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ON THE VOLUME OF THE WIENER SAUSAGE

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Let $W(t, \varepsilon)$ be the ε-Wiener sausage, i.e., the ε-neighborhood of the trace of the Brownian motion up to time t. It is shown that the results of Donsker and Varadhan on the behavior of $E(\exp(-\nu|W(t,\varepsilon)|))$, $\nu > 0$, remain true if ε depends on t and converges to 0 with a certain rate.

1. Introduction. Let β_t, $t \geq 0$, be a standard d-dimensional Brownian motion starting in 0. If $\varepsilon > 0$, the ε-Wiener sausage is the following subset of $\mathbb{R}^d$:

$$W(t,\varepsilon) = \bigcup_{s \leq t} U_\varepsilon(\beta_s).$$

Here $U_\varepsilon(x)$ is the ε-neighborhood of x. The Lebesgue measure of $W(t, \varepsilon)$ is the volume of the Wiener sausage $V(t, \varepsilon)$. Donsker and Varadhan proved in [4] that for $\nu > 0$,

$$\lim_{t\to\infty} t^{-d/(2+d)} \log E(e^{-\nu V(t,\varepsilon)}) = -k(d)\nu^{2/(2+d)}, \tag{1.1}$$

where $k(d)$ is independent of ε and ν:

$$k(d) = \frac{d+2}{d}\left(\frac{2\lambda_d}{d}\right)^{d/(d+2)} \omega_d^{2/(2+d)},$$

λ_d being the smallest eigenvalue of $-\Delta/2$ on the unit ball in $\mathbb{R}^d$ with Dirichlet boundary conditions and ω_d is the volume of the unit ball. This gives information about the probability that $V(t, \varepsilon)$ is untypically small.

If one could let ν vary with t, say $\nu \sim t^\alpha$, then one would obtain broader information on the behavior of $V(t, \varepsilon)$ in the tail. Such information is already contained in [4]. In fact, an inspection of the proof reveals that if ν_t is a positive function in t with

$$\lim_{t\to\infty} \nu_t/t = 0, \qquad \liminf_{t\to\infty} \nu_t > 0,$$

then

$$\lim_{t\to\infty} t^{-d/(2+d)} \nu_t^{-2/(2+d)} \log E(e^{-\nu_t V(t,\varepsilon)}) = -k(d). \tag{1.2}$$

The case where ν is constant is actually the border case where their proof works. It is the aim of this article to show that a certain decay of ν_t to 0 is possible and still (1.2) is true.

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By using Brownian rescaling, one can reformulate this as a result on $E(\exp(-\nu V(t, \varepsilon_t)))$ with fixed value ν and shrinking ε_t. Such an extension appears to be natural: Of course, for $d = 1$, one can take $\varepsilon = 0$ and (1.1) remains true. For $d \geq 2$, $V(t, 0) = 0$ almost surely. In dimension two, Brownian motion just fails to have local times, so one expects that not much smoothing is needed and the natural conjecture is that (1.1) remains true for ε_t decaying with an arbitrary polynomial rate. That is in fact what is proved here.

The problem becomes even more natural when one puts it in the appropriate large deviation setting. By using Brownian rescaling, one obtains

$$E(\exp(-\nu V(t, \varepsilon))) = E\big(\exp(-\nu T V(T, \varepsilon T^{-1/d}))\big),$$

where $T = t^{d/(d+2)}$.

If $\delta > 0$ then $V(T, \delta)$ can be expressed in terms of the empirical measure

$$L_T = \frac{1}{T}\int_0^T \delta_{\beta_s}\, ds,$$

where δ_x is the one point measure at x. If $\delta > 0$ is fixed, we put

$$L_T^{\delta}(x) = (L_T * \chi_\delta)(x) = \int \chi_\delta(y - x) L_T(dy),$$

where $\chi_\delta(x) = \delta^{-d}\omega_d \mathbf{1}_{B_\delta}(x)$, $B_\delta = \{y\colon |y| \leq \delta\}$ and ω_d is the volume of the d-dimensional unit ball. Obviously,

$$V(T, \delta) = |\operatorname{supp}(L_T^{\delta})|.$$

$|\ |$ denotes Lebesgue measure. L_T obeys a large deviation principle (LDP) in the weak topology on the set of probability measures. However, this is not an appropriate topology for discussing supports. Much better is the L_1-topology, on probability densities on which the support is a lower semicontinuous functional. It is not difficult to show that for fixed $\delta > 0$, L_T^{δ} satisfies a LDP in the L_1-topology. What Donsker and Varadhan proved in [3] and [4] is that this LDP remains true with less smoothing, more precisely, when δ decays with $T^{-1/d}$. This then leads to (1.1). One may ask how much smoothing is really needed for a LDP in the L_1-topology. The answer is that $T^{-1/(d-2)}$ is the border case for δ where the LDP starts to fail. We will prove such a result for the Brownian motion on a d-dimensional torus in Section 2. As a consequence of this, we will obtain the following theorem.

THEOREM 1. *Let $d \geq 2$. For any $\nu, \varepsilon, \gamma > 0$, one has*

$$\lim_{T\to\infty} T^{-1} \log E\big(e^{-\nu T V(T, \varepsilon T^{-1/(d-2+\gamma)})}\big) = -k(d)\nu^{2/(2+d)}.$$

REMARKS. 1. The theorem breaks down for $\gamma = 0$ ($d \geq 3$). In fact

$$\begin{aligned}
&\lim_{T\to\infty} T^{-1} \log E\big(e^{-TV(T, \varepsilon T^{-1/(d-2)})}\big)\\
&\quad \geq -\lim_{T\to\infty} E\big(V(T, \varepsilon T^{-1/(d-2)})\big)\\
&\quad = -\lim_{T\to\infty} T^{-d/(d-2)} E\big(V(T^{d/(d-2)}, \varepsilon)\big) = -c(\varepsilon),
\end{aligned}$$

where $c(\varepsilon)$ is the Newtonian capacity of the ball with radius ε in $\mathbb{R}^d$ (see [5]). $c(\varepsilon)$ goes to 0 for $\varepsilon \to 0$, so the statement of Theorem 1 is certainly false. Probably, for $d \geq 3$,

$$\lim_{T\to\infty} T^{-1} \log E\left(e^{-\nu T V(T, \varepsilon T^{-1/(d-2)})}\right) = 0,$$

but I have no proof.

2. The result may be translated into other statements by means of an appropriate rescaling, e.g., for $a < 2/(d^2 - 4)$ and all $\nu, \varepsilon > 0$,

$$\lim_{t\to\infty} t^{-d/(2+d)} \log E\left(e^{-\nu V(t, \varepsilon t^{-a})}\right) = -k(d)\nu^{2/(2+d)},$$

or for $a < 2/d$ and $\nu_t = \nu t^{-a}$ one has (1.2).

Sznitman has recently developed a method for proving extensions of results like (1.1) without relying on large deviation techniques [6].

2. Large deviations in L_1. We give an improvement on the result of Donsker and Varadhan in [3] and [4]. Let B_t be the Brownian motion on the d-dimensional flat torus T_d^R with circumference R. We identify T_d^R with $[0, R)^d$ and put

$$B_t^i = \beta_t^i \mod R,$$

where $\beta_t = (\beta_t^1, \dots, \beta_t^d)$. L_t^δ denotes now the mollified empirical density of B_t. We drop the upper index R in T_d for notational convenience.

THEOREM 2. *Assume $d \geq 2$. If $0 < a < 1/(d-2)$, then for any $\varepsilon > 0$ $L_t^{\varepsilon t^{-a}}$ satisfies a large deviation principle in L_1 with rate function*

$$I(f) = \tfrac{1}{8}\int \frac{|\nabla f|^2}{f} dx,$$

i.e., if A is a measurable subset of $L_1(T_d)$, then

$$-I(\text{int } A) \leq \liminf_{t\to\infty} \frac{1}{t} \log P\left(L_t^{\varepsilon t^{-a}} \in A\right) \leq \limsup_{t\to\infty} \frac{1}{t} \log P\left(L_t^{\varepsilon t^{-a}} \in A\right) \leq -I(\text{cl } A).$$

Here, $I(B) = \inf\{I(f)\colon f \in A)$. int A and cl A refer to the L_1-topology.

The result is proved in [3] and [4] for $a = 1/d$. We extend the argument slightly and prove the following result.

PROPOSITION. *Let $0 < a < a' < 1/(d-2)$, $da' - 2a < 1$. Then for any $\delta > 0$,*

$$\lim_{t\to\infty} \frac{1}{t} \log P\left(\|L_t * \chi_{t^{-a}} - L_t * \chi_{t^{-a'}}\|_1 \geq \delta\right) = -\infty.$$

It is obvious that the proposition leads to Theorem 2: If $a \le 1/d$, the theorem is proved in [4]. If $1/d < a < 1/(d-2)$, we pick a sequence

$$1/d = a_0 < a_1 < \cdots < a_m = a,$$

satisfying $da_{k+1} - 2a_k < 1$ and apply the proposition.

REMARK. In [3], results on the L_1 large deviation principle for mollified empirical processes of much more general processes have been derived. The entropy condition used there can be relaxed using the extension of the method derived here.

PROOF OF THE PROPOSITION. The proof closely follows the arguments of Donsker and Varadhan:

$$P(\|L_t * \chi_{t-a} - L_t * \chi_{t^{-a'}}\|_1 \ge \delta) = P\Big(\sup_{\|g\|_\infty \le 1} |\langle L_t, g * (\chi_{t^{-a}} - \chi_{t^{-a'}})\rangle| \ge \delta\Big).$$

Here $\langle \mu, f \rangle$ denotes the integral $\int f \, d\mu$. Let

$$\mathfrak{G} = \{g * (\chi_{t^{-a}} - \chi_{t^{-a'}}) : \|g\|_\infty \le 1\}.$$

One chooses a finite subset $\mathfrak{G}_\delta$ of the set of bounded measurable functions such that

$$\sup_{f \in \mathfrak{G}} \inf_{h \in \mathfrak{G}_\delta} \|f - h\|_\infty \le \delta/4. \tag{2.1}$$

Such a set exists having less than $N_\delta(t) = \exp(c(\delta)t^{da'})$ elements, where $c(\delta)$ depends on δ but not on t. The argument for this fact is essentially contained in Donsker and Varadhan [4]. It runs as follow: We can choose $\varepsilon > 0$ such that for $|x - x'| \le \varepsilon t^{-a'}$ we have

$$\sup_{\|g\|_\infty \le 1} |\hat{g}(x) - \hat{g}(x')| \le \delta/8,$$

where

$$\hat{g}(x) = g * \chi_{t^{-a}}(x) - g * \chi_{t^{-a'}}(x).$$

We then divide T_d into congruent squares with sidelength $([(\sqrt{d}/\varepsilon)t^{a'}] + 1)^{-1}$. Here $[x]$ denotes the integer part of a number. The number of such squares is bounded by $c_1(\delta)t^{a'd}$. $\mathfrak{G}_\delta$ is then the set of functions $T_d \to [-2, 2]$ which are constant on each of the above defined squares and which take values in some fixed finite subset A of $[-2, 2]$, where each point in A is at distance less than or equal to $\delta/8$ from its nearest neighbors. It is then obvious that (2.1) holds.

For each $h \in \mathfrak{G}_\delta$, we choose a g with $\|g\|_\infty \le 1$ satisfying $\|h - \hat{g}\|_\infty \le \delta/4$ if any such g exists. Collecting all these functions g, we obtain a finite subset

$$\mathfrak{H}_\delta \subset \{f : \|f\|_\infty \le 1\}$$

with the property

$$\sup_{\|f\|_\infty \le 1} \inf_{g \in \mathfrak{H}_\delta} \|\hat{f} - \hat{g}\|_\infty \le \delta/2.$$

$\mathfrak{H}_\delta$ contains at most $N_\delta(t)$ elements. We now obtain

$$\begin{aligned} &P(\|L_t * \chi_{t^{-a}}(x) - L_t * \chi_{t^{-a'}}(x)\|_1 \geq \delta) \\ &\qquad \leq N_\delta(t) \sup_{\|g\|_\infty \leq 1} P(\langle L_t, \hat{g}\rangle \geq \delta/2), \end{aligned} \tag{2.2}$$

$$\begin{aligned} P(\langle L_t, \hat{g}\rangle \geq \delta/2) &= P\left(\int_0^t \hat{g}(B_s)\, ds \geq \delta/2\right) \\ &\leq \exp(-zt\delta/2) E\left(\exp\left(z\int_0^t \hat{g}(B_s)\, ds\right)\right), \end{aligned} \tag{2.3}$$

for an arbitrary $z \geq 0$, which will depend on t later on. It is important that for any bounded measurable function h,

$$E\left(\exp\left(\int_0^t h(B_s)\, ds\right)\right) \leq Ce^{2\|h\|_\infty} e^{t \sup_\mu(\langle \mu, h\rangle - I(\mu))}, \tag{2.4}$$

where C does not depend on h and t and $I(\mu)$ is the entropy function of the process, in our case $I(\mu)$ equals $I(f)$ from Theorem 2 when μ is absolutely continuous with density f and $I(\mu) = \infty$ otherwise.

(2.4) is proved in [3], Lemma 2.2. Applying this to $h = z\hat{g}$ gives

$$E\left(\exp\left(z\int_0^t \hat{g}(B_s)\, ds\right)\right) \leq Ce^z \exp\left(t \sup_\mu (z\langle \mu, \hat{g}\rangle - I(\mu))\right),$$

$$\begin{aligned} \sup_\mu (z\langle \mu, \hat{g}\rangle - I(\mu)) &\leq \sup_{\mu:\, I(\mu)\leq z} z\langle \mu, \hat{g}\rangle = \sup_{\mu:\, I(\mu)\leq z} z\langle g, \mu * \chi_{t^{-a}} - \mu * \chi_{t^{-a'}}\rangle \\ &\leq z \sup_{\mu:\, I(\mu)\leq z} \|\mu * \chi_{t^{-a}} - \mu * \chi_{t^{-a'}}\|_V, \end{aligned}$$

where $\|\ \|_V$ denotes the variation norm. We are going to estimate the right-hand side of this inequality and we estimate $\|\mu - \mu * \chi_{t^{-a}}\|_V$ and $\|\mu - \mu * \chi_{t^{-a'}}\|_V$ separately.

Let ϕ_s be the transition density of the Brownian motion on T_d, i.e.,

$$\phi_s(x) = \sum_{k\in Z} \bar{\phi}_s(x + kR),$$

where

$$\bar{\phi}_s(x) = (2\pi s)^{-d/2} \exp(-|x|^2/2s).$$

We put $s = t^{-2a+\kappa}$, where $\kappa > 0$ will be chosen later on. We estimate

$$\begin{aligned} \|\mu - \mu * \chi_{t^{-a}}\|_V &\leq \|\mu - \mu * \phi_s\|_V + \|\mu * \chi_{t^{-a}} - \mu * \chi_{t^{-a}} * \phi_{2s}\|_V \\ &\quad + \|\mu * \phi_s - \mu * \phi_s * \chi_{t^{-a}} * \phi_s\|_V. \end{aligned} \tag{2.5}$$

We assume $I(\mu) \leq z$. From the convexity of I it follows also that $I(\mu * \chi_{t^{-a}})$ and $I(\mu * \phi_s) \leq z$. Applying now Lemmas 3.1 and 4.1 of [2] to the Brownian motion on T_d, we obtain

$$\|\mu - \mu * \phi_s\|_V \leq \Phi(sz),$$

$$\|\mu * \chi_{t^{-a}} - \mu * \chi_{t^{-a}} * \phi_{2s}\|_V \leq \Phi(2sz),$$

where

$$\Phi(x) = 2 \inf_{a>0} \left(\frac{x + a - \log(1 + a)}{a} \right) \le 2 \inf_{a>0} \frac{x + a^2/2}{a} = \sqrt{8x}\,.$$

We now estimate the third summand on the right-hand side of (2.5). Obviously, there exists a constant c (depending on d only) such that

$$(\chi_{t^{-a}} * \phi_s)(x) \ge (1 - ct^{-\kappa})\phi_s(x),$$

for all $x \in T_d$. Denoting by Q the transition kernel corresponding to convolution with the density ϕ_s and by Q' that corresponding to $\chi_{t^{-a}} * \phi_s$ [i.e., $Qf(x) = (f * \phi_s)(x)$ and $Q'(x) = (f * \phi_s * \chi_{t^{-a}})(x)$], then by Lemma 4.1 of [2], we have

$$\|\mu * \phi_s - \mu * \phi_s * \chi_{t^{-a}} * \phi_s\|_V \le \Phi\big(I_{Q'}(\mu * \phi_s)\big),$$

where

$$I_{Q'}(\mu) = -\inf \int \log \frac{Q'u}{u}\, d\mu;$$

the infimum is over bounded measurable positive functions which are bounded away from 0. Then

$$\begin{aligned}
\int \log \frac{Q'u}{u}\, d\mu &\ge \int \log \frac{(1 - ct^{-\kappa})Qu}{u}\, d\mu \\
&\ge \log(1 - ct^{-\kappa}) + \int \log \frac{Qu}{u}\, d\mu \\
&\ge -2ct^{-\kappa} - I_Q(\mu).
\end{aligned}$$

Therefore

$$I_{Q'}(\mu) \le I_Q(\mu) + 2ct^{-\kappa}$$

and

$$I_{Q'}(\mu * \phi_s) \le 2ct^{-\kappa} + I_Q(\mu * \phi_s) \le 2ct^{-\kappa} + sz.$$

This, of course, works for a' as well as for a and, as we assume $a' > a$, the bound for $\|\mu - \mu * \chi_{t^{-a'}}\|_V$ is even better. Therefore, we obtain

$$\|\mu * \chi_{t^{-a}} - \mu * \chi_{t^{-a'}}\|_V \le \text{const} \cdot \big(\sqrt{zt^{-2a+\kappa}} + t^{-\kappa}\big). \tag{2.6}$$

Collecting now all the estimates obtained so far [(2.2)–(2.4) and (2.6)], we get

$$\begin{aligned}
&P(\|L_t * \chi_{t^{-a}}(x) - L_t * \chi_{t^{-a'}}(x)\|_1 \ge \delta) \\
&\quad \le c_1 \exp\Big\{c(\delta)t^{da'} - zt\delta/2 + c_2 zt\big(\sqrt{zt^{-2a+\kappa}} + t^{-\kappa}\big)\Big\}.
\end{aligned}$$

We choose now $z = z(t) = t^b$ with $a'd - 1 < b < 2a$, $0 < b$, and then κ satisfying $0 < \kappa < 2a - b$. Then we get

$$\limsup_{t\to\infty} t^{-1-b} \log P(\|L_t * \chi_{t^{-a}} - L_t * \chi_{t^{-a'}}\|_1 \ge \delta) < 0,$$

which is much more than what is required. □

PROOF OF THEOREM 1. The theorem is an immediate consequence of Theorem 2 and the results in [4]. The lower bound

$$\liminf_{T\to\infty} T^{-1} \log E\left(e^{-\nu T V(T, \varepsilon T^{-1/(d-2+\gamma)})}\right) \geq -k(d)\nu^{2/(2+d)}$$

follows from the results in [4]. In fact, for $\gamma \geq 2$, this is the result proved there.

The left-hand side of the above inequality is trivially increasing in γ. For the other inequality, i.e.,

$$\limsup_{T\to\infty} T^{-1} \log E\left(e^{-\nu T V(T, \varepsilon T^{-1/(d-2+\gamma)})}\right) \leq -k(d)\nu^{2/(2+d)},$$

one argues as in [4]: The corresponding quantity for the Brownian motion on the torus is an upper bound and then the fact that $|\{f\colon f > 0\}|$ is lower semicontinuous in the L_1-topology leads, together with Theorem 1, to an upper bound on the torus:

$$\limsup_{T\to\infty} T^{-1} \log E\left(e^{-\nu T V(T, \varepsilon T^{-1/(d-2+\gamma)})}\right) \leq -\inf_{f\in L_1}\left(\nu|\{f\colon f > 0\}| + \tfrac{1}{8}\int \frac{|\nabla f|^2}{f}\,dx\right).$$

If R (the circumference of the torus) goes to infinity, then the right-hand side converges to $-k(d)\nu^{2/(2+d)}$. See [1], Chapter 4.3. □

Note added in proof. Sznitman recently derived similar refinements by his method [7]. His approach does not seem to give information on a functional level, like Theorem 2 here. On the other hand, it is more precise as to the speed of the shrinking.

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