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STOCHASTIC VOLTERRA EQUATIONS WITH ANTICIPATING COEFFICIENTS

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Stochastic Volterra equations are studied where the coefficients $F(t, s, x)$ are random and adapted to $\mathscr{F}_{s \vee t}$ rather than the customary $\mathscr{F}_{s \wedge t}$. Such a hypothesis, which is natural in several applications, leads to stochastic integrals with anticipating integrands. We interpret these as Skorohod integrals, which generalize Itô's integrals to the case where the integrand anticipates the future of the Wiener integrator. We shall nevertheless construct an adapted solution, which is even a semimartingale if the coefficients are smooth enough.

1. Introduction. Let Ω denote the space $C(\mathbb{R}_+; \mathbb{R}^k)$ equipped with the topology of uniform convergence on compact sets, $\mathscr{F}$ the Borel σ-field on Ω, P standard Wiener measure, and let $\{W_t(\omega) = \omega(t);\ t \geq 0\}$. For any $t \geq 0$, we define $\mathscr{F}_t = \sigma\{\omega(s);\ s \leq t\} \vee \mathscr{N}$, where $\mathscr{N}$ denotes the class of the elements in $\mathscr{F}$ which have zero P-measure. Our aim in this paper is to study equations of the following form, whose solution $\{X_t\}$ should be an $\mathbb{R}^d$-valued and $\mathscr{F}_t$-adapted process:

$$X_t = X_0 + \int_0^t F(t, s, X_s)\, ds + \int_0^t G_i(H_t; t, s, X_s)\, dW_s^i, \tag{1.1}$$

where we use here and everywhere below the Einstein convention of summation upon repeated indices [i.e., (1.1) should be read with a "$\sum_{i=1}^k$" added in front of the second integral], and $\{H_t\}$ is an $\mathbb{R}^p$-valued and $\mathscr{F}_t$-adapted process, $F(t, s, x)$, maps $\Omega \times \{s, t;\ 0 \leq s \leq t\} \times \mathbb{R}^d$ into $\mathbb{R}^d$ and $F(t, s, x)$ is $\mathscr{F}_t$-measurable, and $G_1(h; t, s, x), \dots, G_k(h; t, s, x)$ map $\Omega \times \mathbb{R}^p \times \{s, t;\ 0 \leq s \leq t\} \times \mathbb{R}^d$ into $\mathbb{R}^d$ and $G_1(h; t, s, x), \dots, G_k(h; t, s, x)$ are $\mathscr{F}_s$-measurable.

A special case of equation (1.1) is a more standard Volterra equation

$$X_t = X_0 + \int_0^t J(t, s) X_s\, ds + \int_0^t K_i(H_t, t, s) X_s\, dW_s^i, \tag{1.2}$$

where $J(t, s)$ and $K_1(h; t, s), \dots, K_k(h; t, s)$ are $d \times d$ matrices. The novelty here is that $G_1(H_t; t, s, x), \dots, G_k(H_t; t, s, x)$ are $\mathscr{F}_t$-adapted, and not $\mathscr{F}_s$-adapted, i.e., the integrands in the stochastic integrals are *anticipating*.

Note that $G_1(H_t; t, s, X_s), \dots, G_k(H_t; t, s, X_s)$ anticipate the increments of $\{W_t\}$ between s and t in a special and restrictive way, namely through H_t. We shall explain below the reason for this restriction.

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Volterra equations with kernels which *anticipate* in the way described above arise in applications (in particular in finance theory, see [4], [5] and Example 3.1) and as such were the motivation for this work.

Clearly the problem in studying the above class of equations is that the integrands in the stochastic integrals are not adapted, and therefore one cannot use as usual the Itô integral to interpret the equation. Our approach is to use the Skorohod integral [20] to interpret the stochastic integrals in (1.1) and (1.2). Recent progress in interpreting the Skorohod integral (see [11], [12] and [16]) have made this possible. We explain our interpretation of the equation in Section 2. Note that recently there has been other work concerning stochastic differential equations where the solution itself is anticipating (which is not the case here). In [19], the Skorohod integral was used to solve a one-dimensional linear equation with an anticipating initial condition. In [13], [14] and [15], another kind of generalized stochastic integral, which generalizes the Stratonovich integral, was used to solve stochastic differential equations with an anticipating initial condition, or with boundary conditions (instead of the usual initial condition).

This article builds on previous work concerning stochastic Volterra equations. Equations where the kernel is adapted to $\mathscr{F}_s$ were studied among others in [2], [7], [17] and [18]; Berger and Mizel considered linear stochastic Volterra equations with anticipating integrands in [3]. Our results differ from theirs since the stochastic integral is not the same, also the discussion in [3] uses in an essential way the linearity of the coefficients. In [17], one of us commented that such equations can also be studied using an "enlargement of filtration" approach, but the technique used in the present paper yields much better (and perhaps more "natural") results.

The paper is organized as follows. Section 2 contains a presentation of some results concerning the Skorohod integral, which will be used later, together with the precise interpretation of equation (1.1). The existence and uniqueness of a solution to equation (1.1) is proved in two steps in Sections 3 and 4. In Section 5, we establish, under additional assumptions, the existence of an a.s. continuous modification of the solution process. This allows us to deduce a weaker existence and uniqueness result, under local Lipschitz conditions. Under still stronger regularity assumptions on the coefficients, we show in Section 6 that the unique continuous solution is a semimartingale.

Let us point out the fact that the reason for restricting ourselves to a Wiener driving process (versus a more general semimartingale) is the fact that the Skorohod integral and the derivations which we will be using below are only defined on Wiener space.

The following notation is used throughout the paper: $c(\alpha, \beta)$ stands for a constant which depends only on α and β, and whose value may vary from one occurrence to another.

2. The Skorohod integral. Most of this section is a review of some basic notions and a few results from [11]. Let again $\Omega = C(\mathbb{R}_+; \mathbb{R}^k)$, $\mathscr{F}$ be its Borel field and P denote Wiener measure on $(\Omega, \mathscr{F})$. $W_t(\omega) = \omega(t)$. Let $\mathscr{F}_t{}^0 =$

$\sigma\{W_s;\ 0 \le s \le t\}$ and $\mathscr{F}_t = \mathscr{F}_t^0 \vee \mathscr{N}$, where $\mathscr{N}$ denotes the class of P-null sets of $\mathscr{F}$. For $h \in L^2(\mathbb{R}_+; \mathbb{R}^k)$, we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^\infty (h(t), dW_t).$$

Let $\mathscr{S}$ denote the dense subset of $L^2(\Omega, \mathscr{F}, P)$ consisting of those classes of random variables of the form:

$$F = f(W(h_1), \dots, W(h_n)), \tag{2.1}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in L^2(\mathbb{R}_+; \mathbb{R}^k)$. If F has the form (2.1), we define its derivative in the direction i as the process $\{D_t^i F;\ t \ge 0\}$ defined by

$$D_t^i F = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(W(h_1), \dots, W(h_n)) h_k^i(t).$$

DF will stand for the k-dimensional process $\{D_t F = (D_t^1 F, \dots, D_t^k F)';\ t \ge 0\}$.

Proposition 2.1. *For $i = 1, \dots, k$, D^i is an unbounded closable operator from $L^2(\Omega)$ into $L^2(\Omega \times \mathbb{R}_+)$. We identify D^i with its closed extension, and denote by $\mathbb{D}_i^{1,2}$ its domain. $\mathbb{D}^{1,2} = \bigcap_{i=1}^k \mathbb{D}_i^{1,2}$ is the domain of D: $L^2(\Omega) \to L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^k)$.*

Note that $\mathbb{D}_i^{1,2}$ (resp. $\mathbb{D}^{1,2}$) is the closure of $\mathscr{S}$ with respect to the norm

$$\|F\|_{i,1,2} = \|F\|_2 + \| \|D^i F\|_{L^2(\mathbb{R}_+)} \|_2$$

(resp. with respect to the norm

$$\|F\|_{1,2} = \|F\|_2 + \sum_{i=1}^{k} \| \|D^i F\|_{L^2(\mathbb{R}_+)} \|_2).$$

D^i is a local operator, in the sense that: $D_t^i F = 0\ dP \times dt$ a.e. on $\{F = 0\} \times \mathbb{R}_+$. We denote by $\mathbb{D}_{i,\mathrm{loc}}^{1,2}$ the set of measurable F's which are such that there exists a sequence $\{(\Omega_n, F_n);\ n \in \mathbb{N}\} \subset \mathscr{F} \times \mathbb{D}_i^{1,2}$ with the two properties

(i) $$\Omega_n \uparrow \Omega \quad \text{a.s., } n \to \infty,$$

(ii) $$F_n|_{\Omega_n} = F|_{\Omega_n}, \qquad n \in \mathbb{N}.$$

For $F \in \mathbb{D}_{i,\mathrm{loc}}^{1,2}$, we define without ambiguity $D_t^i F$ by: $D_t^i F = D_t^i F_n$ on $\Omega_n \times \mathbb{R}_+$, $\forall\ n \in \mathbb{N}$. $\mathbb{D}_{\mathrm{loc}}^{1,2}$ is defined similarly.

For $i = 1, \dots, k$, we define δ_i the Skorohod integral with respect to $\{W_t^i\}$ as the adjoint of D^i, i.e., Dom δ_i is the set of $u \in L^2(\Omega \times \mathbb{R}_+)$ which are such that there exists a constant c with

$$\left| E\int_0^\infty D_t^i F u_t\, dt \right| \le c\|F\|_2, \qquad \forall\ F \in \mathscr{S}.$$

If $u \in$ Dom δ_i, $\delta_i(u)$ is defined as the unique element of $L^2(\Omega)$ which satisfies

$$E(\delta_i(u)F) = E\int_0^\infty D_t^i F u_t\, dt, \qquad \forall\ F \in \mathscr{S}.$$

Let $\mathbb{L}_i^{1,2} = L^2(\mathbb{R}_+; \mathbb{D}_i^{1,2})$. We have that $\mathbb{L}_i^{1,2} \subset \operatorname{Dom} \delta_i$, and for $u \in \mathbb{L}_i^{1,2}$,

$$E\left[\delta_i(u)^2\right] = E\int_0^\infty u_t^2\,dt + E\int_0^\infty\int_0^\infty D_s^i u_t\, D_t^i u_s\,ds\,dt. \tag{2.2}$$

Note that $\{u \in \mathbb{L}^2(\Omega \times \mathbb{R}_+);\ u$ is $\mathscr{F}_t$ progressively measurable$\} \subset \operatorname{Dom} \delta_i$, and for such a u, $\delta_i(u)$ coincides with the usual Itô integral. Note that when u is progressively measurable, $D_s u_t = 0$ for $s > t$, so that (2.2) is consistent with the formula in the adapted case.

REMARK 2.2. From (2.2), the $L^2(\Omega)$ norm of a Skorohod integral can be estimated in terms of the $L^2(\Omega)$ norm of its integrand plus a norm of its derivative. This means that an $L^2(\Omega)$ estimate of the last term in (1.1) introduces the derivative of the solution X. This creates a crucial difficulty if we try to apply standard techniques to study the existence and uniqueness of (1.1). That is the motivation for letting G_i anticipate the increments $\{W_r - W_s;\ s \le r \le t\}$ only through the process $\{H_t\}$.

Note that if $u \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{D}_i^{1,2})$, then for any $T > 0$, $u1_{[0,T]} \in \mathbb{L}_i^{1,2}$ and we can define

$$\int_0^T u_t\,dW_t^i = \delta_i\left(u1_{[0,T]}\right).$$

The Skorohod integral is a local operation on $L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{D}_i^{1,2})$ in the sense that if $u, v \in L^2_{\text{loc}}(\mathbb{R}_+; \mathbb{D}_i^{1,2})$, $\int_0^t u_s\,dW_s^i = \int_0^t v_s\,dW_s^i$ a.s. on $\{\omega;\ u_s(\omega) = v_s(\omega)$ for almost all $s \le t\}$.

Let $\mathbb{L}_{i,\text{loc}}^{1,2}$ denote the set of measurable processes u which are such that for any $T > 0$ there exists a sequence $\{(\Omega_n^T, u_n^T);\ n \in \mathbb{N}\} \subset \mathscr{F} \times \mathbb{L}_i^{1,2}$ such that

(i) $$\Omega_n^T \uparrow \Omega \quad \text{a.s., as } n \to \infty,$$

(ii) $$u = u_n^T\ dP \times dt \quad \text{a.e. on } \Omega_n^T \times [0,T],\ n \in \mathbb{N}.$$

For $u \in \mathbb{L}_{i,\text{loc}}^{1,2}$ we can define its Skorohod integral with respect to W_t^i by

$$\int_0^t u_s\,dW_s^i = \int_0^t u_{n,s}^T\,dW_s^i \quad \text{on } \Omega_n^T \times [0,T].$$

Finally, $\mathbb{L}^{1,2} = \bigcap_{i=1}^k \mathbb{L}_i^{1,2}$, and $\mathbb{L}_{\text{loc}}^{1,2}$ is defined similarly as $\mathbb{L}_{i,\text{loc}}^{1,2}$.

We now introduce the particular class of integrands which we shall use below. Let u: $\mathbb{R}_+ \times \Omega \times \mathbb{R}^p \to \mathbb{R}$ satisfy:

(i) $\forall x \in \mathbb{R}^p$, $\quad (t,\omega) \to u(t,\omega,x)$ is $\mathscr{F}_t$ progressively measurable.

(ii) $\forall (t,\omega) \in \mathbb{R}_+ \times \Omega, \quad u(t,\omega,\cdot) \in C^1(\mathbb{R}^p).$

(iii) For some increasing function φ: $\mathbb{R}_+ \to \mathbb{R}_+$,
$$|u(t,\omega,x)| + |u'(t,\omega,x)| \le \varphi(|x|), \qquad \forall (t,\omega,x) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^p,$$

where $u'(t,x)$ stands for the gradient $(\partial u/\partial x)(t,x)$.

Let θ be a p-dimensional random vector such that

(iv) $$\theta^j \in \mathbb{D}_i^{1,2} \cap L^\infty(\Omega), \qquad j = 1, \dots, p.$$

Let us fix $T > 0$, and consider

$$I^i(x) = \int_0^T u(t, x)\, dW_t^i.$$

Define moreover $v_t = u(t, \theta)$. Under conditions (i)–(iv), the following proposition holds.

PROPOSITION 2.3. *The random field $\{I^i(x);\ x \in \mathbb{R}^p\}$ defined above possesses an a.s. continuous modification, so that we can define the r.v. $I^i(\theta)$. Moreover $v \in \operatorname{Dom} \delta_i$, and the following holds*:

$$\int_0^T v_t\, dW_t^i = I^i(\theta) - \int_0^T u'(t, \theta) D_t^i \theta\, dt. \tag{2.3}$$

We insist that $I^i(\theta)$ stands for $\int_0^T u(t, x)\, dW_t^i|_{x=\theta}$. Similar notation will be used below.

Proposition 2.3 is proved in Nualart and Pardoux [10], under slightly different conditions. We shall need below a localized version of that result.

We replace condition (iv) by

(iv′) $$\theta^j \in \mathbb{D}_{i,\mathrm{loc}}^{1,2}, \qquad j = 1, \dots, p.$$

Under conditions (i), (ii), (iii) and (iv′), $v \in (\operatorname{Dom} \delta_i)_{\mathrm{loc}}$, in the sense that there exists a sequence $\{(\Omega_n, v_n);\ n \in \mathbb{N}\} \subset \mathscr{F} \times \operatorname{Dom} \delta_i$ such that $\Omega_n \uparrow \Omega$ a.s. and $v_n|_{\Omega_n} = v|_{\Omega_n}$. Indeed, let $\{(\Omega_n', \theta_n)\}$ be a localizing sequence for θ in $(\mathbb{D}_i^{1,2})^p$, and $\{\psi_n;\ n \in \mathbb{N}\} \subset C_c^\infty(\mathbb{R}^p; \mathbb{R}^p)$ satisfy $\psi_n(x) = x$ whenever $|x| \le n$. Define

$$v_n(t) = u(t, \psi_n(\theta_n)),$$
$$\Omega_n = \Omega_n' \cap \{|\theta| \le n\}.$$

Then $\{(\Omega_n, v_n);\ n \in \mathbb{N}\}$ satisfies the above conditions.

It is then natural to define the Skorohod integral $\int_0^T v_t\, dW_t^i$ again by formula (2.3), and the latter coincides with $\int_0^T v_n(t)\, dW_t^i$ on Ω_n. Note that our definition of $\int_0^T v_t\, dW_t^i$ does not depend on the localizing sequence of v in $\operatorname{Dom} \delta_i$, provided that sequence is of the form $\{u(\cdot, \theta_n)\}$ with θ_n satisfying (iv).

3. Statement of the problem: Interpretation of equation (1.1). Our aim is to study the equation

$$X_t = X_0 + \int_0^t F(t, s, X_s)\, ds + \int_0^t G_i(H_t; t, s, X_s)\, dW_s^i, \tag{3.1}$$

where we use here and henceforth the convention of summation upon repeated indices. We define $D = \{(t, s) \in \mathbb{R}_+^2;\ 0 \le s \le t\}$. The coefficients F and G are given as follows: F: $\Omega \times D \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable and for each $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $F(\cdot, s, x)$ is $\mathscr{F}_t$ progressively measurable on $\Omega \times [s, +\infty)$. For $i =$

$1,\dots,k$, G_i: $\Omega \times \mathbb{R}^p \times D \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable, for each (h,t,x), $G_i(h;t,\cdot,x)$ is $\mathscr{F}_s$ progressively measurable on $\Omega \times [0,t]$, and for each (ω,t,s,x), $G_i(\cdot;t,s,x)$ is of class C^1.

$\{H_t\}$ is a given progressively measurable p-dimensional process. It will follow from these hypotheses that we shall be able to construct a progressively measurable solution $\{X_t\}$. Therefore, for each t, the process

$$\{G_i(H_t;t,s,X_s);\, s \in [0,t]\}$$

is of the form $v_s = u(s,\theta)$ with $u(s,h) = G_i(h;t,s,X_s)$ and $\theta = H_t$. We shall impose below conditions on G, $\{H_t\}$ and the solution $\{X_t\}$ so as to satisfy the requirements (i), (ii), (iii) and (iv′) of the last section. In particular we shall consider only nonanticipating solutions. Therefore the stochastic integrals in (3.1) will be interpreted according to (2.3), i.e.,

$$\int_0^t G_i(H_t;t,s,X_s)\, dW_s^i = \int_0^t G_i(h;t,s,X_s)\, dW_s^i|_{h=H_t} - \int_0^t G_i'(H_t;t,s,X_s) D_s^i H_t\, ds. \tag{3.2}$$

In other words, we can rewrite (3.1) as

$$X_t = X_0 + \int_0^t \tilde{F}(t,s,X_s)\, ds + \int_0^t G_i(h;t,s,X_s)\, dW_s^i|_{h=H_t}, \tag{3.3}$$

where

$$\tilde{F}(t,s,x) = F(t,s,x) - G_i'(H_t;t,s,x) D_s^i H_t,$$

and the stochastic integrals are now the usual Itô integrals. We shall show below that (3.3) makes sense for any progressively measurable process X which satisfies $X \in \bigcap_{t>0} L^q(0,t)$ a.s., for some $q > p$. We shall find such a solution to (3.3); it will then follow from (3.2) that it is a solution to (3.1). Similarly, uniqueness for (3.1) in the above class will follow from uniqueness for (3.3) in that class.

Let us close this section with an example from finance theory.

EXAMPLE 3.1. Let X_t denote the capital stock of an economic model at time t, and Y_t denote the investment (rate) process, where negative Y_t is possible in some cases. The random evolution of X_t could be governed by a Volterra equation of the form

$$X_t = X_0 + \int_0^t F(t,Y_s,s,X_s)\, ds + \int_0^t G_i(H_t;t,s,X_s)\, dW_s^i.$$

The fact that $f_t = F(t,Y_s,s,X_s)$ is $\mathscr{F}_t$-measurable is natural if we think of f_t as the "payback" rate at time t from investment Y_s at time s, which clearly should depend on the capital stock X_s, as well as random ($\mathscr{F}_t$-measurable) returns to investment. Likewise, $G_i(H_t,t,s,X_s)$ reflects the random effects of depreciation and stochastic growth. A limited example of G_i is given in the

Cox, Ingersoll and Ross [4] model of capital stock growth. More general examples are shown in Duffie and Huang [5].

4. Existence and uniqueness under strong hypotheses. Let us formulate a set of further hypotheses (those stated in Section 3 are assumed to hold throughout the paper), under which we will establish a first result of the existence and uniqueness of a solution of equations of the form (1.1).

Let B be an open bounded subset of $\mathbb{R}^p$, $K > 0$ and $q > p$ s.t.

$$X_0 \in L^q(\Omega, \mathscr{F}_0, P; \mathbb{R}^d), \tag{H.1}$$

$$P(H_t \in B, \forall\, t \geq 0) = 1, \tag{H.2}$$

$$H \in (\mathbb{L}^{1,2})^p, \qquad |D_s H_t| \leq K \quad \text{a.s.}, 0 \leq s \leq t, \tag{H.3}$$

$$|F(t,s,x)| + \sum_{i=1}^{k} |G_i(h;t,s,x)| + \sum_{i=1}^{k} |G_i'(h;t,s,x)| \leq K(1 + |x|), \tag{H.4}$$

for any $0 \leq s \leq t$, $h \in B$, $x \in \mathbb{R}^d$ and a.s.

$$\begin{aligned} &|F(t,s,x) - F(t,s,y)| + \sum_{i=1}^{k} |G_i(h;t,s,x) - G_i(h;t,s,y)| \\ &\quad + \sum_{i=1}^{k} |G_i'(h;t,s,x) - G_i'(h;t,s,y)| \leq K|x-y|, \end{aligned} \tag{H.5}$$

for any $0 \leq s \leq t$, $h \in B$, $x, y \in \mathbb{R}^d$.

Note that from now on q will be a fixed real number s.t. $q > p$ and (H.1) holds. $L^q_{\text{prog}}(\Omega \times (0,t))$ will stand for the space $L^q(\Omega \times (0,t), \mathscr{P}_t, P \times \lambda)$, where $\mathscr{P}_t$ denotes the σ-algebra of progressively measurable subsets of $\Omega \times (0,t)$ and λ denotes the Lebesgue measure on $(0,t)$.

Lemma 4.1. *Let $X \in \bigcap_{t>0} L^q_{\text{prog}}(\Omega \times (0,t))$, where $q > p$, and suppose that* (H.4) *is in force. Then for any $t > 0$ and $i \in \{1, \dots, k\}$, the random field*

$$\left\{ \int_0^t G_i(h;t,s,X_s)\, dW_s^i;\ h \in B \right\}$$

possesses an a.s. continuous modification.

Proof. Using Burkholder and Gundy's and Hölder's inequalities together with (H.4), we obtain

$$\begin{aligned} E&\left(\left| \int_0^t G_i(h;t,s,X_s)\, dW_s^i - \int_0^t G_i(k;t,s,X_s)\, dW_s^i \right|^q \right) \\ &\leq c(t,q) E \int_0^t |G_i(h;t,s,X_s) - G_i(k;t,s,X_s)|^q\, ds \\ &\leq c(t,q) K^q |h-k|^q E \int_0^t (1 + |X_s|)^q\, ds. \end{aligned}$$

The result now follows from the multidimensional generalization of Kolmogorov's lemma (see, e.g., Sznitman [21] or Meyer [9]). □

We can and will from now on assume that for t fixed, the random field

$$\left\{\int_0^t G_i(h;t,s,X_s)\,dW_s^i;\ h\in B\right\}$$

is a.s. continuous in h, provided $X \in L^q_{\text{prog}}(\Omega\times(0,t))$.

From $X\in\bigcap_{t>0}L^q_{\text{prog}}(\Omega\times(0,t))$, define

$$I_t(X,h)=\int_0^t G_i(h;t,s,X_s)\,dW_s^i,\qquad h\in\mathbb{R}^p,\ t>0,$$

$$J_t(X)=\int_0^t \tilde F(t,s,X_s)\,ds+I_t(X,H_t).$$

LEMMA 4.2. *For any $t>0$, $\exists\ c(q,t)$ s.t.*

$$E(|J_t(X)|^q)\le c(q,t)\left(1+E\int_0^t|X_s|^q\,ds\right).$$

PROOF.

$$\left|\int_0^t\tilde F(t,s,X_s)\,ds\right|^q\le c(q,t)\int_0^t|\tilde F(t,s,X_s)|^q\,ds$$

$$\le c(q,t)\left(1+\int_0^t|X_s|^q\,ds\right),$$

where we have used (H.2), (H.3) and (H.4).

$$|I_t(X,H_t)|\le\sup_{h\in B}|I_t(X,h)|.$$

It is easy to show, using in particular (H.4) and Lebesgue's dominated convergence theorem, that the mapping

$$h\to I_t(X,h)$$

from $\mathbb{R}^p$ into $L^q(\Omega)$ is differentiable, and that

$$\frac{\partial I_t(X,h)}{\partial h_j}=\int_0^t\frac{\partial G_i}{\partial h_j}(h;t,s,X_s)\,dW_s^i.$$

Since $q>p$, we can infer from Sobolev's embedding theorem (see, e.g., Adams [1], Theorem 5.4.1c)

$$E\left(\sup_{h\in B}|I_t(X,h)|^q\right)\le c(q)E\int_B\left(|I_t(X,h)|^q+\sum_{j=1}^p\left|\frac{\partial I_t}{\partial h_j}(X,h)\right|^q\right)dh.$$

It then follows from the Burkholder–Gundy inequality that

$$E\Big(\sup_{h\in B}|I_t(X,h)|^q\Big)$$
$$\le c(q,t)\,E\int_B\int_0^t\left(|G_i(h;t,s,X_s)|^q+\sum_{j=1}^p\left|\frac{\partial G_i}{\partial h_j}(h;t,s,X_s)\right|^q\right)ds\,dh$$
$$\le c(q,t)\Big(\int_B dh\Big)\Big(1+E\int_0^t|X_s|^q\,ds\Big),$$

where we have used (H.4) and the relative compactness of B. □

A similar argument, but using (H.5) instead of (H.4), yields the following lemma.

LEMMA 4.3. *For any* $t>0$, $\exists\ c(q,t)$ *s.t.*

$$E\big(|J_t(X)-J_t(Y)|^q\big)\le c(q,t)\,E\int_0^t|X_s-Y_s|^q\,ds.$$

If moreover τ *is a stopping time*,

$$E\big(|J_{t\wedge\tau}(X)-J_{t\wedge\tau}(Y)|^q\big)\le c(q,t)\,E\int_0^{t\wedge\tau}|X_s-Y_s|^q\,ds.$$

We are now in a position to prove the main result of this section. Note that we prove uniqueness only among nonanticipating solutions.

THEOREM 4.4. *Under conditions* (H.1), (H.2), (H.3), (H.4) *and* (H.5), *there exists a unique element* $X\in\bigcap_{t>0}L^q_{\mathrm{prog}}(\Omega\times(0,t))$, *which solves equation* (3.1). *Moreover, if* τ *is a stopping time, uniqueness holds on the random interval* $[0,\tau]$.

PROOF. Equation (3.1) can be rewritten as

$$X_t=X_0+J_t(X),\qquad t\ge 0. \tag{4.1}$$

Uniqueness: Let $X,Y\in\bigcap_{t>0}L^q_{\mathrm{prog}}(\Omega\times(0,t))$ and τ be a stopping time, such that

$$X_t=X_0+J_t(X),\qquad 0\le t\le\tau,$$
$$Y_t=X_0+J_t(Y),\qquad 0\le t\le\tau,$$
$$X_{t\wedge\tau}-Y_{t\wedge\tau}=J_{t\wedge\tau}(X)-J_{t\wedge\tau}(Y).$$

From Lemma 4.3,

$$E\big(|X_{t\wedge\tau}-Y_{t\wedge\tau}|^q\big)\le c(q,t)\,E\int_0^{t\wedge\tau}|X_s-Y_s|^q\,ds$$
$$\le c(q,t)\,E\int_0^t|X_{s\wedge\tau}-Y_{s\wedge\tau}|^q\,ds.$$

The result now follows from Gronwall's lemma.

Existence: Lemmas 4.2 and 4.3 allow us to mimic Itô's classical proof. Let us define a sequence $\{X_t^n,\ t \geq 0,\ n \in \mathbb{N}\}$ as follows:

$$(4.2)\qquad \begin{aligned} X_t^0 &= X_0, && t \geq 0,\\ X_t^{n+1} &= X_0 + J_t(X^n), && t \geq 0,\ n \in \mathbb{N}. \end{aligned}$$

Using Lemma 4.2, we show inductively that

$$X^n \in \bigcap_{t>0} L^q_{\text{prog}}(\Omega \times (0,t)), \qquad n \in \mathbb{N}.$$

It then follows from Lemma 4.3 that

$$E\big(|X_t^{n+1} - X_t^n|^q\big) \leq c(q,t)\int_0^t E\big(|X_s^n - X_s^{n-1}|^q\big)\,ds.$$

A classical argument then shows that

$$E\big(|X_t^{n+1} - X_t^n|^q\big) \leq E\big(|X_0|^q\big)\frac{C_{q,t}^{n+1}t^{n+1}}{(n+1)!}$$

which implies that X^n is a Cauchy sequence in $L^q_{\text{prog}}(\Omega \times (0,t))$; $\forall\ t > 0$. Then there exists X s.t. $X^n \to X$ in $\bigcap_{t>0} L^q_{\text{prog}}(\Omega \times (0,t))$, and using again Lemma 4.3, we can pass to the limit in (4.2), yielding that X solves (4.1). □

5. An existence and uniqueness result under weaker assumptions. Our aim in this section is to "localize" the result of Section 4. We formulate a new set of weaker hypotheses.

(H.1′) X_0 is $\mathscr{F}_0$-measurable.

(H.2′) $H \in (\mathbb{L}^{1,2}_{\text{loc}})^p$, $\{H_t\}$ is a progressively measurable process which can be localized in $(\mathbb{L}^{1,2})^p$ by a progressively measurable sequence.

We assume that there exists an increasing progressively measurable process $\{U_t,\ t \geq 0\}$ with values in $\mathbb{R}_+$, such that

$$(\text{H.3}')\qquad |H_t| + \sum_{i=1}^k |D_s^i H_t| \leq U_t \quad \text{a.s., } 0 \leq s \leq t.$$

Finally, we suppose that for any $N > 0$, there exists an increasing progressively measurable process $\{V_t^N;\ t \geq 0\}$ with values in $\mathbb{R}_+$, such that

$$(\text{H.4}')\qquad \begin{aligned} &|F(t,s,x)| + \sum_{i=1}^k |G_i(h;t,s,x)| + \sum_{i=1}^k |G_i'(h;t,s,x)|\\ &\qquad \leq V_t^N(1+|x|), \qquad \forall\ |h| \leq N,\ 0 \leq s \leq t,\ x \in \mathbb{R}^d, \end{aligned}$$

$$(\text{H.5}')\qquad \begin{aligned} &|F(t,s,x) - F(t,s,y)| + \sum_{i=1}^k |G_i(h;t,s,x) - G_i(h;t,s,y)|\\ &\qquad + \sum_{i=1}^k |G_i'(h;t,s,x) - G_i'(h;t,s,y)| \leq V_t^N|x-y|,\\ &\qquad\qquad \forall\ |h| \leq N,\ 0 \leq s \leq t,\ x,y \in \mathbb{R}^d. \end{aligned}$$

Let again q be a fixed real number, with $q > p$. We have the following theorem.

THEOREM 5.1. *Equation* (3.1) *has a unique solution in the class of progressively measurable processes which satisfy*

$$X \in \bigcap_{t>0} L^q(0,t) \quad a.s.$$

PROOF. (a) Let us first see how equation (3.1) makes sense if $X \in \bigcap_{t>0} L^q(0,t)$ a.s. That is, we have to show that for $t > 0$, fixed,

$$\left\{ \int_0^t G_i(h;t,s,X_s)\,dW_s^i;\ h \in \mathbb{R}^p \right\}$$

is a well-defined random field which possesses an a.s. continuous modification. For that sake, we define

$$\tau_n = \inf\left\{ t;\ \int_0^t |X_s|^q\,ds \ge n \text{ or } V_t^N \ge n \right\}.$$

The argument of Lemma 4.1 can be used to show that

$$h \to \int_0^{t\wedge\tau_n} G_i(h;t,s,X_s)\,dW_s^i$$

possesses an a.s. continuous modification on $\{|h| \le N\}$. Since this is true for any n and N, and $\bigcup_n \{\tau_n \ge t\} = \Omega$ a.s., the result follows.

(b) *Existence*: We want to show existence on an arbitrary interval $[0,T]$ (T will be fixed below). Let $\{H^n;\ n \in \mathbb{N}\}$ denote a progressively measurable localizing sequence for H in $(\mathbb{L}^{1,2})^p$ on $[0,T]$. Since from (H.3′) $\sup_{t\le T}|H_t|$ is a.s. finite, we can and do assume w.l.o.g. that

$$|H_t^n(\omega)| \le n, \qquad \forall\,(t,\omega) \in [0,T]\times\Omega.$$

Note that $H_t(\omega) = H_t^n(\omega)$ a.s. on Ω_n^T, $\forall\ t \in [0,T]$, where $\Omega_n^T \uparrow \Omega$ a.s. as $n \to \infty$.

We moreover define

$$X_0^n = X_0 1_{\{|X_0| \le n\}},$$
$$S_n = \inf\left\{ t;\ \sup_{s\le t} |D_s H_t^n| \vee V_t^n \ge n \right\}.$$

We consider the equation

$$X_t^n = X_0^n + \int_0^t \tilde{F}^n(t,s,X_s^n)\,ds + \int_0^t G_i^n(h;t,s,X_s^n)\,dW_s^i|_{h=H_t^n}, \tag{5.1}$$

where

$$\tilde{F}^n(t,s,x) = 1_{[0,S_n]}(s)\left[F(t,s,x) - G_i'(H_t^n;t,s,x)D_s^i H_t^n\right],$$
$$G_i^n(h;t,s,x) = 1_{[0,S_n]}(s)G_i(h;t,s,x).$$

It is not hard to see that Theorem 4.4 applies to equation (5.1).

Define

$$\bar{S}_n(\omega) = \begin{cases} S_n(\omega) \wedge \inf\left\{t \le T;\ \int_0^t |H_s(\omega) - H_s^n(\omega)|\, ds > 0\right\} & \text{if } |X_0(\omega)| \le n, \\ 0 & \text{otherwise.} \end{cases}$$

$\bar{S}_n$ is a stopping time, and it follows from the uniqueness part of Theorem 4.4 that, if $m > n$

$$X_t^m = X_t^n \quad \text{on } \left[0, \bar{S}_n\right], \quad \text{a.s.}$$

Since moreover $\{\bar{S}_n = T\} \uparrow \Omega$ a.s., we can define the process $\{X_t\}$ on $[0, T]$ by

$$X_t = X_t^n \quad \text{on } \left[0, \bar{S}_n\right], \forall\, n \in \mathbb{N}.$$

Clearly, $X \in \bigcap L^q(0, T)$ a.s. and solves (3.1) on $[0, T]$. Since T is arbitrary, the existence is proved.

(c) *Uniqueness*: It suffices to prove uniqueness on an arbitrary interval $[0, T]$. Let $\{\bar{X}_t,\ t \in [0, T]\}$ be a progressively measurable process s.t. $\bar{X} \in L^q(0, T)$ a.s. and $\bar{X}$ solves (3.1). It suffices to show that $\bar{X}$ coincides with the solution we have just constructed. Let

$$\tilde{S}_n(\omega) = \bar{S}(\omega) \wedge \inf\left\{t \le T;\ \int_0^t |\bar{X}_s(\omega)|^q\, ds \ge n\right\},$$

$$\tilde{X}_t^n = \bar{X}_{t \wedge \tilde{S}_n},$$

$\tilde{X}^n \in L^q(\Omega \times [0, T])$ and it solves equation (5.1) with S_n replaced by $\tilde{S}_n$. Then

$$\tilde{X}_t^n(\omega) = X_t(\omega)\ dt \times dP \quad \text{a.e. on } \left[0, \tilde{S}_n\right].$$

The result follows from the fact that $\{\tilde{S}_n \ge T\} \uparrow \Omega$ a.s. □

Note that the above solution satisfies in fact $X \in \bigcap_{q \ge 1} \bigcap_{t \ge 0} L^q(0, t)$ a.s.

6. Continuity of the solution. We want now to give additional conditions under which the solution of equation (3.1) is an a.s. continuous process.

(H.6) $\forall\, (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $t \to F(t, s, x)$ is a.s. continuous on $(s, +\infty)$.

(H.7) $\{H_t;\ t \ge 0\}$ is a.s. continuous.

(H.8) $\forall\, i \in \{1, \dots, k\}$, $s \in \mathbb{R}_+$, $t \to D_s^i H_t$ is a.s. continuous on $(s, +\infty)$.

(H.9) $\forall\, (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $i \in \{1, \dots, k\}$,
$(t, h) \to G_i'(h; t, s, x)$ is a.s. continuous on $(s, +\infty) \times \mathbb{R}^p$.

We also suppose that there exist $\alpha > 0$, $l > 0$ s.t.

(H.10) $\forall\, N > 0$, $|h| \le N$, $0 \le s \le t \wedge r$, $x \in \mathbb{R}^d$, $|t - r| \le 1$,
there exists an increasing process $\{V_t^N;\ t \ge 0\}$ such that
$|G_i(h; t, s, x) - G_i(h; r, s, x)| \le V_t^N |t - r|^\alpha (1 + |x|^l)$.

THEOREM 6.1. *Under conditions* (H.1′)–(H.5′) *and* (H.6)–(H.10), *the unique solution of equation* (3.1) [*which is progressively measurable and belongs a.s. to* $\bigcap_{q\geq 1}\bigcap_{t>0}L^q(0,t)$] *has an a.s. continuous modification.*

PROOF. We need to show only that whenever $X \in \bigcap_{q\geq 1}\bigcap_{t>0}L^q(0,t)$ a.s., $\{J_t(X);\, t\geq 0\}$ has an a.s. continuous modification.

(a) We first show that $t \to \int_0^t \tilde{F}(t,s,X_s)\,ds$ is a.s. continuous. Note that (H.6), (H.7), (H.8) and (H.9) imply that $\forall\,(s,x)\in \mathbb{R}_+\times\mathbb{R}^d$,

$$(6.1)\qquad t\to \tilde{F}(t,s,x)\quad \text{is a.s. continuous on } (s,+\infty).$$

Moreover, from (H.2′), (H.3′), (H.4′) and the fact that $X \in \bigcap_{q\geq 1}\bigcap_{t>0}L^q(0,t)$ a.s., for any $T>0$, there exists a process $\{Z_s^T;\, s\in[0,T]\}$ such that

$$(6.2)\qquad |\tilde{F}(t,s,X_s)| \leq Z_s^T,\qquad 0\leq s\leq t\leq T \text{ a.s.},$$

$$(6.3)\qquad \int_0^t Z_s^T\,ds < \infty\quad\text{a.s.}$$

Let first $\{t_n;\, n\in\mathbb{N}\}$ be a sequence such that $t_n<t$ for any n and $t_n\to t$ as $n\to\infty$.

$$\begin{aligned}&\int_0^t \tilde{F}(t,s,X_s)\,ds - \int_0^{t_n}\tilde{F}(t_n,s,X_s)\,ds\\ &\qquad = \int_{t_n}^t \tilde{F}(t,s,X_s)\,ds + \int_0^{t_n}\big[\tilde{F}(t,s,X_s)\big]\,ds - \tilde{F}(t_n,s,X_s)\big]\,ds,\end{aligned}$$

$$\left|\int_{t_n}^t \tilde{F}(t,s,X_s)\,ds\right| \leq \int_{t_n}^t Z_s^T\,ds$$

and the latter tends a.s. to 0 as $n\to\infty$.

$$\left|\int_0^{t_n}\big[\tilde{F}(t,s,X_s) - \tilde{F}(t_n,s,X_s)\big]\,ds\right| \leq \int_0^t |\tilde{F}(t,s,X_s) - \tilde{F}(t_n,s,X_s)|\,ds,$$

which tends to 0 from (6.1), (6.2), (6.3) and Lebesgue's dominated convergence theorem. A similar argument gives the same result when $t_n>t$, $t_n\to t$.

(b) We next show that $t\to I_t(X,H_t)$ possesses an a.s. continuous modification. This will follow from (H.7) and

$$(6.4)\qquad (t,h)\to I_t(X,h)\quad \text{has an a.s. continuous modification.}$$

By localization, it suffices to prove (6.4) under the assumptions (H.2)–(H.5) and (H.6)–(H.10), with $V_t^N(\omega)$ in (H.10) replaced by a constant K, and in case $X_0\in\bigcap_{q\geq 1}L^q(\Omega;\mathbb{R}^d)$. It then suffices to show that under the above hypotheses, there exists $c,q>0$ and $\beta>p+1$ s.t.

$$(6.5)\qquad E\big(|I_t(X,h) - I_r(X,k)|^q\big) \leq c\big(|t-r|^\beta + |h-k|^\beta\big)$$

for any $h, k \in \mathbb{R}^p$; $t, r > 0$. Suppose, to fix the ideas, that $0 \le r \le t$,

$$\begin{aligned} I_t(X, h) - I_r(X, k) &= \int_r^t G_i(h; t, s, X_s)\, dW_s^i \\ &\quad + \int_0^r [G_i(h; t, s, X_s) - G_i(h; r, s, X_s)]\, dW_s^i \\ &\quad + \int_0^r [G_i(h; r, s, X_s) - G_i(k; r, s, X_s)]\, dW_s^i. \end{aligned}$$

It follows from the Burkholder–Gundy inequality that

$$\begin{aligned} E\left(\left|\int_r^t G_i(h; t, s, X_s)\, dW_s^i\right|^q\right) &\le c_q \sum_{i=1}^k E\left[\left(\int_r^t |G_i(h; t, s, X_s)|^2\, ds\right)^{q/2}\right] \\ &\le c_q (t - r)^{(q-2)/2} \sum_{i=1}^k E\int_r^t |G_i(h; t, s, X_s)|^q\, ds \\ &\le c_q (t - r)^{(q-2)/2}. \end{aligned}$$

From (H.4) for G_i', we deduce as in Lemma 4.1

$$\begin{aligned} &E\left(\left|\int_0^r [G_i(h; r, s, X_s) - G_i(k; r, s, X_s)]\, dW_s^i\right|^q\right) \\ &\quad \le c_q (h - k)^q \left(1 + E\int_0^r |X_s|^q\, ds\right) \\ &\quad \le c_q (h - k)^q. \end{aligned}$$

From (H.10), and the fact that $X \in \bigcap_{q \ge 1} \bigcap_{t > 0} L^q(\Omega \times (0, t))$,

$$\begin{aligned} &E\left(\left|\int_0^r [G_i(h; t, s, X_s) - G_i(h; r, s, X_s)]\, dW_s^i\right|^q\right) \\ &\quad \le K_{r,q} |t - r|^{\alpha q} \left(1 + E\int_0^r |X_s|^{ql}\, ds\right) \\ &\quad \le c_q |t - r|^{\alpha q}. \end{aligned}$$

(6.5) now follows from the above estimate, provided we chose q such that $\inf((q - 2)/2, \alpha q) > q + 1$. □

7. Semimartingale property of the solution. Under the conditions of Theorem 6.1, there exists a unique (in the sense of Theorem 5.1) continuous solution $\{X_t;\ t \ge 0\}$ of the equation

$$X_t = X_0 + \int_0^t F(t, s, X_s)\, ds + \int_0^t G_i(H_t; t, s, X_s)\, dW_s^i, \tag{3.1}$$

which we rewrite, with the notation of the above sections, as

$$X_t = X_0 + \int_0^t \tilde{F}(t, s, X_s)\, ds + I_t(X, H_t). \tag{7.1}$$

We now want to state conditions under which both $\{\int_0^t \tilde{F}(t, s, X_s); t \geq 0\}$ and $\{I_t(X, H_t); t \geq 0\}$ are semimartingales (and then also $\{X_t; t \geq 0\}$). In order to avoid some technicalities, we shall state some of the conditions in terms of $\tilde{F}$ (and not explicitly in terms of F, G and $\{H_t\}$) for simplicity. In any event, our conditions are easy to check for each example.

Let us first treat the term $\{\int_0^t \tilde{F}(t, s, X_s)\, ds\}$. We shall assume that for any $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the process $\{\tilde{F}(t, s, X_s); t \geq s\}$ can be rewritten in the form

(H.11) $$\tilde{F}(t, s, x) = \tilde{F}(s, s, x) + \int_s^t \Gamma(\theta, s, x)\, d\theta + \int_s^t \Lambda_i(\theta, s, x)\, dW_\theta^i,$$
where $\Gamma, \Lambda_1, \dots, \Lambda_k$ are measurable mappings from $\Omega \times D \times \mathbb{R}^d$ into $\mathbb{R}^d$, and for each (s, x) in $\mathbb{R}_+ \times \mathbb{R}^d$, $\Gamma(\cdot, s, x)$, $\Lambda_1(\cdot, s, x)$, $\cdots, \Lambda_k(\cdot, s, x)$ are progressively measurable on $\Omega \times [s, +\infty)$.

It follows from (H.5′) that $x \to \tilde{F}(s, s, x)$ is continuous for any (ω, s). We suppose moreover that

(H.12) $\quad x \to \Gamma(\theta, s, x)$ is continuous for any $(\omega; \theta, s) \in \Omega \times D$,

and that for any $N > 0$ there exist $\mathbb{R}_+$-valued measurable functions c_N and d_N defined on $\Omega \times D$, which are progressively measurable in (θ, ω) on $\Omega \times [s, +\infty)$ for any fixed s, and such that for some $q > d$ and any $0 \leq s < t$, $N \in \mathbb{N}$,

$$\int_s^t c_N(\theta, s)\, d\theta < \infty \ \text{a.s.}, \qquad E\int_s^t (d_N(\theta, s))^q\, d\theta < \infty, \tag{H.13}$$

$$\sup_{|x| \leq N} \left(|\Gamma(\theta, s, x)| + \sum_{i=1}^k |\Lambda_i(\theta, s, x)|^2 \right) \leq c_N(\theta, s), \tag{H.14}$$

$$\begin{gathered} |\Lambda(\theta, s, x) - \Lambda(\theta, s, y)| \leq d_N(\theta, s)|x - y|, \\ \forall\, x, y \in \mathbb{R}^d \text{ s.t. } |x| \vee |y| \leq N. \end{gathered} \tag{H.15}$$

It then follows that each term in the above decomposition of $\tilde{F}(t, s, x)$ is a.s. continuous in x, after a possible choice of another modification (for the stochastic integral terms, we apply the argument in Lemma 4.1). It is then not hard to show that

$$\tilde{F}(t, s, X_s) = \tilde{F}(s, s, X_s) + \int_s^t \Gamma(\theta, s, X_s)\, d\theta + \int_s^t \Lambda_i(\theta, s, X_s)\, dW_\theta^i.$$

It follows from (H.14) that we can use the Fubini theorems (the one for the stochastic integral terms can be found, e.g., in Jacod [6] Theorem 5.44) to conclude

$$\int_0^t \tilde{F}(t,s,X_s)\,ds = \int_0^t \tilde{F}(s,s,X_s)\,ds + \int_0^t\int_0^\theta \Gamma(\theta,s,X_s)\,ds\,d\theta \tag{7.2}$$
$$+\int_0^t\int_0^\theta \Lambda_i(\theta,s,X_s)\,ds\,dW_\theta^i.$$

We have shown the following proposition.

PROPOSITION 7.1. *Under conditions* (H.1′), (H.2′), (H.3′), (H.4′), (H.5′), (H.11), (H.12), (H.13), (H.14) *and* (H.15), $\{\int_0^t \tilde{F}(t,s,X_s)\,ds;\ t \geq 0\}$ *is a semimartingale whose decomposition is given by* (7.2).

We now consider $I_t(X, H_t)$. Let us assume that $\{H_t\}$ is a semimartingale of the form (here and in the sequel, we use the convention of summation over repeated indices, even when they appear twice as superscripts)

$$H_t = H_0 + \int_0^t K_s\,ds + \int_0^t L_s^i\,dW_s^i, \tag{H.16}$$

where H_0 is a $\mathscr{F}_0$-measurable p-dimensional random vector, $\{K_t, L_t^1,\dots, L_t^k\}$ are progressively measurable p-dimensional random processes with

$$\int_0^t \left(|K_s| + \sum_1^k |L_s^i|^2\right) ds < \infty \quad \text{a.s.}, \forall\, t > 0. \tag{H.17}$$

We suppose moreover that

(H.18) $\forall\,(\omega, s, x)$, $\quad G_i(\cdot\,;\cdot\,, s, x)$ is of class $C^{2,1}$ (C^2 in h and C^1 in t),

and moreover for any $N > 0$; $h, k \in \mathbb{R}^p$ s.t. $|h|, |k| \leq N$; $r, s, t \in \mathbb{R}_+$; $x \in \mathbb{R}^d$; $1 \leq i \leq k$,

$$\left|\frac{\partial G_i}{\partial h}(h;t,s,x)\right| + \left|\frac{\partial^2 G_i}{\partial h^2}(h;t,s,x)\right| + \left|\frac{\partial G_i}{\partial t}(h;t,s,x)\right| \tag{H.19}$$
$$\leq V_{t\vee s}^N(1+|x|),$$

$$\left|\frac{\partial^2 G_i}{\partial h^2}(h;t,s,x) - \frac{\partial^2 G_i}{\partial h^2}(k;r,s,x)\right|$$
$$+\left|\frac{\partial G_i}{\partial t}(h;t,s,x) - \frac{\partial G_i}{\partial t}(k;r,s,x)\right| \tag{H.20}$$
$$\leq V_{t\vee r\vee s}^N(1+|x|)(|h-k| + |t-r|),$$

where again $\{V_t^N;\ t \geq 0\}_{N \in \mathbb{N}}$ is a collection of increasing and progressively measurable $\mathbb{R}_+$-valued processes. Let us now denote by $\{I_t(X, h, r)\}$ the collection of processes indexed by $(h, r) \in \mathbb{R}^p \times \mathbb{R}_+$:

$$I_t(X, h, r) = \int_0^t G_i(h; r, s, X_s)\, dW_s^i.$$

Combining the argument of Lemma 4.1 for $q > p + 1$ with a localization procedure, we obtain the following lemma.

LEMMA 7.2. *For each $t > 0$, $(h, r) \to I_t(X, h, r)$ is a.s. of class $C^{2,1}$, and*

$$\frac{\partial}{\partial h} I_t(X, h, r) = \int_0^t \frac{\partial G_i}{\partial h}(h; r, s, X_s)\, dW_s^i,$$

$$\frac{\partial^2}{\partial h^2} I_t(X, h, r) = \int_0^t \frac{\partial^2 G_i}{\partial h^2}(h; r, s, X_s)\, dW_s^i,$$

$$\frac{\partial}{\partial r} I_t(X, h, r) = \int_0^t \frac{\partial G_i}{\partial r}(h; r, s, X_s)\, dW_s^i,$$

I and its derivatives being jointly continuous in (t, h, r).

Proposition 7.3 then follows from an adaptation of Theorem 1.8.1. in Kunita [8] (see the Appendix).

PROPOSITION 7.3. *Under the above conditions, in particular* (H.16)–(H.20), *$\{I_t(X, H_t, t);\ t \geq 0\}$ is a semimartingale whose decomposition is given by*

$$\begin{aligned} I_t(X, H_t, t) &= \int_0^t G_i(H_s; s, s, X_s)\, dW_s^i \\ &\quad + \int_0^t \left(\frac{\partial I_s}{\partial r}(X, H_s, s) + \frac{\partial I_s}{\partial h}(X, H_s, s) K_s \right) ds \\ &\quad + \int_0^t \frac{\partial I_s}{\partial h}(X, H_s, s) L_s^i\, dW_s^i + \frac{1}{2} \int_0^t \left(\frac{\partial^2 I_s}{\partial h^2}(X, H_s, s) L_s^i, L_s^i \right) ds \\ &\quad + \int_0^t \frac{\partial G_i}{\partial h}(H_s; s, s, X_s) L_s^i\, ds. \end{aligned}$$

We can now conclude this part of the discussion with the following theorem.

THEOREM 7.4. *Assume that conditions* (H.1′)–(H.5′) *and* (H.11)–(H.20) *are in force. Then the unique progressively measurable solution* $\{X_t;\ t \geq 0\}$ *of equation* (3.1) *is a continuous semimartingale which takes the form*

$$\begin{aligned}
X_t = X_0 &+ \int_0^t \tilde{F}(s, s, X_s)\, ds + \int_0^t \int_0^\theta \Gamma(\theta, s, X_s)\, ds\, d\theta \\
&+ \int_0^t \int_0^\theta \Lambda_i(\theta, s, X_s)\, ds\, dW_\theta^i + \int_0^t G_i(H_s; s, s, X_s)\, dW_s^i \\
&+ \int_0^t \left(\frac{\partial I_s}{\partial r}(X, H_s, s) + \frac{\partial I_s}{\partial h}(X, H_s, s) K_s \right) ds \\
&+ \int_0^t \frac{\partial I_s}{\partial h}(X, H_s, s) L_s^i\, dW_s^i + \frac{1}{2} \int_0^t \left(\frac{\partial^2 I_s}{\partial h^2}(X, H_s, s) L_s^i, L_s^i \right) ds \\
&+ \int_0^t \frac{\partial G_i}{\partial h}(H_s; s, s, X_s) L_s^i\, ds.
\end{aligned}$$

APPENDIX

The aim of this Appendix is to prove the following Itô–Ventzell formula, which generalizes Theorem 1.8.1 in Kunita [8]. Sznitman [21] has analogous results for general semimartingales.

THEOREM. *Let* $\{T_t(h, r);\ t \geq 0\}_{(h, r) \in \mathbb{R}^p \times \mathbb{R}_+}$ *be a collection of d-dimensional semimartingales of the form*

$$T_t(h, r) = T_0(h, r) + \int_0^t U_s(h, r)\, ds + \int_0^t V_s^i(h, r)\, dW_s^i, \tag{A.1}$$

where $\{W_t^1, \dots, W_t^k;\ t \geq 0\}$ *are mutually independent* $\mathscr{F}_t$*-Wiener processes defined on* $(\Omega, \mathscr{F}, P)$, $\{U_t(h, r), V_t^1(h, r), \dots, V_t^k(h, r);\ t \geq 0\}_{(h, r) \in \mathbb{R}^p \times \mathbb{R}_+}$ *are progressively measurable processes s.t.* $h \to (V_t^1(h, r), \dots, V_t^k(h, r))$ *is of class* C^1, $\forall\, (\omega, t, r) \in \Omega \times \mathbb{R}_+^2$, *and*

(H.A.1) $\forall\ N \in \mathbb{N},\ t > 0$, *there exists an* R_+*-valued progressively measurable process* $\{\alpha_s^{t, N};\ 0 \leq s \leq t\}$ *such that*

(i) $$\int_0^t \alpha_s^{t, N}\, ds < \infty \quad a.s.,$$

(ii) $$\sup_{r \leq t;\, |h| \leq N} \left(|U_s(h, r)| + \sum_{i=1}^k \left[\left| \frac{\partial V_s^i}{\partial h}(h, r) \right| + |V_s^i(h, r)|^2 \right] \right) \leq \alpha_s^{t, N},$$

$$0 \leq s \leq t,\ \forall\, (\omega, t) \in \Omega \times \mathbb{R}_+,$$

(H.A.2) $$(h, r) \to \left(U_t(h, r), V_t^1(h, r), \dots, V_t^k(h, r), \frac{\partial V_t^1}{\partial h}(h, r), \dots, \frac{\partial V_t^k}{\partial h}(h, r) \right)$$

is continuous.

We assume moreover that

$$\text{(H.A.3)}\qquad \begin{aligned}&\forall\ (\omega,t)\in\Omega\times\mathbb{R}_+,\ (h,r)\to T_t(h,r) \text{ is of class } C^{2,1}, \text{ and } \forall\\ &\omega\in\Omega,\ T,\ \frac{\partial T}{\partial h},\ \frac{\partial^2 T}{\partial h^2},\ \frac{\partial T}{\partial r} \text{ are locally bounded in } (t,h,r).\end{aligned}$$

Let $\{H_t\}$ *be a p-dimensional semimartingale which satisfies* (H.14) *and* (H.15). *Then the following holds*:

$$\begin{aligned}T_t(H_t,t) &= T_0(H_0,0) + \int_0^t U_s(H_s,s)\,ds + \int_0^t V_s^i(H_s,s)\,dW_s^i\\ &\quad + \int_0^t \frac{\partial T_s}{\partial h}(H_s,s)K_s\,ds + \int_0^t \frac{\partial T_s}{\partial r}(H_s,s)\,ds + \int_0^t \frac{\partial T_s}{\partial h}(H_s,s)L_s^i\,dW_s^i\\ &\quad + \frac{1}{2}\int_0^t <\frac{\partial^2 T_s}{\partial h^2}(H_s,s)L_s^i, L_s^i>ds + \int_0^t \frac{\partial V_s^i}{\partial h}(H_s,s)L_s^i\,ds.\end{aligned}\tag{A.2}$$

Proof. Note that each term in (A.1) is $\mathscr{P}\otimes\mathscr{B}_p^+$-measurable, where $\mathscr{P}$ denotes the σ-algebra of progressively measurable subsets of $\Omega\times\mathbb{R}_+$ and $\mathscr{B}_p^+$ denotes the Borel field over $\mathbb{R}^p\times\mathbb{R}_+$. By using a classical localization procedure, it suffices to prove the result in the case where H_t, $T_t(h,r)$, $(\partial T_t/\partial h)(h,r)$, $(\partial^2 T_t/\partial h^2)(h,r)$, $(\partial T_t/\partial r)(h,r)$ and $\int_0^t \alpha_s^{t,N}\,ds$ are uniformly bounded by a constant c which is independent of ω, t, h and r. Therefore we make these assumptions w.l.o.g.

We extend below any function which was defined on $\mathbb{R}_+$ as a function defined on $\mathbb{R}$ by taking it to be 0 on $\mathbb{R}_-$. Let $\varphi\in C_c^\infty(\mathbb{R}^p)$, $\psi\in C_c^\infty(\mathbb{R})$. From Itô's formula,

$$\begin{aligned}\varphi(h-H_t)\psi(r-t) &= \varphi(h-H_0)\psi(r) - \int_0^t \varphi'(h-H_s)K_s\psi(r-s)\,ds\\ &\quad - \int_0^t \varphi(h-H_s)\psi'(r-s)\,ds\\ &\quad - \int_0^t \varphi'(h-H_s)L_s^i\psi(r-s)\,dW_s^i\\ &\quad + \frac{1}{2}\int_0^t <\varphi''(h-H_s)L_s^i, L_s^i>\psi(r-s)\,ds,\end{aligned}$$

and also

$$\begin{aligned}&T_t(h,r)\varphi(h-H_t)\psi(r-t)\\ &\quad = T_0(h,r)\varphi(h-H_0)\psi(r)\\ &\qquad + \int_0^t U_s(h,r)\varphi(h-H_s)\psi(r-s)\,ds\\ &\qquad + \int_0^t V_s^i(h,r)\varphi(h-H_s)\psi(r-s)\,dW_s^i\end{aligned}$$

$$- \int_0^t T_s(h,r) \varphi'(h - H_s) K_s \psi(r-s)\, ds$$

$$- \int_0^t T_s(h,r) \varphi(h - H_s) \psi'(r-s)\, ds$$

$$- \int_0^t T_s(h,r) \varphi'(h - H_s) L_s^i \psi(r-s)\, dW_s^i$$

$$+ \frac{1}{2} \int_0^t T_s(h,r) \langle \varphi''(h - H_s) L_s^i, L_s^i \rangle \psi(r-s)\, ds$$

$$- \int_0^t V_s^i(h,r) \varphi'(h - H_s) L_s^i \psi(r-s)\, ds.$$

We integrate the above identity with respect to $dh\,dr$ over $\mathbb{R}^p \times \mathbb{R}$, and interchange the $dh\,dr$ and the ds (resp. the dW_s^i) integrals, using Fubini's theorem (resp. Theorem 5.44 in Jacod [6]). We moreover integrate by parts all integrals involving derivatives of φ and ψ, yielding

$$\int_{\mathbb{R}^p \times \mathbb{R}} T_t(h,r) \varphi(h - H_t) \psi(r-t)\, dh\,dr$$

$$= \int_{\mathbb{R}^p \times \mathbb{R}} T_0(h,r) \varphi(h - H_0) \psi(r-0)\, dh\,dr$$

$$+ \int_0^t ds \int_{\mathbb{R}^p \times \mathbb{R}} U_s(h,r) \varphi(h - H_s) \psi(r-s)\, dh\,dr$$

$$+ \int_0^t dW_s^i \int_{\mathbb{R}^p \times \mathbb{R}} V_s^i(h,r) \varphi(h - H_s) \psi(r-s)\, dh\,dr$$

$$+ \int_0^t ds \int_{\mathbb{R}^p \times \mathbb{R}} \frac{\partial T_s}{\partial h}(h,r) \varphi(h - H_s) K_s \psi(r-s)\, dh\,dr$$

$$+ \int_0^t ds \int_{\mathbb{R}^p \times \mathbb{R}} \frac{\partial T_s}{\partial r}(h,r) \varphi(h - H_s) \psi(r-s)\, dh\,dr$$

$$+ \int_0^t dW_s^i \int_{\mathbb{R}^p \times \mathbb{R}} \frac{\partial T_s}{\partial h}(h,r) \varphi(h - H_s) L_s^i \psi(r-s)\, dh\,dr$$

$$+ \frac{1}{2} \int_0^t ds \int_{\mathbb{R}^p \times \mathbb{R}} \left\langle \frac{\partial^2 T_s}{\partial h^2}(h,r) L_s^i, L_s^i \right\rangle \varphi(h - H_s) \psi(r-s)\, dh\,dr$$

$$+ \int_0^t ds \int_{\mathbb{R}^p \times \mathbb{R}} \frac{\partial V_s^i}{\partial h}(h,r) \varphi(h - H_s) L_s^i \psi(r-s)\, ds.$$

It remains to replace φ and ψ by sequences $\{\varphi_n\}$ and $\{\psi_n\}$ which converge to the Dirac measure at 0, as $n \to \infty$, and let $n \to \infty$. The convergence follows easily from our hypotheses. □

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