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THE RUSSO–SEYMOUR–WELSH THEOREM AND THE EQUALITY OF CRITICAL DENSITIES AND THE "DUAL" CRITICAL DENSITIES FOR CONTINUUM PERCOLATION ON $\mathbb{R}^2$

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A Russo–Seymour–Welsh (RSW) theorem is established for continuum percolation on $\mathbb{R}^2$. The equality of various definitions of critical densities for the continuum percolation on $\mathbb{R}^2$ is deduced as an application of the RSW theorem. It is also shown that various notions of the size of a cluster yield the same notion of critical density.

1. Introduction. We consider a percolation model defined on $\mathbb{R}^2$. This model should be viewed as a continuum analogue of the discrete site/bond percolation model. Instead of sites/bonds being independently occupied or vacant we have a Poisson process on $\mathbb{R}^2$ with each Poisson point being the centre of an "occupied" disc of random radius. We shall assume that the random variables describing the radii of the discs are i.i.d., strictly positive and bounded above by a positive constant. The model is described in more detail in the next section.

This model was introduced by Gilbert (1961) to model the transmission of radio signals. Hartigan (1981) has also considered this model in a cluster analysis setup. For this model Hall (1985) has shown that under certain moment conditions on the radius random variable the critical densities for phase transition exist. Zuev and Sidorenko (1985) and Men'shikov, Molchanov and Sidorenko (1986) have shown that in any dimension and for the radius random variable bounded above the critical densities arising from occupied clusters are all equal.

In this article, besides considering critical densities defined through "occupied" clusters we also consider critical densities defined through "vacant" clusters, vacant clusters being the analogue of the "dual" in a discrete percolation setup. We show that in $\mathbb{R}^2$ all the critical densities arising from occupied or vacant clusters are equal. In three or more dimensions one cannot expect such a result because, as in discrete percolation, one expects a nondegenerate interval of intensities where infinite occupied and infinite vacant clusters coexist.

Our argument rests crucially on a version of the Russo–Seymour–Welsh theorem for vacant crossings of suitable rectangles in the two-dimensional continuum model. The idea of the proof of the RSW theorem is similar to the idea of the original proof for the two-dimensional discrete model [Russo (1978) and Seymour and Welsh (1978)], but we have to make suitable modifications to

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take into account the dependent structure of the continuum model. In fact, it is this dependent structure which would not allow us to mimic the proof of "RSW for vacant crossings" to obtain "RSW for occupied crossings." Unfortunately, we have not been able to obtain a RSW theorem for occupied crossings. The RSW theorem apart from its intrinsic interest is also expected to yield power estimates for the continuum model as in discrete percolation models.

All these results are motivated by corresponding results in site/bond percolation. For the discrete model Kesten (1980) (for two dimensions), Aizenman and Barsky (1987) and Men'shikov (1986) have shown the equality of various definitions of critical parameters. The original proof of Kesten (1980) and the subsequent modification by Russo (1981) rested on a RSW argument and hence was restricted to two dimensions. We follow a similar line of argument. In this context we remark that in discrete percolation the derivation of many of the power laws and scaling laws are dependent on the RSW argument and as such restricted to two dimensions.

For many of the long technical arguments we sketch the main ideas and refer the reader to Roy (1987) where the details of the proof are presented.

2. The model, definitions and statement of results. Consider a Poisson point process $\xi_1, \xi_2, \dots$ of intensity λ on $\mathbb{R}^2$. Centred at $\xi_1, \xi_2, \dots$ are discs $V(\xi_1), V(\xi_2), \dots$ of radii $\rho_1, \rho_2, \dots$, respectively, where $\rho_1, \rho_2, \dots$ are i.i.d. random variables and have the same distribution as that of a strictly positive random variable ρ. We call this a Poisson system and denote it by (Ξ, λ, ρ).

Given two disjoint regions A and B in $\mathbb{R}^2$, we say that a continuous curve γ is an occupied/vacant connection of A and B in a region S if $\gamma \cap A \neq \varnothing$, $\gamma \cap B \neq \varnothing$, $\gamma \subseteq S$ and $\gamma \subseteq \bigcup_{i=1}^{\infty} V(\xi_i)$ $[\gamma \cap V(\xi_i) = \varnothing$ for all $i \geq 1]$. We denote by $A \rightsquigarrow_o B$ in S ($A \rightsquigarrow_v B$ in S) the existence of an occupied/vacant connection of A and B in the region S. In particular, if $S = [0, l_1] \times [0, l_2]$ and $A = \{0\} \times [0, l_2]$ and $B = \{l_1\} \times [0, l_2]$, the left and right edge, respectively, of S, then any occupied/vacant connection of A and B in S is called a left–right (L–R) occupied/vacant crossing of the rectangle S. The top–bottom (T–B) occupied/vacant crossing of the rectangle S is defined similarly. Finally, if $A = \{a\}$ and $B = \{b\}$ we write $a \rightsquigarrow_o b$ and $a \rightsquigarrow_v b$ to denote, respectively, occupied and vacant connections.

Now we define two regions in $\mathbb{R}^2$, $W(0) := \{x\colon x \rightsquigarrow_o 0 \text{ in } \mathbb{R}^2\}$ and $W^*(0) := \{x\colon x \rightsquigarrow_v 0 \text{ in } \mathbb{R}^2\}$, i.e., the occupied and vacant cluster of the origin, respectively. The crossing probabilities are defined as follows:

$$\sigma((l_1, l_2), 1, \lambda) := P_\lambda\{\exists \text{ a L–R occupied crossing of } [0, l_1] \times [0, l_2]\},$$

$$\sigma^*((l_1, l_2), 1, \lambda) := P_\lambda\{\exists \text{ a L–R vacant crossing of } [0, l_1] \times [0, l_2]\},$$

$$\sigma((l_1, l_2), 2, \lambda) := P_\lambda\{\exists \text{ a T–B occupied crossing of } [0, l_1] \times [0, l_2]\},$$

$$\sigma^*((l_1, l_2), 2, \lambda) := P_\lambda\{\exists \text{ a T–B vacant crossing of } [0, l_1] \times [0, l_2]\}.$$

The critical densities defined by the Lebesgue measure $|W(0)|$ of the occupied cluster $W(0)$ in $\mathbb{R}^2$ are as follows:

$$\lambda_H := \inf\{\lambda: P_\lambda\{|W(0)| = \infty\} > 0\}, \tag{2.1}$$

$$\lambda_T := \inf\{\lambda: E_\lambda[|W(0)| = \infty]\}, \tag{2.2}$$

$$\lambda_S := \inf\left\{\lambda: \limsup_{n\to\infty} \sigma((n, 3n), 1, \lambda) > 0\right\}. \tag{2.3}$$

Hall (1985) has shown that if $E|\rho|^6 < \infty$ and $E|\rho|^8 = \infty$ then $\lambda_T \equiv 0$ and $\lambda_H > 0$. More generally, in d-dimensions, if $E|\rho|^{2(2d-1)} < \infty$ and $E|\rho|^{2(2d)} = \infty$ then $\lambda_T \equiv 0$ and $\lambda_H > 0$. Clearly, we always have $\lambda_T \le \lambda_H$.

The critical densities defined by the Lebesgue measure $|W^*(0)|$ of the vacant cluster $W^*(0)$ in $\mathbb{R}^2$ are

$$\lambda_H^* := \sup\{\lambda: P_\lambda\{|W^*(0)| = \infty\} > 0\}, \tag{2.4}$$

$$\lambda_T^* := \sup\{\lambda: E_\lambda[|W^*(0)|] = \infty\}, \tag{2.5}$$

$$\lambda_S^* := \sup\left\{\lambda: \limsup_{n\to\infty} \sigma^*((n, 3n), 1, \lambda) > 0\right\}. \tag{2.6}$$

These critical densities correspond, in a sense, to the "dual" parameters of the Poisson system.

Another notion of the size of the cluster is $\#W(0)$, the number of Poisson points comprising the occupied cluster $W(0)$. This leads to the following definitions of critical densities:

$$\lambda_\# := \inf\{\lambda: P_\lambda\{\#W(0) = \infty\} > 0\}, \tag{2.7}$$

$$\lambda_N := \inf\{\lambda: E_\lambda[\#W(0)] = \infty\}. \tag{2.8}$$

Clearly, no definition of the dual parameters can be made with this notion.

Men'shikov, Molchanov and Sidorenko (1986) have shown the following for arbitrary dimensions.

THEOREM 2.1 (Men'shnikov, Molchanov and Sidorenko). *In a Poisson system* (Ξ, ρ, λ) *with*

$$0 < \rho \le R \quad a.s. \text{ for some } R > 0, \tag{2.9}$$

$\lambda_\# = \lambda_N = \lambda_S$.

The proof of this theorem is by approximation with percolation models on nonplanar, multiparametric periodic graphs where the equality of the corresponding critical parameters hold.

In addition, if we use diameter as the measure of the size of a cluster, i.e., $d(W(0)) = \sup\{d(x,y): x, y \in W(0)\}$, where $d(\cdot,\cdot)$ represents the Euclidean distance, then we have

$$\lambda_d := \inf\{\lambda: P_\lambda\{d(W(0)) = \infty\} > 0\}, \tag{2.10}$$

$$\lambda_D := \inf\{\lambda: E_\lambda[d(W(0))] = \infty\}, \tag{2.11}$$

$$\lambda_d^* := \sup\{\lambda: P_\lambda\{d(W^*(0)) = \infty\} > 0\}, \tag{2.12}$$

$$\lambda_D^* := \sup\{\lambda: E_\lambda[d(W^*(0))] = \infty\}. \tag{2.13}$$

These various notions of size arise naturally [see Kesten (1987)]. We shall show that the critical densities remain unaltered regardless of the definition of size we adopt. In particular, we prove the following theorem.

THEOREM 2.2. *In a Poisson system* (Ξ, ρ, λ) *on* $\mathbb{R}^2$ *where* (2.9) *holds, we have* (i) $\lambda_H = \lambda_\# = \lambda_d$, (ii) $\lambda_T = \lambda_N = \lambda_D$, (iii) $\lambda_H^* = \lambda_d^*$ *and* (iv) $\lambda_T^* = \lambda_D^*$.

Although we prove the above theorem for two dimensions, the proof extends to any dimensions.

Next we prove the equality of all the critical densities in $\mathbb{R}^2$. To this end we first obtain the RSW theorem for the Poisson system on $\mathbb{R}^2$.

THEOREM 2.3 (RSW). *Consider a Poisson system* (Ξ, ρ, λ) *on* $\mathbb{R}^2$, *where* (2.9) *holds. If for some constants* $\delta_1 > 0$ *and* $\delta_2 > 0$ *and for some* $l_1, l_2 > 4R$ *and* $2R < l_3 < 3l_1/2$,

$$\sigma^*((l_1, l_2), 1, \lambda) \geq \delta_1 \text{ and } \sigma^*((l_3, l_2), 2, \lambda) \geq \delta_2,$$

then for any integer $k \geq 1$,

$$\sigma^*((kl_1, l_2), 1, \lambda) \geq K_k(\lambda, R) f_k(\delta_1, \delta_2),$$

where $K_k(\lambda, R) > 0$ *is independent of* δ_1, δ_2 *and* $f_k(\delta_1, \delta_2)$ *is independent of* λ *and* R.

The importance of the above theorem is in constructing vacant circuits around the origin [see Chapter 6 of Kesten (1982)]. Unfortunately, we have not been able to obtain the RSW theorem for occupied paths. Nonetheless, the above theorem allows us to prove the following theorem.

THEOREM 2.4. *In a Poisson system* (Ξ, ρ, λ) *on* $\mathbb{R}^2$ *where* (2.9) *holds the critical densities defined in* (2.1)–(2.8) *and* (2.10)–(2.13) *are all equal.*

In view of Theorems 2.1 and 2.2 to prove Theorem 2.4 it suffices to show $\lambda_H = \lambda_H^*$.

REMARK. An alternate proof of Theorem 2.1 for two dimensions using Russo's pivotal point argument and the above RSW Theorem 2.3 can be obtained in Roy (1987).

3. Proof of Theorem 2.2. First we state two preliminary results. The first lemma, a version of the FKG inequality, needs some groundwork.

Consider the space $\mathscr{S} = \{-1, 1\}^{\mathbb{R}^d \times \mathbb{R}_+}$, where $\mathbb{R}_+ = (0, \infty)$, and let $\mathscr{F}$ denote the Borel σ-field on $\mathscr{S}$. On $(\mathscr{S}, \mathscr{F})$ we assign the probability measure corresponding to the Poisson system (Ξ, ρ, λ), i.e., for any set $A \subseteq \mathbb{R}^d \times \mathbb{R}_+$ the number of points $(z, r) \in A$ $(z \in \mathbb{R}^d,\ r \in \mathbb{R}_+)$ with $\omega(z, r) = 1$, for some configuration $\omega \in \mathscr{S}$, has a Poisson distribution with mean $(l_\lambda \times \mu)(A)$, l_λ being the Lebesgue measure on $\mathbb{R}^d$ which assigns mass λ to the unit cube in

$\mathbb{R}^d$ and μ is the probability distribution of the radius random variable ρ. Intuitively, $\omega(z, r) = 1$ means that there is a Poisson point of radius r centred at z.

Let ω and ω' be two configurations in $\mathscr{S}$. We say that $\omega \le \omega'$ if for any $z \in \mathbb{R}^d$ and $r \in \mathbb{R}_+$, $\omega'(z, r) = 1$ whenever $\omega(z, r) = 1$. A function $f: \mathscr{S} \to \mathbb{R}$ is said to be increasing (decreasing) if $f(\omega) \le f(\omega')$ $[f(\omega) \ge f(\omega')]$ for every $\omega \le \omega'$. An event $A \in \mathscr{F}$ is said to be increasing (decreasing) if the indicator function 1_A is an increasing (decreasing) function.

LEMMA 3.1 (FKG inequality). *If A and B are both increasing or both decreasing events in $\mathscr{F}$, then $P(A \cap B) \ge P(A)P(B)$.*

The proof of this lemma follows from a lattice approximation together with the martingale convergence theorem and the standard FKG inequality on a partially ordered lattice [see Kemperman (1977)]. For more details see Roy (1987).

For any bounded region S in $\mathbb{R}^2$ an easy application of the FKG inequality yields $E_\lambda(d(W^*(S))) \le C(\lambda, S)E_\lambda(d(W^*(0)))$, where $C(\lambda, S)$ is some positive constant and $W^*(S) = \bigcup_{x \in S} W^*(0)$.

The next lemma is a version of Theorem 5.1 of Kesten (1982) for continuum percolation. Its proof follows, after minor adjustments, from the proof of the original site percolation version. This lemma provides exponential bounds on the probabilities of the growth of occupied (vacant) clusters at the origin when the probability of occupied (vacant) crossing of a suitable rectangle is small. Although the lemma holds for arbitrary dimension, we state here only for two dimensions.

LEMMA 3.2. *Consider a Poisson system (Ξ, ρ, λ) and suppose (2.9) holds. If for some (N_1, N_2) with $N_1, N_2 \ge R$ and for some $\kappa < (25e)^{-121/4}$, we have*

$$\sigma((N_1, 3N_2), 1, \lambda) \le \kappa \quad \text{and} \quad \sigma((3N_1, N_2), 2, \lambda) \le \kappa, \tag{3.1}$$

then there exist positive constants C_1, C_2, C_3, C_4 depending on λ such that the following hold:

$$P_\lambda\{|W(0)| \ge a\} \le C_1 \exp(-C_2 a) \quad \text{for all } a > 0, \tag{3.2}$$

$$P_\lambda\{d(W(0)) \ge b\} \le C_3 \exp(-C_4 b) \quad \text{for all } b > 0. \tag{3.3}$$

Also, if for some (M_1, M_2) with $M_1, M_2 \ge R$ and for some $\kappa^* < (25e)^{-121/4}$,

$$\sigma^*((M_1, 3M_2), 1, \lambda) \le \kappa^* \quad \text{and} \quad \sigma^*((3M_1, M_2), 2, \lambda) \le \kappa^*, \tag{3.4}$$

then there exist positive constants C_5, C_6, C_7, C_8 depending on λ such that the following hold:

$$P_\lambda\{|W^*(0)| \ge a\} \le C_5 \exp(-C_6 a) \quad \text{for all } a > 0, \tag{3.5}$$

$$P_\lambda\{d(W^*(0)) \ge b\} \le C_7 \exp(-C_8 b) \quad \text{for all } b > 0. \tag{3.6}$$

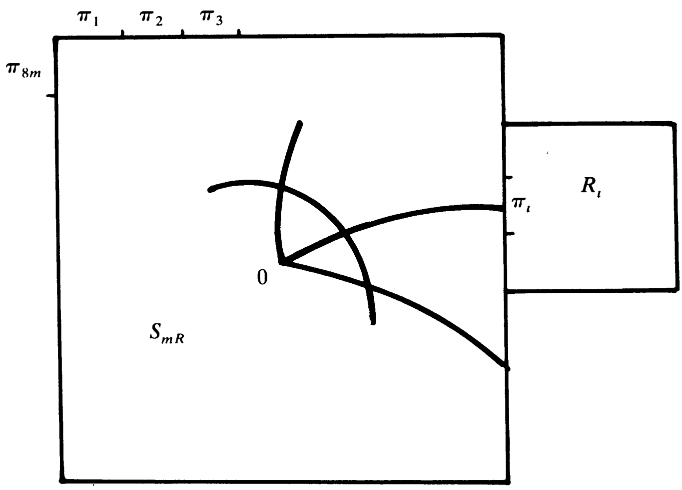

FIG. 1. *The segments* $\pi_1, \pi_2, \dots, \pi_{8m}$ *and the region* R_i.

Now we proceed to prove Theorem 2.2. Although the proof presented below is for two dimensions it can easily be extended to any dimension.

First we prove (iii), i.e., $\lambda_H^* = \lambda_d^*$. For any integer $m \geq 1$ let S_m denote the square $\{|x| < m, |y| < m\}$ and let $D_m = \{|W^*(0) \cap S_{mR}^c| \geq R^2\}$.

We show that for any $\lambda > 0$ and some constant $C(\lambda) > 0$,

$$P_\lambda\{D_m \text{ occurs}\} \geq C(\lambda) P_\lambda\{d(W^*(0)) = \infty\}. \tag{3.7}$$

Indeed, let $\pi_1, \pi_2, \dots, \pi_{8m}$ be disjoint segments of length R on the perimeter of S_{mR} (see Figure 1). For $1 \leq j \leq 8m$, let $R_j = \{(x, y)\colon |x - a| \leq 3R, |y - b| \leq 3R$ for some $(a, b) \in \pi_j\}$ and define the random variable $I := \min\{j\colon$ $\exists$ a continuous curve γ in S_{mR} with one endpoint at the origin and the other endpoint on π_j and such that for any Poisson point ξ situated in S_{mR}^c, $\gamma \cap V(\xi) = \varnothing\}$. I is well-defined on both $\{d(W^*(0)) = \infty\}$ and on $\{|W^*(0)| = \infty\}$. We note here that the continuous curve γ in the definition of I may intersect an occupied disc centred outside the square S_{mR}.

The events $\{I = i\}$, $i = 1, 2, \dots, 8m$, being mutually disjoint, we have

$$\begin{aligned} P_\lambda\{D_m \text{ occurs}\} &\geq \sum_{i=1}^{8m} P_\lambda\{R_i \text{ has no Poisson point} | I = i\} P_\lambda\{I = i\} \\ &\geq C(\lambda) P_\lambda\{d(W^*(0)) = \infty\}. \end{aligned}$$

Here the first inequality follows because the event $\{I = i\}$ depends only on the Poisson points centred in S_{mR}, while the event $\{R_i$ has no Poisson point$\}$ depends on Poisson points situated in R_j which is disjoint from S_{mR}. Thus (3.7) is true.

Similarly, for any $\lambda > 0$, we can show

$$P_\lambda\{D_m \text{ occurs}\} \geq C(\lambda) P_\lambda\{|W^*(0)| = \infty\} \tag{3.8}$$

for some constant $C(\lambda) > 0$. (3.7) and (3.8) easily imply (iii) of Theorem 2.2.

The above proof can be modified to yield

$$\lambda_H = \lambda_d. \tag{3.9}$$

Now we prove (iv), i.e., $\lambda_T^* = \lambda_D^*$.

Let $S_{2R}(i) = S_{2R} + (0, i4R) := \{(x, y)\colon (x, y - i4R) \in S_{2R}\}$ and $W^*(S_{2R}(i)) = \bigcup_{x \in S_{2R}(i)} W^*(x)$. By translation invariance we have, for any $k \geq 1$,

$$\begin{aligned}
&P_\lambda\{\exists \text{ a vacant L–R crossing of } [0, 3^k] \times [0, 3^{k+1}]\}\\
&\quad \leq P_\lambda\big(\bigcup\{d(W^*(S_{2R}(i))) \geq 3^k\big)\\
&\quad \leq \sum P_\lambda\{d(W^*(S_{2R}(i))) \geq 3^k\}\\
&\quad \leq 3^{k+1} P_\lambda\{d(W^*(S_{2R}(0))) \geq 3^k\}/4R,
\end{aligned}$$

where the union and the sum are over all integers i between 0 and $3^{k+1}/4R$.

If $\lambda > \lambda_D^*$, then $\sum_{k \geq 1} 3^{k+1} P_\lambda\{d(W^*(0)) \geq 3^k\} < \infty$ and also, by the application of the FKG inequality mentioned earlier, $\sum_{k \geq 1} 3^{k+1} P_\lambda\{d(W^*(S_{2R(0)})) \geq 3^k\} < \infty$. Thus, $\sum_{k \geq 1} P_\lambda\{\exists$ a vacant L–R crossing of $[0, 3^k] \times [0, 3^{k+1}]\} < \infty$, and so we can find an integer $k_0 \geq 0$, such that for all $k \geq k_0$, $P_\lambda\{\exists$ a vacant L–R crossing of $[0, 3^k] \times [0, 3^{k+1}]\} < 25e^{-121/4}$. An application of Lemma 3.2 now yields $\lambda \geq \lambda_T^*$. This shows $\lambda_T^* \leq \lambda_D^*$.

Similarly we can show $\lambda_T^* \geq \lambda_D^*$. This completes the proof of (iv).

Again, the proof above can be easily modified to yield

$$\lambda_T = \lambda_D. \tag{3.10}$$

Thus in view of (3.9) and (3.10), to prove (i) and (ii), it remains to show

$$\lambda_\# = \lambda_d \quad \text{and} \quad \lambda_N = \lambda_T. \tag{3.11}$$

Clearly, $d(W(0)) \leq 2R(\# W(0))$, i.e., $\lambda_\# \leq \lambda_d$.

Conversely, suppose $\lambda_\# < \lambda < \lambda_d$. Then, for some $\delta > 0$,

$$P_\lambda\{\# W(0) = \infty\} = \delta > 0. \tag{3.12}$$

Thus, for any $m > 0$, $P_\lambda\{\#(W(0) \cap S_m^c) = \infty\} = \delta$. In particular, for any m,

$$P_\lambda\{W(0) \cap S_m^c \neq \varnothing\} = \delta > 0. \tag{3.13}$$

But this contradicts the fact that $\lambda < \lambda_d$. Hence $\lambda_d \leq \lambda_\#$. This completes the proof of the first part of (3.11).

To prove $\lambda_N = \lambda_T$, we first observe that $|W(0)| \leq V(R) \# W(0)$, where $V(a) = \pi a^2$, the area of a disc of radius a. So, $\lambda_N \leq \lambda_T$.

The reverse inequality is proved in the following two cases.

CASE 1. Suppose there exists $\eta > 0$ such that $\rho \geq \eta$ a.s. We partition $\mathbb{R}^2$ by the integer lattice $\mathbb{Z}^2$ and let C be a cell of this lattice. Let $\Xi(C)$ be the Poisson process Ξ restricted to the region outside C. Let $W_C(0)$ denote the maximal connected, occupied region in $\mathbb{R}^2$ containing 0 and formed by the Poisson process $\Xi(C)$ with the radius random variable ρ. Let $\delta(x)$ denote the Euclidean distance of $W_C(0)$ from the point x in $\mathbb{R}^2$. For all cells C at a distance of at least $\mathbb{R}$ from the origin we have

$$E(\#(W(0) \cap C)|\Xi(C))$$
$$\le \sum_{k \ge 1} kP_\lambda\{\exists\ k \text{ Poisson points in } C \text{ with at least one of them having}$$
$$\text{an associated disc which intersects } W_C(0)|\Xi(C)\}$$
$$\le \sum_{k \ge 1} ke^{-\lambda}\lambda^k\left\{1 - \left[\left(\int_C P(\rho < \delta(x_1))\,dx_1\right)\cdots\left(\int_C P(\rho < \delta(x_k))\,dx_k\right)\right]\right\}/k!$$
$$\le \sum_{k \ge 1} ke^{-\lambda}\lambda^k\left\{k - \left(\int_C P(\rho < \delta(x))\,dx\right)\right\}/k!$$
$$\le k\left(\int_C P(\rho \ge \delta(x))\,dx\right),$$

for some constant $k > 0$.

W.l.o.g. assume $\eta < 1/2$. Let $v = \min(V(\eta), 1)$, where $V(\eta) = \pi\eta^2$. We have

$$E_\lambda(|W(0) \cap C||\Xi(C)) \ge ve^{-\lambda}\lambda\left(\int_C P(\rho \ge \delta(x)\,dx)\right).$$

This is because if a disc is centred in a cell at least $V(\eta)$ of its volume area be inside the cell. Thus, from the above two inequalities, for some constant $c(\lambda) > 0$,

$$(3.14) \qquad E_\lambda(\#[W(0) \cap C]|\Xi(C)) \le c(\lambda)E_\lambda(|W(0) \cap C|\,|\Xi(C)).$$

Also, for any cell C at a distance less than R from the origin we have the trivial bound $E_\lambda(\#[W(0) \cap C]) \le E_\lambda(\#[C]) = \lambda$. So taking expectations on both sides of (3.14) and summing over all C, we have $E_\lambda(\#W(0)) \le \lambda R^2 + c(\lambda)E_\lambda(|W(0)|)$. This shows that $\lambda_T \le \lambda_N$.

CASE 2. Now suppose that there does not exist any $\eta > 0$ such that $\rho \ge \eta$ a.s. Then, for any $\lambda < \lambda_T$, setting $\beta = (\lambda_T - \lambda)/\lambda_T$, there exists $0 < \alpha < \beta$ and $r_0 > 0$ with $P(\rho < r_0) = \alpha$. Let $\bar{\lambda} > \lambda$ be such that $\alpha = (\bar{\lambda} - \lambda)/\bar{\lambda}$. Since $0 < \alpha < \beta$, we have $\lambda < \bar{\lambda} < \lambda_T$. Thus if we set $\mu = \bar{\lambda} - \lambda$, then

$$P(\rho < r_0) = \mu/(\lambda + \mu).$$

Now let ρ_1 and ρ_2 be two random variables whose probability distributions are as follows:

$$P(\rho_1 \ge r) = P(\rho \ge r|\rho \ge r_0),$$
$$P(\rho_2 \ge r) = P(\rho \ge r|\rho < r_0).$$

Let (Ξ_1, λ, ρ_1) and (Ξ_2, μ, ρ_2) be two Poisson systems of intensities λ and μ and radius random variables ρ_1 and ρ_2, respectively. The superposition of the above two Poisson systems is a Poisson system $(\Xi', \bar{\lambda}, \rho)$ of intensity $\bar{\lambda}$ and radius random variable ρ. This is because $P_\lambda\{\xi \in \Xi_1|\xi \in \Xi_1 \cup \Xi_2\} =$

$\lambda/(\lambda+\mu)$. If $W_1(0)$ and $W'(0)$ denote the occupied clusters of 0 in the systems (Ξ_1, λ, ρ_1) and $(\Xi', \bar{\lambda}, \rho)$, respectively, then we have $W_1(0) \subseteq W'(0)$.

Let $(\Xi_3, \bar{\lambda}, \rho)$ be a Poisson system independent of all other Poisson systems. Then, since $\bar{\lambda} < \lambda_T$, for the occupied cluster $W_3(0)$ in $(\Xi_3, \bar{\lambda}, \rho)$, we have $E_\lambda(|W_3(0)|) < \infty$. But $(\Xi_3, \bar{\lambda}, \rho)$ and $(\Xi', \bar{\lambda}, \rho)$ are equivalent in law, so we have $E_\lambda(|W'(0)|) < \infty$. Thus, $E_\lambda(|W_1(0)|) \le E_\lambda(|W'(0)|) < \infty$. But $\rho_1 \ge r_0$ a.s. and $W_1(0)$ is the occupied cluster in (Ξ_1, λ, ρ_1), so we have from case 1, $E_\lambda(\#W_1(0)) < \infty$. Again $P(\rho_1 \ge r) \ge P(\rho \ge r)$ for any $r > 0$, so if $W(0)$ is the occupied cluster of 0 in the Poisson system (Ξ, λ, ρ) then we must have $E_\lambda(\#W(0)) \le E_\lambda(\#W_1(0)$. This shows that $E_\lambda(\#W(0)) < \infty$, thereby proving $\lambda_T \le \lambda_N$. This completes the proof of Theorem 2.2. □

4. Proof of the RSW theorem (Theorem 2.3). The proof of the RSW theorem is quite technical. Here we present the main steps of the proof.

Consider a lattice $\mathbb{L}_n = a_n\mathbb{Z} \times a_n\mathbb{Z}$, for some $a_n > 0$. We first prove a "discrete version" of the RSW theorem for l_1 and l_2 which are positive integer multiples of $4a_n$. Since we eventually let $a_n \to 0$, by the monotonicity property of the crossing probabilities [i.e., $\sigma^*((a,b),1,\lambda) \ge \sigma^*((c,b),1,\lambda)$ and $\sigma^*((a,b),2,\lambda) \ge \sigma^*((a,d),2,\lambda)$ for $0 < a \le c < \infty$ and $0 < b \le d < \infty$] this restriction on l_1 and l_2 will not affect the proof of the RSW theorem.

By a cell in the lattice $\mathbb{L}_n$ we will mean the closed cell, i.e., we include the perimeter of the cell. Two cells in this lattice are said to be adjacent if they have an edge in common. A cell C in $\mathbb{L}_n$ will be called vacant (occupied) if $C \cap (\bigcup_{i\ge 1} V(\xi_i)) = \varnothing$ [$C \cap (\bigcup_{i \ge 1} V(\xi_i)) \ne \varnothing$]. A (vacant/occupied) $\mathbb{L}_n$-path is a sequence of (vacant/occupied) adjacent cells. A (vacant/occupied) L–R $\mathbb{L}_n$-crossing r of the rectangle $[0,l_1]\times[0,l_2]$ is a (vacant/occupied) $\mathbb{L}_n$-path lying in the rectangle $[0,l_1]\times[0,l_2]$ with one end-cell of r having an edge on $\{0\}\times[0,l_2]$ and the other end-cell of r having an edge on $\{l_1\}\times[0,l_2]$. The T–B $\mathbb{L}_n$-crossings of the rectangle $[0,l_1]\times[0,l_2]$ can be defined similarly. We denote the $\mathbb{L}_n$-crossing probabilities as follows:

$$\sigma_n^*((l_1,l_2),1,\lambda) := P_\lambda\{\exists \text{ a vacant L–R } \mathbb{L}_n\text{-crossing of the rectangle } [0,l_1]\times[0,l_2]\},$$

$$\sigma_n^*((l_1,l_2),2,\lambda) := P_\lambda\{\exists \text{ a vacant T–B } \mathbb{L}_n\text{-crossing of the rectangle } [0,l_1]\times[0,l_2]\}.$$

Let r be a L–R $\mathbb{L}_n$-crossing of the rectangle $[0,l_1]\times[0,l_2]$. Let $\tilde{r}$ denote the piece of r in $[l_1/4, l_1]\times[0,l_2]$ after its "last intersection" with the line $\{l_1/4\}\times[0,l_2]$ and let $m(\tilde{r})$ denote the reflection of $\tilde{r}$ on $\{l_1\}\times[0,l_2]$. Also define $J(r) := \{(x_1,x_2) \in [l_1/4, 7l_1/4]\times[0,l_2]\colon (x_1,x_2)$ can be connected by a continuous curve γ to $[l_1/4, 7l_1/4]\times\{l_2\}$ such that $\gamma \subseteq [l_1/4, 7l_1/4]\times[0,l_2]$ and $\gamma \cap (\tilde{r} \cup m(\tilde{r})) = \varnothing\}$. Let $Y(r) := \sup\{x_2\colon (l_1/4, x_2) \in \tilde{r} \cap \{l_1\}\times[0,l_2]\}$ (see Figure 2).

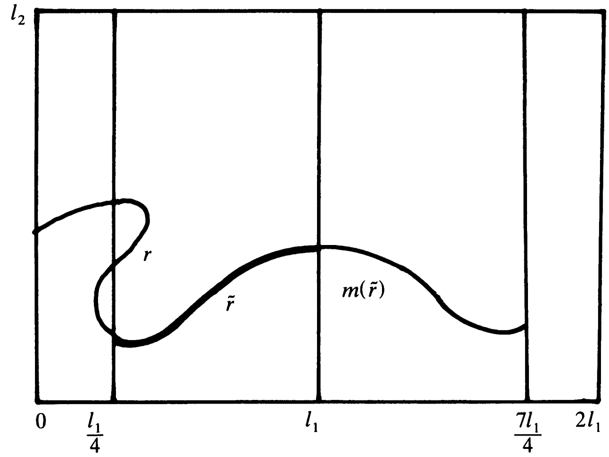

FIG. 2. *The* L–R *crossing r of* $[0, l_1] \times [0, l_2]$. *The segment $\tilde{r}$ of r after the last intersection with the line* $\{l_1/4\} \times [0, l_2]$ *is depicted by the thick line. The reflection of $\tilde{r}$ on* $\{l_1\} \times [0, l_2]$ *is $m(\tilde{r})$. The region $J(r)$ is the region above $\tilde{r} \cup m(\tilde{r})$ in the rectangle* $[l_1/4, 7l_1/4] \times [0, l_2]$.

LEMMA 4.1. $P_\lambda\{\exists$ *a vacant L–R $\mathbb{L}_n$-crossing r of the rectangle* $[0, l_1] \times [0, l_2]$ *with* $Y(r) \le l_2/2$ *and a vacant $\mathbb{L}_n$-path s with* $s \cap \tilde{r} \ne \emptyset$, $s \subseteq J(r)$ *and* $s \cap ([l_1/4, 7l_1/4] \times \{l_2\}) \ne \emptyset\} \ge K(\lambda, R, n)\sigma_n^*((l_1, l_2), 1, \lambda)\sigma_n^*((l_3, l_2), 2, \lambda)/4$, *where* $K(\lambda, R, n) = \exp(-\lambda(3R + a_n)(2R + a_n)$ *and* l_i, $i = 1, 2, 3$, *are as in Theorem* 2.3.

This lemma does not follow from the RSW lemma for bond/site percolation on the lattice because the events $\{r$ is a vacant $\mathbb{L}_n$-crossing of the rectangle $[0, l_1] \times [0, l_2]\}$ and $\{\exists$ a vacant $\mathbb{L}_n$-path s with $s \cap \tilde{r} \ne \emptyset$, $s \subseteq J(r)$ and $s \cap ([l_1/4, 7l_1/4] \times \{l_2\}) \ne \emptyset\}$ are not independent. This dependence is handled by the construction of a suitable L–R vacant $\mathbb{L}_n$-crossing of the rectangle $[l_1 - 2R, l_1] \times [0, l_2]$ which starts on $\{l_1 - 2R\} \times [0, l_2]$ at the end of the "lowest" L–R vacant $\mathbb{L}_n$-crossing of the rectangle $[0, l_1 - 2R] \times [0, l_2]$. Taking this vacant $\mathbb{L}_n$-path as the L–R vacant $\mathbb{L}_n$-crossing r of the rectangle $[0, l_1] \times [0, l_2]$, we can decompose the event $\{r$ is the lowest vacant $\mathbb{L}_n$-crossing of the rectangle $[0, l_1] \times [0, l_2]\}$ into two events A and B (say), such that the event $E := \{\exists$ a vacant $\mathbb{L}_n$-path s with $s \cap \tilde{r} \ne \emptyset$, $s \subseteq J(r)$ and $s \cap ([l_1/4, 7l_1/4] \times \{l_2\}) \ne \emptyset\}$ is positively correlated (in the FKG sense) with one of the events A (say), while conditioned on A the events E and B are independent. The details of the proof of this lemma being rather tedious and technical we refer the reader to Roy (1987).

Now let $\{a_n\}_{n \ge 1}$ be a sequence decreasing to 0 and such that $l_1/4$, l_2 and l_3 are all integer multiples of a_n for all $n \ge 1$. As $n \to \infty$, $\sigma_n^*((l_1, l_2), 1, \lambda) \to \sigma^*((l_1, l_2), 1, \lambda)$ and $\sigma_n^*((l_3, l_2), 2, \lambda) \to \sigma^*((l_3, l_2), 2, \lambda)$. Thus from Lemma 4.1

we have

$$P_\lambda\{\exists \text{ a vacant L–R crossing } \gamma \text{ of the rectangle } [0, l_1] \times [0, l_2] \text{ with } Y(\gamma) \le l_2/2 \text{ and there is a vacant path } \gamma' \text{ with } \gamma' \cap \tilde{\gamma} \ne \varnothing, \gamma' \subseteq J(\gamma) \text{ and } \gamma' \cap ([l_1/4, 7l_1/4] \times \{l_2\}) \ne \varnothing\} \ge \exp(-6R^2)\sigma^*((l_1, l_2), 1, \lambda)\sigma^*((l_3, l_2), 2, \lambda)/4.$$

Here for any L–R crossing γ of $[0, l_1] \times [0, l_2]$, $\tilde{\gamma}$, $Y(\gamma)$, $m(\tilde{\gamma})$ and $J(\gamma)$ are defined as before.

An iteration argument as in Lemma 6.1, Kesten (1982) completes the proof of the theorem. □

5. Proof of Theorem 2.4. We shall show

$$\lambda_D \le \lambda_d^* \le \lambda_D^* \le \lambda_S^* \le \lambda_H. \tag{5.1}$$

This together with Theorem 2.1 will establish Theorem 2.4.

First we clearly have

$$\lambda_d^* \le \lambda_D^*. \tag{5.2}$$

Moreover, (3.6) of Lemma 3.2 directly implies

$$\lambda_D^* \le \lambda_S^*. \tag{5.3}$$

To show

$$\lambda_D \le \lambda_D^*, \tag{5.4}$$

we observe that as in the proof of (iv) of Theorem 2.3 in Section 3, if $\lambda < \lambda_D$, we obtain

$$\sum_{k \ge 1} P_\lambda\{\exists \text{ an occupied L–R crossing of } [0, 3^k] \times [0, 3^{k+1}]\} < \infty. \tag{5.5}$$

Since there exists an occupied L–R crossing of $[0, 3^k] \times [0, 3^{k+1}]$ if and only if there does not exist a vacant T–B crossing of $[0, 3^k] \times [0, 3^{k+1}]$, from (5.5) and on an application of the Borel–Cantelli lemma we have

$$P_\lambda\{\exists \text{ a vacant L–R crossing } l_k \text{ of } [0, 3^{k+1}] \times [0, 3^k] \text{ for all large } k\} = 1,$$

$$P_\lambda\{\exists \text{ a vacant T–B crossing } t_k \text{ of } [0, 3^{k+1}] \times [0, 3^{k+2}] \text{ for all large } k\} = 1.$$

A horizontal crossing l_k of $[0, 3^{k+1}] \times [0, 3^k]$ and a vertical crossing T_k of $[0, 3^{k+1}] \times [0, 3^{k+2}]$ must intersect. Also, t_k and l_{k+1} must intersect. So the vacant crossings $\{l_k\}_{k \ge 1}$ and $\{t_k\}_{k \ge 1}$ combine to yield $P_\lambda\{\exists$ an unbounded connected vacant region in the first quadrant$\} = 1$, i.e., $P_\lambda\{d(W^*(x)) = \infty$ for some $x = (x_1, x_2)$, x_1, x_2 both rational$\} = 1$. This along with translation invariance implies $P_\lambda\{d(W^*(0)) = \infty\} > 0$, which establishes (5.4).

We remark here that the above proof is essentially two dimensional and fails in more dimensions.

Finally, we use the RSW theorem to show $\lambda_S^* \le \lambda_H$.

First we establish the following proposition.

PROPOSITION 5.1. *If* $\lambda < \lambda_S^*$, *then there is a sequence of integers* $0 \le n_1 \le n_2 \le \cdots$ *with* $n_k \uparrow \infty$ *as* $k \uparrow \infty$ *such that for every* $k \ge 1$ *and for some* $\delta > 0$ *the following hold*: (a) $(5n_{2k-1}/4) > n_{2k}$, (b) $\sigma^*((n_{2k-1}, n_{2k}), 1, \lambda) \ge \delta$ *and* (c) $\sigma^*((5n_{2k-1}/4, n_{2k}), 2, \lambda) \ge \delta$.

PROOF. We begin with a lemma whose proof is simple and can be obtained in Roy (1987), Lemma 4.8.1.

LEMMA 5.1. *Let* $n, k > 0$ *and* $\zeta > 0$ *be such that* $\sigma^*((n, (1 + 2k)n), 1, \lambda) > \zeta$. *Then for any* $t > 0$ *and for some* $f(t, k, \zeta) > 0$, $\sigma^*((n, (1 + 2t)n), 1, \lambda) > f(t, k, \zeta)$.

Since $\lambda < \lambda_S^*$, there is an increasing sequence $\{m_k\}_{k \ge 1}$ of positive reals with $m_k \uparrow \infty$ as $k \uparrow \infty$ and some $\delta_1 > 0$ such that for all $k \ge 1$, $\sigma^*((m_k, 3m_k), 1, \lambda) > \delta_1$. Now taking $n_{2k-1} = 5m_{2k}/6$ and $n_{2k} = m_k$ and applying the monotonicity property of the crossing probabilities and Lemma 5.1, we have for some $0 < \delta < \delta_1$, both (b) and (c) of Proposition 5.1 hold. This proves Proposition 5.1. □

Suppose $\lambda < \lambda_S^*$ and $\{n_k\}_{k \ge 1}$ and $\delta > 0$ as in Proposition 5.1. Let $j_k = n_{2k-1}$ and $l_k = n_{2k-1} + n_{2k}$ for all $k \ge 1$. W.l.o.g. assume $j_{k+1} > 3j_k$ for all $k \ge 1$. By the RSW theorem and the FKG lemma, for every $k \ge 1$, $P_\lambda\{\exists$ a vacant circuit surrounding the origin and lying in the annuli $S_{l_k} \setminus S_{j_k}\} \ge C(\lambda)g(\delta)$, where $C(\lambda) > 0$ is independent of k and δ and $g(\delta) > 0$ is independent of λ.

An application of the Borel–Cantelli lemma yields $P_\lambda\{\exists$ infinitely many vacant circuits surrounding the origin$\} = 1$. Thus $P_\lambda\{|W(0)| = \infty\} = 0$, i.e., $\lambda \le \lambda_H$. This completes the proof of Theorem 2.4.

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