V(t, x) = 0\}$ can be considered as the wave front position at the time t. Some results about behavior of the $\lim_{\varepsilon \downarrow 0} u^{\varepsilon}(t, x)$ when condition (N) is not fulfilled are also available in [6] and [7].

Evans and Sauganidis in [3] gave an analytical proof of some results of [6]. Moreover, they suggested a game-theoretical approach to the problem which, together with the small viscosity solution method, gives the description of the wave front without condition (N). In this section, we explain how the general case can be treated by a slight modification of the methods of [4] and [6]. We use the same approach in the following section for studying the systems of RDE's.

Let us consider the heat process (t_s, X_s^ε). The first component is the deterministic motion with the speed -1: $t_s = t_0 - s$. The second component X_s^ε is the Markov process in R^r, corresponding to the operator εL. The phase space of the heat process is $\mathscr{H} = (-\infty, \infty) \times R^r$. Let F be a closed subset of $\mathscr{H}$. Define the functional $\tau = \tau_F(t, \varphi)$ on $(-\infty, \infty) \times C_{0,\infty}$ with values in $[0, \infty]$ by the formula

$$\tau = \tau_F(t, \varphi) = \inf\{s: (t - s, \varphi_s) \in F\}.$$

It is clear that $\tau_F(t, X^\varepsilon)$ is the first time when the heat process touches F; τ_F is a Markov time with respect to the family of σ-fields $\{\mathscr{F}_s, s \geq 0\}$; $\mathscr{F}_s$ is the minimal σ-field in the probability space, such that X_s^ε is $\mathscr{F}_s$-measurable for any $s_1 \leq s$. The functionals τ_F we call Markov functionals. Denote θ the set of all Markov functionals.

Let us introduce the function $V^*(t, x)$, $t > 0$, $x \in R^r$,

$$V^*(t,x) = \inf_{\tau \in \theta} \sup\left\{\int_0^{\tau \wedge t}\left[c(\varphi_s) - \frac{1}{2}\sum_{i,j=1}^r a_{ij}(\varphi_s)\dot{\varphi}_s^i\dot{\varphi}_s^j\right] ds: \varphi \in C_{ot},\right.$$
$$\left.\varphi_0 = x, \epsilon_t \in G_0\right\}.$$

It is clear that $V^*(t, x) \leq (0 \wedge V(t, x))$.

Since $\tau = \tau(t, X^\varepsilon)$ and $\tau \wedge t$ are Markov times, using the strong Markov property of the process X_t^ε, we derive from (7) that the following equation is fulfilled for the function $u^\varepsilon(t, x)$:

$$(10) \quad \begin{aligned} &u^\varepsilon(t,x) \\ &= E_x u^\varepsilon(t - (\tau \wedge t), X_{\tau\wedge t}^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^{\tau\wedge t} c(X_s^\varepsilon, u^\varepsilon(t - s, X_s^\varepsilon))\, ds\right\}. \end{aligned}$$

Of course, (10) is true for any Markov time τ with respect to the family of σ-fields $\mathscr{F}_s$, not only for the defined above functionals of the heat process. Consideration of (10) instead of (7) is actually the main modification which allows us to describe the motion of the wave fronts in the general situation without condition (N).

LEMMA 1. *If* $V^*(t, x) < 0$, *then* $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^\varepsilon(t, x) < 0$ *and*

$$\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = 0$$

uniformly in (t, x) *from any compact set* $F \subset \{(s, y), s > 0, V^*(s, y) < 0\}$.

PROOF. Since $V^*(t, x) < 0$, there exists $\tau^* \in \theta$ such that

$$\sup\left\{\int_0^{\tau^*\wedge t}\left[c(\varphi_s) - \frac{1}{2}\sum_{i,j=1}^r a_{ij}(\varphi_s)\dot{\varphi}_s^i\dot{\varphi}_s^j\right] ds: \varphi \in C_{ot}, \varphi_0 = x, \varphi_t \in G_0\right\}$$
$$= -\beta < 0.$$

Taking into account that $c(x,u) \le c(x) = c(x,0)$, for every integer $n > 0$, we have from (10)

$$u^\varepsilon(t,x) \le \sum_{k=1}^{n} E_x\chi_k u^\varepsilon(t-\tau^*\wedge t, X^\varepsilon_{t\wedge\tau^*})\exp\left\{\frac{1}{\varepsilon}\int_0^{\tau^*\wedge t} c(X_s^\varepsilon)\,ds\right\} \tag{11}$$
$$+ E_x\chi_{\tau\ge t} g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_s^\varepsilon)\,ds\right\},$$

where χ_k is the indicator of the set $\{(t(k-1)/n) \le \tau^* < kt/n\}$, $k = 1,\dots,n$, and $\chi_{\tau\ge t}$ is the indicator of the set $\{\tau^* \ge t\}$. Using the Laplace asymptotic formula for functional integrals, we get

$$E_x\chi_k u^\varepsilon(t-(\tau^*\wedge t), X^\varepsilon_{t\wedge\tau^*})\exp\left\{\frac{1}{\varepsilon}\int_0^{\tau^*\wedge t} c(X_s^\varepsilon)\,ds\right\}$$
$$\le \left[1+\sup_x g(x)\right] E_x\chi_k \exp\left\{\frac{1}{\varepsilon}\int_0^{tk/n} c(X_s^\varepsilon)\,ds\right\} \tag{12}$$
$$\approx \frac{1}{\varepsilon}\sup\left\{\int_0^{tk/n} c(\varphi_s^\varepsilon)\,ds - S_{o,tk/n}(\varphi):\varphi_0 = x,\right.$$
$$\left.\frac{k-1}{n}t \le \tau^*(\varphi) \le \frac{kt}{n}\right\},\qquad \varepsilon\downarrow 0.$$

Here the sign $\approx$ means logarithmic equivalence for $\varepsilon\downarrow 0$. Note that

$$\sup\left\{\int_0^{tk/n} c(\varphi_s^\varepsilon)\,ds - S_{o,tk/n}(\varphi):\varphi_0 = x,\ \frac{t(k-1)}{n} \le \tau^*(t,\varphi) \le \frac{kt}{n}\right\}$$
$$\le \sup\left\{\int_0^{\tau^*\wedge t}\left[c(\varphi_s) - \frac{1}{2}\sum_{i,j=1}^{r} a_{ij}(\varphi_s)\dot\varphi_s^i\dot\varphi_s^j\right]ds:\right. \tag{13}$$
$$\left.\frac{(k-1)t}{n} \le \tau^*(t,\varphi) \le \frac{kt}{n}\right\}$$
$$+\frac{t}{n}\sup_{x\in R^r} c(x) \le \beta + \frac{t}{n}\sup_{x\in R^r} c(x),$$

for $k = 1,\dots,n$. Now choosing $n > (1/\beta)2t\sup_{x\in R^r} c(x)$, we derive from (12) and (13) that

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\ln E_x\chi_k u^\varepsilon(t-(\tau^*\wedge t), X^\varepsilon_{t\wedge\tau^*})\exp\left\{\frac{1}{\varepsilon}\int_Q^{\tau^*\wedge t} c(X_s^\varepsilon)\,ds\right\} \le -\frac{\beta}{2}. \tag{14}$$

A similar bound holds for the last term in (11):

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\ln E_x\chi_{\tau^*\ge t} g(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c(X_s^\varepsilon)\,ds\right\} \le -\frac{\beta}{2}. \tag{15}$$

From (11), (14) and (15) we have the first statement of Lemma 1. The second statement follows from the first one and uniformity of the convergence in (14) and (15). □

LEMMA 2. *Suppose that* $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^\varepsilon(t_0, x_0) = 0$, $t_0 > 0$. *Then a constant* A *exists such that* $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = 1$ *uniformly in* (t,x) *from any compact subset of the cone* $K^A_{t_0,x_0} = \{(s,y)\colon s > t_0, |x - x_0| < A(s - t_0)\}$.

PROOF. Using the a priori bound for the Hölder norm of a bounded solution of a uniformly parabolic equation with bounded coefficients (see [9]), we can derive from the conditions of Lemma 2 that for any $\delta > 0$, there exist $\varepsilon_0, \delta_1 > 0$ such that

$$u^\varepsilon(t_0, x) > e^{-\delta/\varepsilon} \quad \text{for} \quad |x - x_0| \le e^{-\delta_1/\varepsilon}, \qquad 0 < \varepsilon < \varepsilon_0.$$

Now using the properties 1 and 2 of (6), we conclude that $u^\varepsilon(t,x) \ge \tilde{u}^{\varepsilon,\delta}(t - t_0, x)$, where $\tilde{u}^{\varepsilon,\delta}(t,x)$ is the solution of (6) with the initial function

$$g = g^{\varepsilon,\delta}(x) = \begin{cases} e^{-\delta/\varepsilon}, & \text{for } |x - x_0| \le e^{-\delta_1/\varepsilon}, \\ 0, & \text{for } |x - x_0| > e^{-\delta_1/\varepsilon}, \end{cases} \tag{16}$$

and $c(x,u)$ replaced by $\tilde{c}(u) = \inf_{x\in R^r} c(x,u)$.

In the case $c = \tilde{c}(u)$ independent of x, we can use Theorem 6.2.2 from [7]. We should take into account that our initial function depends on ε. It does not influence the proof of the upper bound. So we can conclude that $\lim_{\varepsilon\downarrow 0} \tilde{u}^{\varepsilon,\delta}(t,x) = 0$ for $t > 0$ and $\rho(x, x_0) > t\sqrt{2\tilde{c}(0)}$, where ρ is the Riemannian metric corresponding to the form $ds^2 = \sum_{i,j=1}^r a_{ij}(x)\,dx^i\,dx^j$, $(a_{ij}(x)) = (a^{ij}(x))^{-1}$. The convergence of $\tilde{u}^{\varepsilon,\delta}(t,x)$ to zero is uniform in any compact subset of the set $\mathscr{E} = \{(s,y)\colon s > 0, \rho(x,y) > s\sqrt{2\tilde{a}(0)}\}$.

To prove that $\lim_{\varepsilon\downarrow 0} \tilde{u}^\varepsilon(t,x) = 1$ for (t,x) such that $t > 0$, $\rho(x, x_0) < t\sqrt{2\tilde{c}(0)}$, we use the following bound for the transition density $p_\varepsilon(t,x,y)$ of the process X_t^ε (see [9]): for any $\delta_1, t > 0$ there exist $\varepsilon_0, \delta_0 > 0$ such that

$$p_\varepsilon(t,x,y) > e^{-\delta_1/\varepsilon} \quad \text{for } |x - y| < \delta_0, 0 < \varepsilon < \varepsilon_0. \tag{17}$$

From (16) and (17), we derive that for any $\delta_2 > 0$ one can find $s_1 \in (0, \delta_2)$ and $\delta, \varepsilon_0, \delta_3 > 0$ such that the following bound holds:

$$\tilde{u}^{\varepsilon,\delta}(s,y) > e^{-\delta_2/\varepsilon} \quad \text{for } |y - x_0| < \delta_3, 0 < \varepsilon < \varepsilon_0. \tag{18}$$

Now we can prove that $\lim_{\varepsilon\downarrow 0} \varepsilon \ln \tilde{u}^{\varepsilon,\delta}(t,x) = 0$ for points (t,x) with $t > 0$, $\rho(x, x_0) < t\sqrt{2\tilde{c}(0)}$, as was done in Theorem 6.2.2 of [7], and then we can check that $\lim_{\varepsilon\downarrow 0} \tilde{u}^{\varepsilon,\delta}(t,x) = 1$ uniformly on any compact subset of the set $\{(t,x), \ \rho(x,x_0) < t\sqrt{2\tilde{c}(0)}\}$. Since $\{(t,x)\colon \ \rho(x,x_0) < t\sqrt{2\tilde{c}(0)}\} \supset \{(t,x)\colon |x - x_0| < At\}$ for some $A > 0$, and $u^\varepsilon(t,x) \le \tilde{u}^{\varepsilon,\delta}(t - t_0, x)$, we derive from here the statement of Lemma 2. □

LEMMA 3. *Assume that* $\lim_{\varepsilon'\downarrow 0} u^{\varepsilon'}(t_0, x_0) = 0$ *for some sequence* $\varepsilon' \downarrow 0$. *Then there exists* $A > 0$ *such that* $\limsup_{\varepsilon'\downarrow 0} \varepsilon' \ln u^{\varepsilon'}(t,x) < 0$ *for any point*

$$(t,x) \in D^A_{t_0,x_0} = \{(s,y)\colon 0 < s < t_0, |x_0 - y| < A(t_0 - s)\}.$$

Let $\mathscr{E}^{(\varepsilon')} = \{(t,x): \lim_{\varepsilon' \downarrow 0} u^{\varepsilon'}(t,x) = 0,\ t > 0\}$. *For every compact F belonging to the interior* $(\mathscr{E}^{(\varepsilon')})$ *of* $\mathscr{E}^{(\varepsilon')}$, $\lim_{\varepsilon' \downarrow 0} u^{\varepsilon'}(t,x) = 0$ *uniformly in* $(t,x) \in F$.

PROOF. The first statement follows immediately from Lemma 2. To prove the second statement note that compact F can be covered by a finite number of cones $D^{A/2}_{t_k, x_k}$ with vertices $(t_k, x_k) \in (\mathscr{E}^{(\varepsilon')}) \setminus F$. The uniformity follows from the uniformity of the bound in Lemma 2. □

REMARK. It follows from Lemma 3 that the set $\mathscr{E}^{(\varepsilon')}$ belongs to the closure of its interior $(\mathscr{E}^{(\varepsilon')})$. If $(t,x) \in \mathscr{E}^{(\varepsilon')}$, then $(t-h, x) \in (\mathscr{E}^{(\varepsilon')})$ for small $h > 0$.

LEMMA 4. *Let F be a compact subset of the interior* (M) *of the set* $M = \{(t,x): t > 0,\ x \in R^r,\ V^*(t,x) = 0\}$. *Then* $\lim_{\varepsilon \downarrow 0} \varepsilon \ln u^{\varepsilon}(t,x) = 0$ *uniformly in* $(t,x) \in F$.

PROOF. Suppose that for a point $(t,x) \in (M)$, there exists a sequence $\varepsilon' \downarrow 0$ such that $\lim_{\varepsilon' \downarrow 0} \varepsilon' \ln u^{\varepsilon'}(t,x) = -\beta < 0$. Then $\lim_{\varepsilon' \downarrow 0} u^{\varepsilon'}(t,x) = 0$ and $(t,x) \in \mathscr{E}^{(\varepsilon')}$, where $\mathscr{E}^{(\varepsilon')}$ was introduced in Lemma 3. Without loss of generality we can assume that $(t,x) \in (\mathscr{E}^{(\varepsilon')})$. If this is not true, one can take a point $(t-h, x)$ with small enough $h > 0$. This new point belongs to $(\mathscr{E}^{(\varepsilon')})$ according to the remark above, and belongs to (M) since (M) is open.

Define the Markov functional, corresponding to the complement of the set $(\mathscr{E}^{(\varepsilon')})$,

$$\tau = \tau(t, \varphi) = \min\{s: (t-s, \varphi_s) \notin (\mathscr{E}^{(\varepsilon')})\}.$$

Since $(t,x) \in M$,

$$\sup\left\{\int_0^{\tau \wedge t}\left[c(\varphi_s) - \tfrac{1}{2}\sum_{i,j=1}^{r} a_{ij}(\varphi_s)\dot{\varphi}_s^i \dot{\varphi}_s^j\right] ds : \varphi_0 = x,\ \varphi_t \in G_0\right\} \ge 0.$$

Therefore for any $\delta > 0$, there exist φ_s^δ, $s \in [0,t]$, $\varphi_0^\delta = x$, $\varphi_t^\delta \in G_0$ such that

$$R_{0, \tau \wedge t}(\varphi^\delta) = \int_0^{\tau \wedge t}\left[c(\varphi_s^\delta) - \frac{1}{2}\sum_{i,j=1}^{r} a_{ij}(\varphi_s^\delta)\dot{\varphi}_s^{\delta,i}\dot{\varphi}_s^{\delta,j}\right] ds \ge -\frac{\delta}{4},$$

$(t-s, \varphi_s^s) \in \mathscr{E}^{(\varepsilon')}$ for $s \in [0, \tau(\varphi^\delta))$, and $(t - (t \wedge \tau), \varphi^\delta_{\tau(\varphi^\delta) \wedge t}) \in \partial \mathscr{E}^{(\varepsilon')}$.

Now we define a reconstruction of the φ^δ. For any small $\lambda_1, \lambda_2 > 0$, we introduce the function $\varphi_s^{\delta, \lambda_1, \lambda_2}$,

$$\varphi_s^{\delta,\lambda_1,\lambda_2} = \begin{cases} \varphi_0^\delta, & \text{for } s \in [0, \lambda_1], \\ \varphi^\delta_{((s-\lambda_1)(T-\lambda_1))/(T-2\lambda_1)}, & \text{for } s \in [\lambda_1, T-\lambda_1], \\ \varphi^\delta_{T-\lambda_1+(s-T+\lambda_1)/(1-\lambda_2)}, & \text{for } s \in [T-\lambda_1, T-\lambda_1\lambda_2]. \end{cases}$$

Here $T = \tau(t, \varphi^\delta) \wedge t$; the function $\varphi_s^{\delta,\lambda_1,\lambda_2}$ is defined for $s \in [0, T - \lambda_1\lambda_2]$.

The second reconstruction is defined by formula (h is again a small positive number, $h < \lambda_1\lambda_2$)

$$\bar{\varphi}_S = \bar{\varphi}_s^{\delta_1, \lambda_1, \lambda_2, h} = \begin{cases} \varphi_0^{\delta, \lambda_1, \lambda_2}, & \text{for } s \in [0, h], \\ \varphi_{s-h}^{\delta, \lambda_1, \lambda_2}, & \text{for } s \in [h, T - \lambda_1\lambda_2 + h]. \end{cases}$$

Denote $\bar{T} = T - \lambda_1\lambda_2 + h$, $z = \bar{\varphi}_{\bar{T}} = \varphi_T^\delta$. The positive numbers λ_1, λ_2, h one can choose so small that

$$(19)\qquad \begin{aligned} &\int_0^{\bar{T}} \left[c(\bar{\varphi}_s) - \frac{1}{2} \sum_{i,j=1}^{r} a_{ij}(\bar{\varphi}_s)\dot{\bar{\varphi}}_s^i \dot{\bar{\varphi}}_s^j \right] ds \geq -\frac{\delta}{2}, \\ &\int_{\bar{T}-h}^{\bar{T}} \left[\sum_{i,j=1}^{r} a_{ij}(\bar{\varphi}_s)\dot{\bar{\varphi}}_s^i \dot{\bar{\varphi}}_s^j \right] ds \leq \frac{\delta}{8}. \end{aligned}$$

Note that the set $\{(t - s, \bar{\varphi}_s)\colon s \in [h, \bar{T} - h]\}$ is a compact subset of $(\mathscr{E}^{(\varepsilon')})$. Therefore, as it follows from Lemma 3, $u^{\varepsilon'}(t - s, \bar{\varphi}_s) \to 0$ when $\varepsilon' \downarrow 0$ uniformly in $s \in [h, \bar{T} - h]$.

Since $(t - T, z) \notin \mathscr{E}^{(\varepsilon')}$ and $\bar{T} < T$, we have from Lemma 2

$$\lim_{\varepsilon'' \downarrow 0} u^{\varepsilon''}(t - \bar{T}, z) = 1,$$

at least for a subsequence $\{\varepsilon''\}$ of the sequence $\{\varepsilon'\}$. Moreover, $\lim_{\varepsilon'' \downarrow 0} u^{\varepsilon''}(s, y) = 1$ uniformly in a neighborhood of the point $(t - \bar{T}, z)$.

Let α_1 be the Euclidian distance between the set $\{(s, y),\ s \in [h, t - h],\ y = \bar{\varphi}_s\}$ and the compliment of the set $(\mathscr{E}^{(\varepsilon')})$; α_2 is the size of the neighborhood of the point $(t - \bar{T}, z)$, where $\lim_{\varepsilon'' \downarrow 0} u^{\varepsilon''}(s, y) = 1$. Denote $t_y = \min\{s\colon |\bar{\varphi}_s - \bar{\varphi}_{\bar{T}}| < y\}$, $y > 0$, and let $\alpha_3 > 0$ be so small that

$$\max_x c(x)\left(\bar{T} - t_{\alpha_3}\right) + \int_{t_{\alpha_3}}^{\bar{T}} \sum_{i,j=1}^{r} a_{ij}(\bar{\varphi}_s)\dot{\bar{\varphi}}_s^i \dot{\bar{\varphi}}_s^j \, ds \leq \frac{\delta}{8};$$

$\alpha_4 > 0$ is such that $|c(x) - c(y)| \leq \delta/8t$ for $|x - y| < \alpha_4$. Put

$$\Gamma = \left\{(s, y)\colon |s - \bar{T}| + |y - z| \leq \tfrac{1}{2}(\alpha_2 \wedge \alpha_3)\right\}, \qquad \alpha = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4,$$

$$\zeta(t, \varphi) = \min\{s\colon (t - s, \varphi_s) \in \Gamma\},\ \zeta = \zeta(t, X_\cdot^\varepsilon).$$

Using (10) we have

$$(20)\qquad \begin{aligned} u^\varepsilon(t, x) &= E_x u\left(t - \zeta \wedge t,\ X_{\zeta \wedge t}^\varepsilon\right) \exp\left\{\frac{1}{\varepsilon} \int_0^{\zeta \wedge t} c\left(X_s^\varepsilon,\ u^\varepsilon(t - s,\ X_s^\varepsilon)\right) ds\right\} \\ &\geq E_x u^\varepsilon\left(t - \zeta \wedge t,\ X_{\zeta \wedge t}^\varepsilon\right)\chi_\alpha \exp\left\{\frac{1}{\varepsilon} \int_0^{t \wedge \zeta} c\left(X_s^\varepsilon,\ u^\varepsilon(t - s,\ X_s^\varepsilon)\right) ds\right\} \\ &= I_1, \end{aligned}$$

where χ_α is the indicator of the set $\{\sup_{0 \leq s \leq \bar{T}} |X_s^\varepsilon - \bar{\varphi}_s| < \alpha\} = B_\alpha$.

For small enough $\varepsilon > 0$, $u^\varepsilon(s,y) > \frac{1}{2}$ in the α-neighborhood of the point $(\overline{T},z)$ and

$$\int_0^{t\wedge\zeta} c(X_s^\varepsilon, u^\varepsilon(t-s, X_s^\varepsilon))\,ds \ge \int_0^{\overline{T}} c(X_s^\varepsilon)\,ds - \frac{\delta}{8} \quad \text{for } X_\cdot^\varepsilon \in B_\alpha.$$

Therefore, using the lower bound for $P_x\{B_\alpha\}$ given by the large deviation principle, we get that

$$(21)\qquad \begin{aligned} I_1 &\ge \frac{1}{2} E_x \chi_\alpha \exp\left\{\frac{1}{\varepsilon}\int_0^{\overline{T}} c(\overline{\varphi}_s)\,ds\right\}\exp\left\{-\frac{\delta}{4\varepsilon}\right\} \\ &\ge \exp\left\{\frac{1}{\varepsilon}\left[\int_0^{\overline{T}}\left[c(\overline{\varphi}_s) - \frac{1}{2}\sum_{i,j=1}^r a_{ij}(\overline{\varphi}_s)\dot{\overline{\varphi}}_s^i\dot{\overline{\varphi}}_s^j\right]ds - \frac{\delta}{2}\right]\right\}, \end{aligned}$$

for $\varepsilon > 0$ small enough. From (19), (20) and (21) we have that $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^\varepsilon(t,x) \ge -\delta$. Since δ is arbitrary positive number, taking into account that $\limsup \varepsilon \ln u^\varepsilon(t,x) \le 0$, we conclude that $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^\varepsilon(t,x) = 0$.

Uniformity of the convergence $u^\varepsilon(t,x)$ to 0 for points $(t,x) \in F \subset (M)$ follows from the fact that set F can be covered by finite number of the cones $D_{tk,xk}^{A/2}$, introduced in Lemma 3, with the vertices outside F. □

THEOREM 1. *Let $u^\varepsilon(t,'x)$ be the solution of* (6). *Then* $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = 0$ *uniformly for (t,x) belonging to any compact set $F_1 \subset [(s,y)\colon V^*(s,y) < 0\}$. For any compact subset F_2 of the interior of the set $\{(s,y),\ s > 0,\ V^*(s,y) = 0\}$, $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = 1$ uniformly in $(t,x) \in F_2$.*

PROOF. The first statement follows from Lemma 1 while the second statement follows from Lemmas 3 and 4. □

REMARK. In general $V^*(t,x) \le (V(t,x) \wedge 0)$ and $\{(t,x),\ V(t,x) < 0) \subseteq \{(t,x),\ V^*(t,x) < 0\}$. The inclusion may be strict; see the corresponding example in the end of the next section. If the condition (N) is fulfilled, the inclusion becomes an equality. One can give a bound from below for the interior of the set $\{(t,x)\colon V^*(t,x) = 0\}$:

$$(22)\qquad ((\{(t,x)\colon V^*(t,x) = 0\})) \supseteq \{(t,x)\colon \hat{\rho}(x, G_0) < t\},$$

where $\hat{\rho}$ is the Riemannian metric corresponding to the form

$$ds^2 = \frac{1}{2c(x)}\sum_{i,j=1}^r a_{ij}(x)\,dx^i\,dx^j.$$

The proof of (22) follows from Theorem 1 and Lemma 6.2.4 from [7]. In the case $c(x) = c = \text{const}$, (22) becomes an equality. Under condition (N), the interior of the set $\{(t,x)\colon t > 0,\ V^*(t,x) = 0\}$ is equal to the set $\{(t,x),\ V(t,x) > 0\}$.

3. Weakly coupled RDE. In this section we consider the system

$$\frac{\partial u_k^\varepsilon(t,x)}{\partial t} = \varepsilon L_k u_k^\varepsilon(t,x) + \frac{1}{\varepsilon} f_k(x, u_k^\varepsilon(t,x)) + \sum_{j=1}^{r} d_{kj}\big(u_j^\varepsilon(t,x) - u_k^\varepsilon(t,x)\big), \tag{23}$$

$$u_k^\varepsilon(0,x) = g_k(x), \qquad k = 1,\dots,n,\ t > 0,\ x \in R^r.$$

We assume that

$$L_k = \frac{1}{2}\sum_{i,j=1}^{r} a_k^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j}, \qquad k = 1,\dots,n,$$

are uniformly elliptic operators with bounded smooth coefficients (say, $a_k^{ij} \in c^3$), $d_{kj} > 0$. The nonlinear terms $f_k(x,\cdot)$ are elements of $\mathscr{F}_1$ for any $x \in R^r$ and $k = 1,\dots,n$. Assumptions on initial functions g_k are the same as in the case of a single equation. We denote by G_0 the support of the function $\sum_{k=1}^n g_k(x)$. Since $g_k(x) \geq 0$, G_0 is equal to the union of the supports of g_k.

A Markov process (X_t^ε, ν_t) in the phase space $R^r \times \{1,\dots,n\}$ can be connected with the system (23). The component ν_t of this process is the right-continuous Markov process with n states such that $P\{\nu_{t+\Delta} = j|_{\nu_t} = i\} = d_{ij}\Delta + O(\Delta)$, $\Delta \downarrow O$, $i \neq j$. The first component X_t^ε is defined by the stochastic differential equation

$$dX_t^\varepsilon = \sqrt{\varepsilon}\,\sigma_{\nu_t}(X_t^\varepsilon)\,dW_t, \qquad \sigma_k(x)\sigma_k^*(x) = \big(a_k^{ij}(x)\big),$$

where W_t is an r-dimensional Wiener process. Using the Itô formula, it is not difficult to check that the generator A of the process (X_t^ε, ν_t) on functions $f(x,k)$, $x \in R^r$, $k = 1,\dots,n$, having uniformly continuous bounded second derivatives in x, has the form

$$Af(x,k) = \varepsilon L_k f(x,k) + \sum_{j=1}^{r} d_{kj}\big(f(x,j) - f(x,k)\big).$$

Taking this into account one can write down the probabilistic representation for the solution of (23) in the linear case when $f_k = \tilde{c}_k(t,x)u_k$, $k = 1,2,\dots,n$. In particular, the generalized Feynman–Kac formula for the solution of (23) in this case has the form

$$u_k^\varepsilon(t,x) = E_{x,k}\, g_{\nu_t}(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t \tilde{c}_{\nu_s}(t-s, X_s^\varepsilon)\,ds\right\}. \tag{24}$$

Using (24) we get the following integral equation for the solution of (23) in the nonlinear case $f_k = c_k(x, u_k)u_k$:

$$u_k^\varepsilon(t,x) = E_{x,k}\, g_{\nu_t}(X_t^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t c_{\nu_s}\big(X_s^\varepsilon, u_{\nu_s}^\varepsilon(t-s, X_s^\varepsilon)\big)\,ds\right\}, \tag{25}$$

$$x \in R^r,\ t \geq 0,\ k = 1,\dots,n.$$

Using the strong Markov property of the process (X_t^ε, ν_t), we can write down the equation

$$u_k^\varepsilon(t,x) = E_{x,k} u_{\nu_{\tau\wedge t}}^\varepsilon(t-(\tau\wedge t), X_{\tau\wedge t}^\varepsilon) \times \exp\left\{\frac{1}{\varepsilon}\int_0^{\tau\wedge t} c_{\nu_s}\big(X_s^\varepsilon, u_{\nu_s}^\varepsilon(t-s, X_s^\varepsilon)\big)\,ds\right\}, \tag{26}$$

where τ is arbitrary Markov time with respect to the filtration $\{\mathscr{F}_t, t \geq 0\}$, $\mathscr{F}_t = \sigma(X_s^\varepsilon, \nu_s^\varepsilon, 0 \leq s \leq t)$.

LEMMA 5. *The following properties of the solutions of* (23) *hold*:

(i) $0 \leq u_k^\varepsilon(t,x) \leq 1 \vee \sup_{x,k} g_k(x)$.

(ii) $\limsup_{\varepsilon\downarrow 0} u_k^\varepsilon(t,x) \leq 1$ *for* $t > 0$, $X \in R^r$, $k = 1,2,\dots,n$.

(iii) *Let* $\{u_k'(t,'x)\}$ *be the solution of* (23) *for an initial function* $g'(x) = (g_1'(x),\dots,g_n'(x))$, *and let* $\{u_k''(t,x)\}$ *be the solution with an initial function* $g''(x) = (g_1''(x),\dots,g_n''(x))$. *Suppose that* $g_k'(x) \geq g_k''(x)$ *for* $x \in R^r$, $k = 1,\dots,n$. *Then* $u_k'(t,x) \geq u_k''(t,x)$ *for all* $t > 0$, $x \in R^r$, $k = 1,\dots,n$.

(iv) *Let* $\{u_k'(t,x)\}$ *and* $\{u_k''(t,x)]$ *be the solutions of* (23) *with* $f_k = f_k'(x,u)$ *and* $f_k = f_k''(x,u)$, $k = 1,\dots,n$. *Suppose that* $f_k'(x,u) \geq f_k''(x,u)$ *for all* $x \in R^r$, $k = 1,\dots,n$. *Then* $u_k'(t,x) \geq u_k''(t,x)$, *for* $x \in R^r$, $t > 0$, $k = 1,\dots,n$.

PROOF. (i) Suppose that the set $G = \{(s,y,i)\colon u_i^\varepsilon(s,y) > 1 \vee \sup_{x,l} g_l(x)\}$ contains a point (t,x,k) and put $\zeta_1 = \inf\{s\colon (t-s, X_s^\varepsilon, \nu_s) \notin G\}$. Since ζ_1 is a Markov time and $\zeta_1 < t$ with probability 1, we have from (26)

$$u_k^\varepsilon(t,x) = E_{x,k} u_{\nu_{\zeta_1}}^\varepsilon\big(t-\zeta_1, X_{\zeta_1}^\varepsilon\big)\exp\left\{\frac{1}{\varepsilon}\int_0^{\zeta_1} c_{\nu_s}\big(X_s^\varepsilon, u_{\nu_s}^\varepsilon(t-s, X_s^\varepsilon)\big)\,ds\right\}. \tag{27}$$

According to the definition of ζ_1,

$$u_{\nu_{\zeta_1}}^\varepsilon\big(t-\zeta_1, X_{\zeta_1}^\varepsilon\big) \leq 1 \vee \sup_{1,x} g_1(x). \tag{28}$$

Since

$$\inf_{0\leq s\leq\zeta_1} u_{\nu_s}^\varepsilon(t-s, X_s^\varepsilon) \geq 1$$

for trajectories (X_s^ε, ν_s) starting from (x,k), and $c(x,u) \leq 0$ for $u \geq 1$, the integrand in the exponent of (27) is nonpositive. Thus from (27) and (28), we have that $u_k^\varepsilon(t,x) \leq 1 \vee \sup_{1,x} g_1(x)$, and the set G must be empty.

(ii) Assume that for some $t > 0$, $x \in R^r$, $k \in \{1,\dots,n\}$, there is a sequence $\varepsilon' \downarrow 0$ such that $\lim_{\varepsilon'\downarrow 0} u_k^{\varepsilon}{}'(t,x) = 1 + 2\alpha > 1$. Denote

$$D = D_\varepsilon^\alpha = \{(s,y,i)\colon s < t, u_i^\varepsilon(t-s,y) > 1+\alpha\},$$

$$\zeta_2 = \inf\{s\colon s \leq t, u_{\nu_s}^\varepsilon(t-s, X_s^\varepsilon) = 1+\alpha\}.$$

Let K be a compact in R^r. Denote $Q = \{(i,x,u)\colon i \in \{1,\dots,n\}\}$, $x \in K$, $1 + \alpha \leq u \leq 1 + \sup_{1,x} g_1(x)$. Replacing ζ_1 by ζ_2 in (27) and taking into account that $\max_{(i,x,u)\in Q} c_1(x,u) < -\beta < 0$, we get

$$(29)\quad \begin{aligned} u_k^\varepsilon(t,x) &\le E_{x,k} u_{\nu_{\zeta_2}}(t-\zeta_2, X_{\zeta_2}^\varepsilon)\chi_{\zeta_2<t} \\ &\quad + E_{x,k} g_{\nu_t}(X_s^\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_0^t c_{\nu_s}(X_s^\varepsilon)\,ds\right)\chi_{\zeta_2=t} \\ &\le (1+\alpha)P_{x,k}\{\zeta_2<t\} + P_{x,k}\{\zeta_2=t\}, \end{aligned}$$

if ε is so small that $\sup_{y,l} g_l(y)e^{-\beta t/\varepsilon} < 1$. From (29), one can see that $u_k^\varepsilon(t,x) \le 1+\alpha$. This contradiction proves the second statement.

(iii) The difference $v_k(t,x) = u_k'(t,x) - u_k''(t,x)$ satisfies the system

$$\frac{\partial v_k}{\partial t} = \varepsilon L_k v_k + \frac{1}{\varepsilon}\hat{c}_k(x,u_k',u_k'')v_k + \sum d_{kj}(v_j - v_k),$$
$$v_k(0,x) = g_k'(x) - g_k''(x) = \delta_k(x) \ge 0, \qquad k=1,\dots,n,$$

where

$$\hat{c}_k(x,u_k',u_k'') = \frac{f_k(x,u_k') - f_k(u,u_k'')}{u_k' - u_k''}.$$

Then for $v_k(t,x)$, the following equality holds:

$$(30)\quad \begin{aligned} v_k(t,x) = E_{x,k}\delta_{\nu_t}(X_s^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^t \hat{c}_{\nu_s}\left(X_s^\varepsilon, u_{\nu_s}'(t-s,X_s^\varepsilon),\right.\right. \\ \left.\left. u_{\nu_s}''(t-s,X_s^\varepsilon)\right)ds\right\}. \end{aligned}$$

Since $\delta_k(x) \ge 0$, we derive from (30) that $v_k(t,x) \ge 0$.

(iv) The functions $W_k(t,x) = u_k'(t,x) - u_k''(t,x)$ satisfy the system

$$\frac{\partial W_k(t,x)}{\partial t} = \varepsilon L_k W_k + h(t,x) + \hat{\hat{c}}_k(t,x)W_k + \sum_{j=1}^n d_{kj}(W_j - W_k),$$
$$W_k(0,x) = 0, \qquad k=1,\dots,n,$$

where

$$h_k(t,x) = f_k'(x,u_k'(t,x)) - f_k''(x,u_k'(t,x)),$$
$$\hat{\hat{c}}(t,x) = \frac{f_k''(x,u_k'(t,x)) - f_k''(x,u_k''(t,x))}{u'(t,x) - u''(t,x)}.$$

One can write down the following equation for $W_k(t,x)$:

$$(31\text{a})\quad W_k(t,x) = E_x\int_0^t h_{\nu_s}(t-s,X_s^\varepsilon)\exp\left\{\frac{1}{\varepsilon}\int_0^s \hat{\hat{c}}_k(t-s_1,X_{s_1}^\varepsilon)\,ds_1\right\}ds.$$

Since $h_k(t,x) \ge 0$ for all t, x and k, we have from (31a) that $W_k(t,x) \ge 0$. □

Denote by H_{ot} the set of all right-continuous step-functions on $[0,t]$, $t<\infty$, with values from the set $\{1,\dots,n\}$ having finite number of jumps. For any

$\alpha \in H_{ot}$, define the functional

$$S^{\alpha}_{ot}(\varphi) = \frac{1}{2}\int_0^t \sum_{i,j=1}^{r} a^{\alpha_s}_{ij}(\varphi_s)\dot{\varphi}^i_s\dot{\varphi}^j_s\, ds$$

for absolutely continuous $\varphi \in C_{ot}$, and $S^{\alpha}_{ot}(\varphi) = +\infty$ for the rest of C_{ot}. It is known that $S^{\alpha}_{ot}(\varphi)$ is the action functional for the family of processes $X^{\alpha,\varepsilon}_t$, defined by the stochastic differential equation

$$dX^{\alpha,\varepsilon}_t = \sqrt{\varepsilon}\,\sigma_{\alpha_t}(X^{\alpha,\varepsilon}_t)\,dW_t, \qquad \sigma_k(x) = \left(a^{ij}_k(x)\right)^{1/2},$$

with W_t a Wiener process in R^r (see [8] and [11], where it is proved for continuous coefficients, but in the case of finite number of discontinuities the proof preserves). But we need uniform in $\alpha \in H_{ot}$ bounds of the probabilities of large deviations. Such bounds are given by Lemma 6.

LEMMA 6. *For any $\varphi \in C_{ot}$, $\varphi_0 = x$ and any $\gamma, \delta > 0$, one can find $\varepsilon_0 > 0$ such that for all $\alpha \in H_{ot}$,*

$$P_x\left\{\sup_{0\le s\le t}|X^{\alpha,\varepsilon}_s - \varphi_s| < \delta\right\} > \exp\left\{-\frac{1}{\varepsilon}\left[S^{\alpha}_{ot}(\varphi) + \gamma\right]\right\},$$

provided $\varepsilon \in (0, \varepsilon_0]$.

For any $s, \delta, \gamma > 0$, one can find $\varepsilon_0 > 0$ such that for all $\alpha \in H_{ot}$ and $\varepsilon \in (0, \varepsilon_0]$,

$$P_x\{\rho_{ot}(X^{\alpha,\varepsilon}_{\cdot}, \phi^{\alpha}_s) \ge \delta\} \le \exp\left\{-\frac{1}{\varepsilon}(S - \gamma)\right\},$$

where $\phi^{\alpha}_s = \{\varphi \in C_{ot},\ \varphi_0 = x,\ S^{\alpha}_{ot}(\varphi) \le S\}$, ρ_{ot} is the uniform metric in C_{ot}.

PROOF. The first statement one can get from the standard proof of the lower bound φ (see [11] and [8]).

To prove the upper bound usually the process $X^{\alpha,\varepsilon}_{\delta_1}(t)$ is considered:

$$dX^{\alpha,\varepsilon}_{\delta_1}(t) = \sqrt{\varepsilon}\,\sigma_{\alpha_t}\left(X^{\alpha,\varepsilon}_{\delta_1}(\pi_\delta(t))\right)dW_t, \qquad \pi_{\delta_1}(t) = [t/\delta_1]\delta_1, \qquad 0 < \delta_1 \ll 1$$

(see [11], Section 6). For any $\alpha \in H_{ot}$, the process $X^{\alpha,\varepsilon}_{\delta_1}(s)$ is a continuous transformation $\Gamma = \Gamma_{\alpha,\delta_1}$ of the process $\sqrt{\varepsilon}\,W_s$, $0 \le s \le t$. Thus the action functional for $X^{\alpha,\varepsilon}_{\delta_1}(t)$ can be calculated using the contraction principle. For small enough δ_1 one can have good bounds for $\rho_{o,t}(X^{\alpha,\varepsilon}_{\cdot}, X^{\alpha,\varepsilon}_{\delta_1})$. Combining the bounds for $X^{\alpha,\varepsilon}_{\delta_1}$ and for $\rho_{o,t}(X^{\alpha,\varepsilon}, X^{\alpha,\varepsilon}_{\delta_1})$, the upper bound included in the large deviation principle can be proved. But the transformation Γ_{α,δ_1} is not uniformly continuous with respect to the number of jumps of the function α_s, $s \in [0,t]$. To have uniform bounds, let us introduce the family of processes $\overline{X}^{\alpha,\varepsilon}_{\delta_1}(s)$, which are defined as follows. Denote $h_i = h_i[\pi_{\delta_1}(t), \pi_{\delta_1}(t) + \delta_1]$, the time which the function α_s spent in the state i during the time interval $[\pi_{\delta_1}(t), \pi_{\delta_1}(t) + \delta_1]$. Put $h_0 = 0$,

$$\tilde{\sigma}(t,x) = \sigma_k(x) \quad \text{for } t \in [\pi_{\delta_1}(t) + h_0 + \cdots + h_{k-1},\ \pi_{\delta_1}(t) + h_0 + \cdots + h_k),\ k = 1,\ldots,n.$$

The number of discontinuities of the function $\tilde{\sigma}(s,x)$ on time interval $[0,t]$ is not bigger than $t\delta^{-1}n$. The process $\overline{X}_{\delta_1}^{\alpha,\varepsilon}(s)$ is defined as the solution of the equation

$$d\overline{X}_{\delta_1}^{\alpha,\varepsilon}(s) = \sqrt{\varepsilon}\,\tilde{\sigma}\big(s, \overline{X}_{\delta_1}^{\alpha,\varepsilon}(\pi_{\delta_1}(s))\big)\,dW_s.$$

This equation defines a map $\sqrt{\varepsilon}\,W \to \overline{X}_{\delta_1}^{\alpha,\varepsilon}$ in C_{ot}, which is uniformly continuous in $\alpha \in H_{ot}$, and the contraction principle gives us bounds which are uniform with respect to $\alpha \in H_{ot}$. At the same time, since $\overline{X}_{\delta_1}^{\alpha,\varepsilon}(s) = X_{\delta_1}^{\alpha,\varepsilon}(s)$ for $s = k\delta_1$, k an integer, the distance $\rho_{o,t}(X_{\delta_1}^{\alpha,\varepsilon}, \overline{X}_{\delta_1}^{\alpha,\varepsilon})$ can be properly bounded for small enough δ_1. Together with the bound of $\rho_{o,t}(X^{\alpha,\varepsilon}, X_{\delta_1}^{\alpha,\varepsilon})$ from Section 6 of [11], it gives us a uniform upper bound for

$$P_x\{\rho_{o,t}(X^{\alpha,\varepsilon}, \phi^s) \geq \delta\}. \qquad \square$$

Define the functional $R_{ot}(\varphi,\alpha)$, $\varphi \in C_{ot}$, $\alpha \in H_{ot}$, by the formula

$$R_{ot}(\varphi,\alpha) = \int_0^t c_{\alpha_s}(\varphi_s)\,ds - S_{ot}^{\alpha}(\varphi).$$

For any fixed $\alpha \in H_{ot}$, the functional $R_{ot}(\varphi,\alpha)$ is semicontinuous from above in $\varphi \in C_{ot}$.

Introduce the function $V(t,x)$, $t \geq 0$, $x \in R^r$,

$$V(t,x) = \sup\{R_{ot}(\varphi,\alpha) : \varphi \in C_{ot},\ \varphi_0 = x,\ \varphi_t \in G_0,\ \alpha \in H_{ot}\},$$

where G_0 is the support of $\sum_{k=1}^n g_k(x)$ and $\{g_k(x)\}$ are the initial functions in (23). It is easy to check that the function $V(t,x)$ is continuous. Denote $\mathscr{E}_- = \{(t,x) : t > 0,\ x \in R^r,\ V(t,x) \leq 0\}$.

We say that condition (N) is fulfilled for (23), if for any $(t,x) \in \mathscr{E}_-$,

$$V(t,x) = \sup\{R_{ot}(\varphi,\alpha) : \varphi \in C_{ot},\ \varphi_0 = x,\ \varphi_t \in G_0,\ \alpha \in H_{ot},\ V(t-s,\varphi_s) < 0 \text{ for } 0 < s < t\}.$$

THEOREM 2. *The following statements hold for the solution* $\{u_k^{\varepsilon}(t,x)\}$ *of* (23):

(i) *Let* F_1 *be a compact subset of the set* $\{(t,x) : t > 0,\ x \in R^r,\ V(t,x) < 0\}$. *Then* $\lim_{\varepsilon\downarrow 0} u_k^{\varepsilon}(t,x) = 0$ *uniformly in* $(t,x) \in F_1$ *and* $k = 1,\dots,n$.

(ii) *Assume that condition* (N) *is fulfilled. Let* F_2 *be a compact subset of the set* $\{(t,x) : t > 0,\ x \in R^r,\ V(t,x) > 0\}$. *Then* $\lim_{\varepsilon\downarrow 0} u_k^{\varepsilon}(t,x) = 1$ *uniformly in* $(t,x) \in F_2$ *and* $k = 1,\dots,n$.

PROOF. The proof is similar to the proof of Theorem 1 from [6], and we merely point out the differences.

Since $c_k(x) = c_k(x,0) = \max_{0\leq u} c_k(x,u)$, we have from (25)

$$\begin{aligned} u_k^{\varepsilon}(t,x) &\leq E_{x,k}\, g_{\nu_t}(X_t^{\varepsilon})\exp\left\{\frac{1}{\varepsilon}\int_0^t c_{\nu_s}(X_s^{\varepsilon})\,ds\right\} \\ &= E_{x,k}E_{x,k}\left[g_{\nu_t}(X_t^{\varepsilon})\exp\left\{\frac{1}{\varepsilon}\int_0^t c_{\nu_s}(X_s^{\varepsilon})\,ds\right\}\middle|\,\nu_s,\ 0 \leq s \leq t\right]. \end{aligned} \tag{31b}$$

We can calculate the asymptotics of the conditional expectation in (31b) under the condition $\nu_s = \alpha_s$, using the large deviation principle for the family $\{X_s^{\alpha,\varepsilon}\}$,

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln E_x g_{\nu_s}(X_t^{\alpha,\varepsilon}) \exp\left\{ \frac{1}{\varepsilon} \int_0^t c_{\nu_s}(X_s^{\alpha,\varepsilon})\, ds \right\} \tag{32}$$
$$= \sup\{R_{ot}^{\alpha}(\varphi): \varphi \in C_{ot},\ \varphi_0 = x,\ \varphi_t \in G_{\alpha_t}\},$$

where G_{α_t} is the support of the function $g_{\alpha_t}(x)$. The convergence in (32) is uniform with respect to x from any compact set $F \subset R^r$, and due to Lemma 6, with respect to any $\alpha \in H_{ot}$. From (31b) and (32), the first statement of Theorem 2 follows.

The proof of the second statement uses the same arguments as the proof of the similar statement in the Theorem 1 from [6]: Using condition (N), the first statement and the lower bound from Lemma 6, one can prove that for every $\delta > 0$ and any compact set $F \subset \{(t,x): v(t,x) = 0\}$, there exist $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $(t,x) \in F$ and $k = 1, \dots, n$,

$$u_k^{\varepsilon}(t,x) \geq \exp\left\{ -\frac{\delta}{\varepsilon} \right\}. \tag{33}$$

Then the second statement of Theorem 2 can be proved with the help of (25) and the strong Markov property as it was done in the case of one equation ([4], [6] and [7]).

Now we consider an example where condition (N) is fulfilled and Theorem 2 can be used. This example will also be helpful in a general situation, when condition (N) does not hold.

Let $c_k(x) = c$ be independent of $x \in R^r$ and k. In this case,

$$V(t,x) = ct - \tfrac{1}{2} \inf\left\{ \int_0^t \sum_{i,j=1}^{r} a_{ij}^{\alpha_s}(\varphi_s) \dot{\varphi}_s^i \dot{\varphi}_s^j\, ds: \varphi_0 = x,\ \varphi_t \in G_0,\ \alpha \in H_{ot} \right\}. \tag{34}$$

Calculate

$$D(t,x,y) = \inf\left\{ \int_0^t \sum_{i,j=1}^{r} a_{ij}^{\alpha_s}(\varphi_s) \dot{\varphi}_s^i \dot{\varphi}_s^j\, ds: \varphi_0 = x,\ \varphi_t = y,\ \alpha \in H_{ot} \right\}$$
$$= \inf_{\substack{N \\ i_0, i_1, \dots, i_N \in \{1,\dots,n\}}} \ \inf_{\substack{z_1,\dots,z_{n-1} \in R^r \\ z_0 = x,\ z_N = y}} \ \inf_{\substack{0 < t_0, \dots, t_{N-1} \\ \Sigma_0^{N-1} t_k = t}} \tag{35}$$
$$\inf_{\substack{\varphi_0 = x,\ \varphi_{t_0} = z_1, \dots, \varphi_{t_0 + \cdots + t_{N-1}} = Z_n \\ \alpha \in H_{ot}:\ \alpha_s = \alpha_{i_k} \text{ for} \\ S \in [t_0 + \cdots + t_k,\ t_0 + \cdots + t_k + t_{k+1})}} \int_0^t \sum_{i,j=1}^{r} a_{ij}^{\alpha_s}(\varphi_s) \dot{\varphi}_s^i \dot{\varphi}_s^j\, ds.$$

Note that

$$\inf\left\{ \int_0^t \sum_{i,j=1}^{r} a_{ij}(\varphi_s) \dot{\varphi}_s^i \dot{\varphi}_s^j\, ds: \varphi_0 = a,\ \varphi_t = b \right\} = \frac{1}{t} \rho^2(a,b),$$

where ρ is the Riemannian metric corresponding to the form $ds^2 = \sum_{i,j=1}^r a_{ij}(x)\,dx^i, dx^j$ (see, for example, [7], Chapter 6). Then the inner inf in (35) is equal to

$$\begin{aligned} D_1 &= D_1(t, x; z_1, \ldots, z_{N-1}; y; t_0, t_1, \ldots, t_{N-1}, t; i_0, i_1, \ldots, i_{N-1}) \\ &= \sum_{k=0}^{N-1} \frac{\rho_{i_k}^2(z_k, z_{k+1})}{t_k}, \end{aligned}$$

where ρ_k is the Riemannian metric corresponding to $(a_{ij}^k, (x)) = (a_k^{ij}(x))^{-1}$.

Consider now $\inf_{t_0, \ldots, t_{N-1}} D_1$. Using the Cauchy inequality, we have

$$\begin{aligned} D_1 &= \sum_{k=0}^{N-1} \frac{\rho_{i_k}^2(z_k, z_{k+1})}{t_k} = \sum_{k=0}^{N-1} \left(\frac{\rho_{i_k}(z_k, z_{k+1})}{\sqrt{t_k}} \right)^2 \sum_{k=0}^{N-1} \left(\frac{\sqrt{t_k}}{\sqrt{t}} \right)^2 \\ &\geq \left(\sum_{k=0}^{N-1} \frac{\rho_{i_k}(z_k, z_{k+1})}{\sqrt{t}} \right)^2 = \frac{1}{t} \left(\sum_0^{N-1} \rho_{i_k}(z_k, z_{k+1}) \right)^2. \end{aligned}$$

On the other hand, if

$$t_k = t\rho_{i_k}(z_k, z_{k+1}) \left(\sum_0^{N-1} \rho_{i_k}(z_k, z_{k+1}) \right)^{-1},$$

then $D_1 = t^{-1}(\sum_0^{N-1} \rho_{i_k}(z_k, z_{k+1}))^2$. Thus

$$\inf_{\substack{t_0 + \cdots + t_{N-1} = t \\ t_i > 0}} D_1 = \frac{1}{t} \left(\sum_0^{N-1} \rho_{i_k}(z_k, z_{k+1}) \right)^2.$$

Now we assume for a moment that all operators L_k have coefficients independent of x. Then we can unite all time intervals on which the same metric ρ_{i_k} is considered. Using the triangle axiom for metric, we get that

$$D(t, x, y) = \frac{1}{t} \inf_{\substack{z_1, \ldots, z_{n-1} \\ z_0 = x,\, z_n = y}} \left(\sum_{k=1}^n \rho_k(z_{k-1}, z_k) \right)^2,$$

where n is the number of equations in our system. Taking into account that in the space-homogeneous case a Riemannian metric ρ has the property that $\rho(\alpha x, \alpha y) = |\alpha| \rho(x, y)$ for any real α, it is easy to check that the set

$$\left\{ x: \inf_{\substack{z_1, \ldots, z_{n-1} \in R^r \\ z_0 = x,\, z_n = y}} \left(\sum_{k=1}^n \rho_k(z_{k-1}, z_k) \right) \leq d \right\}$$

is the convex envelope of Riemannian spheres

$$S_k = \{x: \rho_k(x, y) \leq d\}, \qquad k = 1, 2, \ldots, n.$$

To describe $D(t, x, y)$ in the case of space nonhomogeneous coefficients, let us define the function $d(x, y)$, $x, y \in R^r$, by the following conditions:

1. $d(x, \alpha y) = |\alpha| d(x, y)$ for any real α.

2. $d(x,y) = 1$ on the boundary of the convex envelope of the sets

$$\left\{y \in R^r: \sum_{i,j=1}^{r} a_{ij}^k(x)y^i, y^j \le 1\right\} = S_k, \qquad k = 1,2,\dots,n.$$

Put

$$\bar{\rho}(x,y) = \inf\left\{\int_0^t d(\varphi_s, \dot{\varphi}_2)\, ds: \varphi_0 = x,\ \varphi_t = y\right\}.$$

It is easy to check that this infimum is independent of the parameter t and defines a metric $\bar{\rho}$ in R^r (Finsler metric).

One can calculate that

$$D(t,x,y) = \frac{1}{t}\bar{\rho}^2(x,y); \tag{36}$$

in the space-homogeneous case, it was proved above. The general case can be considered by the approximation of the metric by a piecewise space-homogeneous one.

From (34), (35) and (36) we derive

$$V(t,x) = ct - \frac{\bar{\rho}^2(x,G_0)}{2t}.$$

Now we can check that condition (N) is fulfilled: In our case $\mathscr{E}_- = \{(t,x): t > 0,\ V(t,x) \le 0\} = \{x: \rho(x,G_0) \ge t\sqrt{2c}\}$. For a point $(t,x) \in \mathscr{E}_-$, choose a small $h > 0$ and consider the function φ_s^h such that $\varphi_s^h = x$ for $s \in [0,h]$ and $\varphi_s^h = \hat{\varphi}(s-h)$ for $s \in [h,t]$, where $\hat{\varphi}(s)$ is the minimal geodesics of the metric $\bar{\rho}$, connecting x and G_0 with the parameterization proportional to the length, $\hat{\varphi}(0) = x$, $\hat{\varphi}_{t-h} \in G_0$. The point $(t-s, \varphi_s^h)$ for all $s \in (0,t)$ belongs to the set $\{(s,y), V(s,y) < 0\}$,

$$\sup\{R_{ot}(\varphi^h, \alpha): \alpha \in H_{ot}\} = ct - \frac{\bar{\rho}^2(x,G_0)}{t-h}.$$

Therefore $V(t,x) = \lim_{h \downarrow 0} \sup\{R_{ot}(\varphi^h,\alpha),\ \alpha \in H_{ot}\}$ and condition (N) is fulfilled. □

From Theorem 2 we have the following result.

THEOREM 3. *Let $c_k(x,0) = c$, for all $x \in R^r$, $k \in \{1,2,\dots,n\}$. Let $\bar{\rho}(x,y)$, $x,y \in R^r$, denote the Finsler metric corresponding to the kernel $d(x,y)$, which is defined by conditions* 1 *and* 2 *above.*

Then

$$\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = \begin{cases} 1, & \text{for } \bar{\rho}(x,G_0) < t\sqrt{2c}, \\ 0, & \text{for } \bar{\rho}(x,G) > t\sqrt{2c}. \end{cases}$$

The convergence is uniform in (t,x) for (t,x) in any compact set F such that $F \cap \{(t,x)\colon \bar{\rho}(x, G_0) = t\sqrt{2c}\} = \phi$.

The statement of Theorem 3 means that the propagation of the wave front is governed by the Huygens principle. The corresponding velocity field is homogeneous and isotropic in the Finsler metric $\bar{\rho}$.

LEMMA 7. *Suppose that for some $x_0 \in R^r$ and $k_0 \in \{1, \dots, n\}$ and for any $\delta_1, \delta_2 > 0$, there exists $\varepsilon_0 > 0$ such that*

$$(37) \qquad g_{k_0}(x) = g_k^\varepsilon(x) \geq e^{-\delta_1/\varepsilon} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0,\ |x - x_0| \leq e^{-\delta_2/\varepsilon}.$$

Then a constant $A > 0$ exists such that $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 1$ for $t > 0$, $|x - x_0| < At$ and any $k = 1, \dots, n$. The convergence is uniform in any compact subset of the cone $\{(t,x)\colon t > 0,\ |x - x_0| < At\}$.

PROOF. Because of Lemma 5 it is sufficient to consider the case when $g_k(x) \equiv 0$ for $k \neq k_0$ and $\tilde{g}_{k_0}^\varepsilon(x) = g_{k_0}^\varepsilon(x)$ for $|x - x_0| \leq \exp\{-\delta_2/\varepsilon\}$ and $\tilde{g}_{k_0}^\varepsilon(x) = 0$ for $|x - x_0| > \exp\{-\delta_2/\varepsilon\}$. Moreover, we can confine ourselves to the case $c(x,u) = c(u)$.

From Theorem 3 one can derive that $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 0$, for $(t,x) \in \{(s,y)\colon s > 0,\ \bar{\rho}(x, x_0) > s\sqrt{2c(0)}\}$. Here $\bar{\rho}$ is a corresponding Finsler metric. As it was explained when we proved Theorem 3, for every point (t,x), $t > 0$, $\bar{\rho}(x, x_0) > t\sqrt{2c(0)}$, and any $\delta > 0$, there exists a function φ_s^δ, $0 \leq s \leq t$, $\varphi_0^\delta = x$, $\varphi_t^\delta = x_0$, $\bar{\rho}(x_0, \varphi_s^\delta) > s\sqrt{2c(0)}$ for $0 < s < t$ and $\alpha^\delta \in H_{ot}$ such that

$$(38) \qquad \int_0^t \left[c_{\alpha_s^\delta}(\varphi_s^\delta) - \frac{1}{2} \sum_{i,j=1}^{r} a_{ij}^{\alpha_s^\delta}(\varphi_s^\delta) \dot{\varphi}_s^{\delta,i} \dot{\varphi}_s^{\delta,j} \right] ds > -\frac{\delta}{2}.$$

Without loss of generality we can assume that $\alpha_s^\delta = k_0$ for $s \in [t-h, t]$ for small enough $h > 0$. Taking into account (38), the lower bound for Lemma 6, (37) and the bound (17) for the transition density of the process corresponding to the operator L_{k_0}, we get that

$$(39) \qquad \lim_{\varepsilon \downarrow 0} \varepsilon \ln u_k^\varepsilon(t,x) = 0$$

for $t > 0$, $\bar{\rho}(x, x_0) \leq t\sqrt{2c(0)}$, and any $k = 1, \dots, n$. From (39) it follows that $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 1$ in the interior points of the set $\{(t,x), t > 0,\ \bar{\rho}(x,x_0) < t\sqrt{2c(0)}\}$. One can carry out the proof of the last statement in the same way as the proof of Theorem 1 from [6].

To get the statement of Lemma 7, note that

$$\{(t,x), t > 0, |x - x_0| < At\} \subset \left\{(t,x), t > 0, \bar{\rho}(x, x_0) < t\sqrt{2c(0)}\right\},$$

for some $A > 0$. □

Let $\tau = \tau(t, \varphi)$, $t \in (-\infty, \infty)$, $\varphi \in C_{0\infty}$, be a Markov functional introduced in Section 2 and θ be the set of all Markov functionals.

Denote

$$V^*(t,x) = \inf_{\tau \in \theta} \sup \left\{ \int_0^{t\wedge\tau} \left[c_{\alpha_s}(\varphi_s) - \frac{1}{2} \sum_{i,j=1}^{r} a_{ij}^{\alpha_s}(\varphi_s)\dot{\varphi}_s^i \dot{\varphi}_s^j \right] ds : \varphi \in C_{ot},\ \varphi_0 = x,\ \varphi_t \in G_0,\ \alpha \in H_{ot} \right\}.$$

LEMMA 8. *If $V^*(t,x) < 0$, then $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 0$ for any $k = 1, \dots, n$. The convergence is uniform in (t,x) for (t,x) in any compact subset of the set $\{(s,y): V^*(s,y) < 0\}$.*

Taking into account equation (26), Lemma 5 and the upper bound from Lemma 6, the proof of this lemma is similar to the proof of Lemma 1 and thus we omit it.

The following lemma is a corollary of Lemma 7.

LEMMA 9. *Let $\varepsilon' \downarrow 0$, $\mathscr{E}^{(\varepsilon')} = \{(t,x): \lim_{\varepsilon' \downarrow 0} u_{k_0}^{\varepsilon'}(t,x) = 0$ for some $k_0 \in \{1, \dots, n\}\}$ and $(t_0, x_0) \in \mathscr{E}^{(\varepsilon')}$. Then there exists $A > 0$ such that $\limsup_{\varepsilon' \downarrow 0} \varepsilon' \ln u_k^{\varepsilon'}(t,x) < 0$ for any $k = 1, \dots, n$ and $(t,x) \in D_{t_0,x_0}^A$, where*

$$D_{t_0,x_0}^A = \{(s,y): 0 < s < t_0,\ |x_0 - y| \le A(t_0 - s)\}.$$

LEMMA 10. *Let F be a compact subset of the interior (M) of the set $M = \{(t,x): V^*(t,x) = 0\}$. Then $\lim_{\varepsilon \downarrow 0} \varepsilon \ln u_k^\varepsilon(t,x) = 0$ for $k = 1, \dots, n$ uniformly in $(t,x) \in F$.*

PROOF. The proof of this lemma is similar to the proof of Lemma 4, and we omit it. □

THEOREM 4. *Let $(u_1^\varepsilon(t,x), \dots, u_n^\varepsilon(t,x))$ be the solution of (25).*

For any compact subset F_1 of the set $\{(s,y): s > 0, V^*(s,y) < 0\}$, $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 0$ for $k = 1, \dots, n$ uniformly in $(t,x) \in F_1$.

For any compact subset F_2 of the interior of the set $\{(s,y): s > 0, V^*(s,y) = 0\}$, $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 1$ for $k = 1, \dots, n$ uniformly in $(t,x) \in F_2$.

PROOF. The proof follows from Lemmas 5–9. □

In Theorem 3 we considered the case where $c_k(x) = c$ is independent of k and x. Now we consider as an example (25) when $L_k = L$ for all k:

$$\frac{\partial u_k^\varepsilon(t,x)}{\partial t} = Lu_k^\varepsilon + c_k(x, u_k^\varepsilon)u_k^\varepsilon + \sum_{j=1}^{n} d_{k_j}(u_j^\varepsilon - u_k^\varepsilon), \tag{40}$$

$$u_k^\varepsilon(0,x) = g_k(x), \qquad k = 1, \dots, n.$$

The function $V^*(t,x)$ for (40) has the form

$$V^*(t,x) = \inf_{\tau\in\theta} \sup \left\{ \int_0^{t\wedge\tau} \left[c_{\alpha_s}(\varphi_s) - \frac{1}{2} \sum_{i,j=1}^{r} a_{ij}^{\alpha_s}(\varphi_s)\dot\varphi_s^i \dot\varphi_s^j \right] ds : \varphi_0 = x,\ \varphi_t \in G_0,\ \alpha \in H_{ot} \right\}$$

$$= \inf_{\tau\in\theta} \sup_{\substack{\varphi_0 = x\\ \varphi_t\in G_0}} \left\{ \int_0^{t\wedge\tau} \left[\max_{1\le k\le n} c_k(\varphi_s) - \frac{1}{2} \sum_{i,j=1}^{r} a_{ij}(\varphi_s)\dot\varphi_s^i \dot\varphi_s^j \right] ds \right\}.$$

Thus the function $V^*(t,x)$ and the law of the wave front propagation will be the same as in the case of the single equation

$$(41) \qquad \frac{\partial u^\varepsilon(t,x)}{\partial t} = \varepsilon L u^\varepsilon + \hat c(x) u^\varepsilon(1-u^\varepsilon), \qquad u^\varepsilon(0,x) = \sum_{k=1}^{n} g_k(x),$$

where $\hat c(x) = \max_k c_k(x)$.

In particular, let $r=1$, $a^{11}(x) \equiv 1$ and $G_0 = \{x \le 0\}$ and let the function $\hat c(x)$ decrease when x increases for $x > 0$. Denote by ψ_s the solution of the equation $\dot\psi_s = \sqrt{2\hat c(\psi_s)}$, $\psi_0 = 0$, for $s \ge 0$. It follows from Example 2.2 of Section 6.2 of [7] that

$$\lim_{\varepsilon\downarrow 0} u_k^\varepsilon(t,x) = \lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x) = \begin{cases} 1, & x < \psi_t,\ t > 0,\\ 0, & x > \psi_t,\ t > 0.\end{cases}$$

One can check in this case that condition (N) is not fulfilled, and the infimum in the definition of the function $V^*(t,x)$ is reached on the Markov functional $\tau^*(t,\varphi) = \min\{s\colon \psi_{t-s} = \varphi_s\}$.

As in the case of the single equation, the wave front may have jumps. For example, if in the case under consideration an interval $(\alpha,\beta) \subset R^+$ exists, where the function $\hat c(x)$ increases fast enough, then the front will have jumps. It follows from Example 3 in [6].

In the conclusion of this section we formulate some results on the upper and lower bounds for the domains where $u_k^\varepsilon(t,x)$ are close to 0 or 1.

Denote $V^{k*}(t,x)$ as the V^*-function defined for the single equation $\partial u/\partial t = \varepsilon L_k u + (1/\varepsilon) c_k(u) u$ with the initial function $u(0,x) = \sum_{k=1}^n g_k(x)$.

The following inclusions are a simple implication of the definition of the V^*-function:

$$\{(t,x)\colon t>0,\ V^*(t,x) = 0\} \supseteq \bigcup_{k=1}^{n} \{(t,x),\ t>0,\ V^{k*}(t,x) = 0\}.$$

If for some $k_0 \in \{1,\dots,n\}$,

$$\sum_{i,j=1}^{r} a_{ij}^{k_0}(x)\lambda^i\lambda^j \le \sum_{i,j=1}^{r} a_{ij}^{k}(x)\lambda^i\lambda^j \quad \text{for any } \lambda, x \in R^r,\ k = 1,\dots,n,$$

and $c_{k_0}(x) \ge c_k(x)$ for $x \in R^r$, $k = 1,\dots,n$, then $V^*(t,x) = V^{k_0 *}(t,x)$.

One can have more explicit bounds from below for the set $\{(t, x)\colon t > 0, V^*(t, x) = 0\}$ in the following way. Denote by $h(x, y)$, $x, y \in R^r$, the function defined by the properties: $h(x, y) = |t| h(x, y)$ for any real t; $h(x, y) = 1$ on the boundary of the convex envelope of the ellipsoids

$$S_k^x = \left\{y\colon \Sigma a_{ij}^k(x)(2c_k(x))^{-1} y^i y^j \le 1\right\}, \qquad k = 1, \dots, n.$$

Then

$$\{(t, x)\colon t > 0, V^*(t, x) = 0\} \supseteq \{(t, x), t > 0, \bar{\rho}(x, G_0) < t\},$$

where $\bar{\rho}(x, y) = \inf\{\int_0^t h(\varphi_s, \dot{\varphi}_s)\, ds\colon \varphi \in C_{ot}, \varphi_0 = x, \varphi_t = y\}$ is the Finsler metric with the kernel $h(x, y)$. Taking into account Lemma 5 and Theorem 3, one can carry out the proof of this inclusion in the same way that Lemma 6.2.4 from [7] was proved.

4. Space-homogeneous isotropic case. In this section we give explicit description of the wave front motion for the space-homogeneous isotropic system. For brevity, we consider the case of two equations,

$$\begin{aligned} \frac{\partial u_1^\varepsilon(t, x)}{\partial t} &= \frac{\varepsilon a_1}{2} \Delta u_1^\varepsilon + \frac{1}{\varepsilon} c_1(u_1^\varepsilon) u_1^\varepsilon + d_1(u_2^\varepsilon - u_1^\varepsilon), \\ \frac{\partial u_2^\varepsilon(t, x)}{\partial t} &= \frac{\varepsilon a_2}{2} \Delta u_2^\varepsilon + \frac{1}{\varepsilon} c_2(u_2^\varepsilon) u_2^\varepsilon + d_2(u_1^\varepsilon - u_2^\varepsilon), \\ & x \in R^r, t > 0, u_1^\varepsilon(0, x) = g(x), u_2^\varepsilon(0, x) = g_2(x). \end{aligned} \tag{42}$$

We make the usual assumptions on the nonlinear terms and the initial functions, $c_k = c_k(0)$, $G_0 = \operatorname{supp}(g_1 + g_2)$. Without loss of generality we assume that $c_1 \ge c_2$. Otherwise we change the indexing.

THEOREM 5. *Let $\rho(\cdot, \cdot)$ be the Euclidean distance in R^r and $c_1 \ge c_2$. Then for $k = 1, 2$, $t > 0$, $x \in R^r$,*

$$\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t, x) = \begin{cases} 1, & \text{if } \rho(x, G_0) < vt, \\ 0, & \text{if } \rho(x, G_0) > vt, \end{cases}$$

uniformly in (t, x) for (t, x) in any compact $F \subset \{(t, x)\colon t > 0, x \in R^r\}$ such that

$$F \cap \{(t, x)\colon t > 0, x \in R^r, \rho(x, G_0) = vt\} = \phi.$$

The speed v is given by the formulas

$$v = \begin{cases} \sqrt{2a_1c_1}, & \text{if } a_1 \ge a_2, \\ \sqrt{2a_1c_1}, & \text{if } a_1 < a_2, 2a_1c_1 \ge c_1a_2 + a_1c_2, \\ \dfrac{c_1a_2 - c_2a_1}{\sqrt{2(a_2 - a_1)(c_1 - c_2)}}, & \\ & \text{if } a_1 < a_2, 2a_1c_1 \vee 2a_2c_2 < c_1a_2 + a_1c_2, \\ \sqrt{2a_2c_2}, & \text{if } a_1 < a_2, 2a_2c_2 \ge c_1a_2 + a_1c_2. \end{cases} \tag{43}$$

PROOF. If $a_1 \geq a_2$, then taking into account our assumption $c_1 \geq c_2$, we have $V(t,x) = c_1 t - \rho^2(x, G_0)/2a_1 t$, where ρ is Euclidean distance. It is obvious that condition (N) is fulfilled in this case, and our statement follows from Theorem 2.

Consider now the case $a_1 < a_2$. Denote

$$V_0(t,x) = \sup\left\{\int_0^t \left[c_{\alpha_s} + \frac{|\dot{\varphi}_s|^2}{2a_{\alpha_s}}\right] ds \colon \varphi \in C_{ot}, \varphi_0 = x, \varphi_t = 0, \alpha \in H_{ot}\right\}.$$

Because of the homogeneity in space,

$$V(t,x) = \sup_{y \in G_0} V_0[t, x-y]. \tag{44}$$

It is easy to check that

$$V_0(t,x) = \max_{0 \leq p \leq 1}\left[c_1 pt + c_2(1-p)t - \frac{1}{2}\min_z\left(\frac{|z|^2}{pta_1} + \frac{|x-z|^2}{(1-p)ta_2}\right)\right].$$

Taking into account that

$$\min_z\left[\left(\frac{|z|^2}{pta_1} + \frac{|x-z|^2}{(1-p)ta_2}\right)\right] = \frac{|x|^2}{t(pa_1 + (1-p)a_2)},$$

we have from (44)

$$V_0(t,x) = t\max_{0 \leq p \leq 1}\left[c_1 p + c_2(1-p) - \frac{|x|^2}{2t^2(pa_1 + (1-p)a_2)}\right]. \tag{45}$$

Denote by $f(P,M)$ the function under the max sign in (45), $M = |x|/t\sqrt{2}$. Solving the equation $df/dp = 0$, and taking into account that the smallest of the roots corresponds to the maximum, we see that $\max_{0 \leq p \leq 1} f(P,M)$ is reached at the point

$$P_0 = \frac{a_2}{a_2 - a_1} - \frac{M}{\sqrt{(c_1 - c_2)(a_2 - a_1)}},$$

if $P_0 \in [0,1]$. If $P_0 > 1$, the maximum is reached at the point $P = 1$ and $\max_{0 \leq p \leq 1} f(P,M) = f(0,M)$ in the case $P_0 < 0$.

Thus we get the following expression for $\bar{f}(M) = \max_{0 \leq p \leq 1} f(P,M)$:

$$\bar{f}(M) = \begin{cases} c_1 - \dfrac{M^2}{a_1}, & M \leq a_1\sqrt{\dfrac{c_1 - c_2}{a_2 - a_1}}, \\ \dfrac{c_1 a_2 - c_2 a_1}{a_2 - a_1} - 2M\sqrt{\dfrac{c_1 - c_2}{a_2 - a_1}}, & a_1\sqrt{\dfrac{c_1 - c_2}{a_2 - a_1}} < M < a_2\sqrt{\dfrac{c_1 - c_2}{a_2 - a_1}}, \\ c_2 - \dfrac{M^2}{a_2}, & M \geq a_2\sqrt{\dfrac{c_1 - c_2}{a_2 - a_1}}. \end{cases}$$

From (44) and (45) we get

$$V(t,x) = t\bar{f}\left(\frac{\rho(x,G_0)}{t\sqrt{2}}\right).$$

Since the condition (N) is fulfilled for the function $V(t,x)$, we get from Theorem 2 that the position of the wave front at time t is defined by the equation $\bar{f}(\rho(x,G_0)/t\sqrt{2}) = 0$. Solving this equation, we find that $\rho(x,G_0) = tv$, where v is defined by the formulas in (43). □

Consider the case when a_2 and c_1 are fixed and $a_1, c_2 \to 0$. Then Theorem 5 gives the following expression for the speed v:

$$v = \sqrt{\frac{c_1 a_2}{2}} + o(1), \qquad a_1, c_2 \downarrow 0.$$

The speeds in separated equations, when $d_1 = d_2 = 0$, will be $\sqrt{2a_1c_1}$ and $\sqrt{2a_2c_2}$. We see that in this case the speed of the front in the coupled system is bigger than in separated equations.

In the case $a_1 = c_2 = 0$ (42) has the form

$$\begin{aligned} \frac{\partial u_1^\varepsilon(t,x)}{\partial t} &= \frac{1}{\varepsilon}c_1(u_1^\varepsilon)u_1^\varepsilon + d_1(u_2^\varepsilon - u_1^\varepsilon), \qquad u_1^\varepsilon(0,x) = g_1(x), \\ \frac{\partial u_2^\varepsilon(t,x)}{\partial t} &= \frac{\varepsilon a_2}{\varepsilon}\Delta u_2^\varepsilon + d_2(u_1^\varepsilon - u_2^\varepsilon), \qquad u_2^\varepsilon(0,x) = g_2(x). \end{aligned} \tag{46}$$

(46) is formally excluded from our considerations because of the degeneration of the first equation. But (25) is fulfilled and only minor changes should be made in the proof to show that

$$\lim_{\varepsilon\downarrow 0} u_k^\varepsilon(t,x) = \begin{cases} 1, & \text{if } \rho(x,G_0) < t\sqrt{\dfrac{c_1a_2}{2}}, \\ 0, & \text{if } \rho(x,G_0) > t\sqrt{\dfrac{c_1a_2}{2}}. \end{cases}$$

5. Remarks and generalizations.

REMARK 1. Note that the law of the motion of the wave front is independent of the constants d_{ij}. The only assumption is that these constants are positive. If $d_{ij} = d_{ij}(x) > 0$, then the results also hold. To consider this case, one can, for example, use the fact that the measure in the space of trajectories of the process $(X_t^\varepsilon, \nu_t^\varepsilon)$, corresponding to the equations with $d_{ij} = d_{ij}(x)$ is absolutely continuous with respect to the measure of the process (X_t^ε, ν_t), corresponding to the case when all $d_{ij} = 1$. The density can easily be written down explicitly. We can use the same approach in the case $d_{ij} = d_{ij}(x,u) > 0$. Using explicit formulas for the density, one can consider the case when constants d_{ij} are negative (see [7]). If some $d_{ij}(x)$ are 0 in a domain $D \subset R^r$, new effects may appear.

REMARK 2. Let $D \subset R^r$ be a domain with smooth enough boundary. One can consider the mixed problem for the coupled RDE's

$$\begin{aligned}\frac{\partial u_k^\varepsilon(t,x)}{\partial t} &= \varepsilon L_k u_k^\varepsilon + \frac{1}{\varepsilon} f_k(x, u_k^\varepsilon)\\ &\quad + \sum_{i,j=1}^{r} d_{kj}(u_j^\varepsilon - u_k^\varepsilon), \qquad x \in D, t>0, \qquad (47)\\ \left.\frac{\partial u_2^\varepsilon(t,x)}{\partial n}\right|_{\partial D} &= 0, \qquad u_k^\varepsilon(0,x) = g_k(x), \qquad k = 1,\dots,n.\end{aligned}$$

The probabilistic approach allows us to describe the behavior of the solution of (47) and $\varepsilon \downarrow 0$. One should take into account the form of the action functional for the family of processes $(\tilde{X}_t^\varepsilon, \nu_t)$ with reflection on the boundary (see, for example [8]). The Dirichlet conditions and more general boundary conditions can also be considered in a similar way.

REMARK 3. Consider a system of weakly coupled RDE's with a drift. For brevity, we confine ourselves to a space-homogeneous system of two equations. After rescaling of the space and time as it was explained in Section 1, we get the system

$$\begin{aligned}\frac{\partial u_1^\varepsilon(t,x)}{\partial t} &= \frac{\varepsilon a_1}{2}\Delta u_1^\varepsilon + (b_1, \Delta u_1^\varepsilon) + \frac{1}{\varepsilon} c_1(u_1^\varepsilon) u_1^\varepsilon + d_1(u_2^\varepsilon - u_1^\varepsilon),\\ \frac{\partial u_2^\varepsilon(t,x)}{\partial t} &= \frac{\varepsilon a_2}{2}\Delta u_2^\varepsilon + (b_2, \Delta u_1^\varepsilon) + \frac{1}{\varepsilon} c_2(u_2^\varepsilon) u_2^\varepsilon + d_2(u_1^\varepsilon - u_2^\varepsilon), \qquad (48)\\ u_k^\varepsilon(0,x) &= g_1(x), u_2^\varepsilon(0,x) = g_2(x), G_0 = \operatorname{supp}(g_1 + g_2).\end{aligned}$$

For brevity, let G_0 be a compact subset of $\mathbb{R}^r$. If $b_1 = b_2 = b$, we can introduce new variables $\tilde{u}_k^\varepsilon(t,x) = u_k^\varepsilon(t, x + bt)$. The functions $(\tilde{u}_1^\varepsilon, \tilde{u}_2^\varepsilon)$ satisfy the system without drift. On can get the wave front for (48) by translating the front for (48) with $b = 0$ and the same initial function on vector $-bt$. If $|b| < v$, where v is the speed of the front from Theorem 5, then the domain, where the solution of (48) is close to 1, will expand in all directions for small ε. If $|b| = v$, a half space exists in R^r, where $u_k^\varepsilon(t,x)$ will be always close to zero. In the case $|b| > v$, $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) = 0$ for any $x \in R^r$, $k = 1, 2$, and t large enough.

In the general case when $b_1 \neq b_2$, the domains were $u_k^\varepsilon(t,x)$ tends to 1 and 0 are defined by a function $V(t,x)$: $\lim_{\varepsilon \downarrow 0} u_k^\varepsilon(t,x) \doteq 1$ if $V(t,x) > 0$, and $\lim u_k^\varepsilon(t,x) = 0$ if $V(t,x) < 0$. The function $V(t,x) = \min_{y \in G_0} V_0(t, x-y)$, where

$$V_0(t,x) = t \max_{0 \le p \le 1} \left\{ c_1 p + c_2(1-p) - \frac{1}{2} \frac{|(x/t) + p b_1 + (1-p) b_2|^2}{a_1 p + a_2(1-p)} \right\}.$$

Calculating the maximum and solving the equation $V(t, x) = 0$, we can describe the motion of the wave front.

REMARK 4. Consider the initial-boundary problem for a single equation,

$$\frac{\partial u^\varepsilon(t,x,y)}{\partial t} = \frac{\varepsilon}{2}\sum_{i=j=1}^{r} a^{ij}(x,y)\frac{\partial^2 u^\varepsilon}{\partial x^i \partial x^j} + \frac{1}{2}\frac{\partial^2 u^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon}f(u^\varepsilon),$$

(49) $$x \in R^r,\ |y| < 1,$$

$$\left.\frac{\partial u^\varepsilon(t,x,y)}{\partial y}\right|_{y=\pm 1} = 0, \qquad u^\varepsilon(0,x,y) = g(x) \geq 0, \qquad f \in \mathscr{F}_1.$$

Here the y-variable plays, in a sense, the same part as the number k in the case of system. To describe the behavior of the $u^\varepsilon(t,x,y)$ for $\varepsilon \downarrow 0$, let us introduce the Finsler metric $\tilde{\rho}$ in R^r,

$$\tilde{\rho}(x_1, x_2)\inf\left\{\int_0^t l(\varphi_s, \dot{\varphi}_s)\,ds,\ \varphi_0 = x_1,\ \varphi_t = x_2\right\}, \qquad x_1, x_2 \in R^r,$$

where $l(x,z)$ satisfies the properties: $l(x,tz) = |t|l(x,z)$, $l(x,z) = 1$ on the boundary of the convex envelope of the family of the ellipses $S_y^x = \{z \in R^r\colon \sum a_{ij}(x,y)z^1z^j \leq 1\}$, $|y| \leq 1$. Then one can prove that $\lim u^\varepsilon(t,x,y) = 1$ in the domain $\{(x,y)\colon\ x \in R^r,\ \tilde{\rho}(x, G_0) < t\sqrt{2f'(0)},\ |y| \leq 1\}$ and $\lim_{\varepsilon\downarrow 0} u^\varepsilon(t,x,y) = 0$ outside the closure of this domain.

REMARK 5. Consider a family of dynamical systems in R^r depending on a point z of a measurable space H as a parameter,

(50) $$\dot{u}_t = f(z, u_t), \qquad z \in H,\ f = (f_1, \ldots, f_n).$$

Suppose that a Markov process (X_t, ν_t) in the phase space $H \times \{1,\ldots,n\}$ is defined. Let A be the generator of the process (X_t, ν_t). Consider the system of "partial differential equations"

(51) $$\frac{\partial u^\lambda(t,k,x)}{\partial t} = f_k(x, u^\lambda) + \lambda A u^\lambda(t,k,x),$$

$$u^\lambda(0,k,x) = g_k(x), \qquad x \in H,\ k = 1,\ldots,n,\ t > 0,$$

where λ is a numerical parameter, $g(x) = (g_1(x),\ldots,g_n(x))\colon H \to R^n$.

(51) defines a semiflow in the space of functions on $H \times \{1,\ldots,n\}$, which we call coupling of the dynamical systems (50) by the Markov process (X_t, ν_t). If $\lambda > 0$ is small, we speak about weak coupling, and in the case of large λ we speak about strong coupling.

For example, a single equation

$$\frac{\partial u(t,x)}{\partial t} = f(x,u) + \lambda L u, \qquad x \in R^r,\ u \in R^1,$$

which we considered in Section 2, can be looked upon as the result of coupling of the one-dimensional dynamical systems $\dot{u} = f(x, u)$, depending on $x \in R^r = H$, by the Markov process in R^r, corresponding to the elliptic operator λL.

In general, (50) can be a family of semiflows in a linear space. For example, (23) can be considered as the result of coupling of the semiflows

$$\frac{\partial u_k}{\partial t} = L_k u_k + f_k(x, u_k), \qquad x \in R^r, \tag{52}$$

in the space of continuous functions on R^r, depending on a parameter $k \in H = \{1, \dots, n\}$, by the Markov process in H.

Using the generalized Feynman–Kac formula or other formulas for probabilistic representation of the solutions of linear problems, such a point of view allows us to write down an integral equation for the solutions of the nonlinear problems. This integral equation can be used for an asymptotic study of the solutions. In particular, in the case of weak coupling ($\lambda \downarrow 0$) after proper rescaling of the space and time, some results on wave front propagation can be proved in a general set up.

In the case of strong coupling (when $\lambda \to \infty$), one can expect that some averaging principle describes the asymptotic behavior of the coupled equations. For example, let (52) be coupled by the Markov process ν_t^λ in $\{1, \dots, n\}$ such that $P\{\nu_{t+\Delta}^\lambda = j | \nu_t^\lambda = i\} = \lambda d_{ij}\Delta + o(\Delta)$, $\Delta \downarrow 0$, $d_{ij} > 0$, $i \neq j$. The corresponding coupled system has the form

$$\begin{aligned} \frac{\partial u_k^\lambda(t, x)}{\partial t} &= L_k u_k^\lambda + f_k(x, u_k^\lambda) + \lambda \sum_{j=1}^n d_{kj}\left(u_j^\lambda - u_k^\lambda\right), \\ u_k^\lambda(0, x) &= g_k(x), \qquad k \in \{1, \dots, n\}, x \in R^r, t > 0. \end{aligned} \tag{53}$$

If now $\lambda \to \infty$, under some minor assumptions, $\lim_{\lambda \to \infty} u_k^\lambda(t, x) = u(t, x)$, $k = 1, \dots, n$, exists independent of k and is the solution of the Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} = \bar{L}u + \bar{f}(x, u), \qquad u(0, x) = \bar{g}(x),$$

where the bar means averaging with respect to the stationary distribution $(q_1, \dots, q_n)$ of the process ν_t^ε:

$$\bar{L} = \sum_1^n q_k L_k, \qquad \bar{f} = \sum_1^n q_k f_k(x, u), \qquad \bar{g} = \sum_1^n q_k g_k(x).$$

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