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# LARGE DEVIATIONS FOR THE MAXIMUM LOCAL TIME OF STABLE LÉVY PROCESSES

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Let $X(t)$ be a strictly stable Lévy process of index $1 < \alpha \le 2$ and skewness index $|h| \le 1$. Let $L_t^x$ be its local time and $L_t^* = \sup_x L_t^x$ the maximum local time. We show that $\lim_{\lambda \to +\infty} \lambda^{-\alpha} \log P(L_1^* > \lambda) = -C_{\alpha h}$, where $C_{\alpha h}$ is a known constant. In the case that $X(t)$ is a standard Brownian motion, $C_{\alpha h} = 1/2$ and the result is due to Perkins.

**1. Introduction.** Let $X(t)$ be a strictly stable Lévy process of index $1 < \alpha \le 2$. That is $X(0) = 0$, $X$ has stationary independent increments and characteristic function

$$E(e^{izX(t)}) = \exp(-t\psi(z)), \tag{1.1}$$

where $\psi(z) = C_0|z|^\alpha(1 - ih\,\mathrm{sgn}(z)\tan(\pi\alpha/2))$. Here $C_0 > 0$ and $|h| \le 1$. As is well known [Boylan (1964)], $X_t$ has a jointly continuous local time $L_t^x$, which we may normalize so that for all measurable $B \subset \mathbb{R}$,

$$\int_0^t 1_B(X(s))\,ds = \int_B L_t^x\,dx.$$

The maximum local time is $L_t^* = \sup_x L_t^x$.

In this article, we estimate $P(L_1^* > \lambda)$ as $\lambda \to +\infty$. Our results are sharp, in that they give the exact order of exponential decay. As background, we recall asymptotics for $P(L_1^0 > \lambda)$ as $\lambda \to +\infty$, which are known, and essentially due to Hawkes (1971). Set $\beta = \alpha/(\alpha - 1)$. For $t > 0$, we have

$$L_t^0 =_d t^{1/\beta} L_1^0 \quad \text{and} \quad L_t^* =_d t^{1/\beta} L_1^*. \tag{1.2}$$

The right-continuous inverse of $L^0$, $\tau(t)$, is a stable subordinator of index $\beta$. In particular, setting

$$\rho^{-1/\beta} = \pi^{-1}\Gamma(1 + 1/\alpha)\Gamma(1/\beta)C_0^{-1/\alpha}\,\mathrm{Re}\big[(1 - h\tan(\pi\alpha/2))^{-1/\alpha}\big],$$

we have $Ee^{-\lambda\rho^{-1}\tau(t)} = e^{-t^{1/\beta}}$. This implies probability estimates for $P(\tau(t) < \lambda)$ as $\lambda \to 0^+$. [See Lemma 1 of Hawkes (1971).] Then, using (1.2), we obtain

$$P\big(L_1^0 > \lambda\big) \sim C_1\lambda^{-\alpha/2}\exp(-C_{\alpha h}\lambda^\alpha) \quad \text{as } \lambda \to +\infty, \tag{1.3}$$

where $C_1 > 0$, $C_{\alpha h} = \alpha^{-1}(\rho/\beta)^{\alpha/\beta}$ and $f(\lambda) \sim g(\lambda)$ means $\lim_{\lambda \to +\infty} f(\lambda)/g(\lambda) = 1$. Our result shows that the exponential decay of $P(L_1^* > \lambda)$ is the same as for $L_1^0$.

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THEOREM 1.4.

$$\log P(L_1^* > \lambda) \sim -C_{\alpha h}\lambda^{\alpha} \quad as\ \lambda \to +\infty.$$

To put this result in perspective, for the standard Brownian motion, ($\alpha = 2$), we have [Perkins (1986)]

$$\log P(L_1^* > \lambda) \sim -\frac{\lambda^2}{2} \quad \text{as } \lambda \to +\infty. \tag{1.5}$$

Until recently though, the only known proof used the Ray–Knight theorem, and so could not be extended to other processes. Recently, the author [Lacey (1989)] developed a new proof of (1.5), which also gave new information about local times of the Brownian sheet. The techniques used there are reasonably general, and we will adapt them to the current situation in proving Theorem 1.4.

Three ingredients are needed for the proof. Good estimates for $P(L_1^x > \lambda)$ for all $x \in \mathbb{R}$; relatively weak estimates for $|L_1^x - L_1^y|$; and a covering argument for the range of $X(t)$, for $0 \le t \le 1$. For the first ingredient, we will use (1.3) and the strong Markov property of $X(t)$; for the second, we use a (deep) inequality from Barlow (1985), Lemma 3.4 below, and combine it with Dudley's theorem [Dudley (1967), also Section 2 below]. We emphasize that the use of Barlow's inequality is purely a matter of convenience; a cruder estimate, such as the one contained in Getoor and Kesten (1972), could be employed, at the cost of a more complicated argument. The third ingredient is easily supplied if $X(t)$ is a Brownian motion, or even a Lévy process with bounded jumps. For a stable process, however, it is more delicate, and occupies the bulk of the proof in Section 3 below.

There is some interest in providing lower-order estimates in Theorem 1.4; exact estimates are known for the Brownian motion, see Czáki (1989). However, it seems that our approach will not give this information. There is also interest in establishing the analog of Theorem 1.4 for other Lévy processes, such as those considered in the final section of Perkins (1986). For the processes considered therein, our approach, like Perkins' approach to the modulus of continuity of $L^*$, is not as exact.

The interest in Theorem 1.4 is that sample path properties of $L^*$ can be deduced from it. Standard arguments and Theorem 1.4 prove the law of the iterated logarithm and modulus of continuity for $L^*$, due to Donsker and Varadhan (1972) and Perkins (1986), respectively. There is also a connection with Barlow–Yor inequalities for $L^*$. Davis (1986) observed that for all stopping times $\tau$ and $p \ge 1$,

$$C_p\|\tau^{1/\beta}\|_p \le \|L_\tau^*\|_p \le D_p\|\tau^{1/\beta}\|_p.$$

Using his argument, which uses good-$\lambda$ inequalities, and Theorem 1.4, one can show that the optimal choice of $D_p$ satisfies $C^{-1}p^{1/\alpha} \le D_p \le Cp^{1/\alpha}$, for some

$C > 0$. Likewise, using the fact that

$$C^{-1}\lambda^{-\beta} \le \log P(L_1^* < \lambda) \le C\lambda^{-\beta} \quad \text{as } \lambda \downarrow 0.$$

[In Griffin (1985) apply scaling to line 6, page 276.] The optimal choice of $C_p$ is seen to satisfy $C^{-1}p^{-1/\beta} \le C_p \le Cp^{-1/\beta}$.

The remainder of the paper is organized as follows. In Section 2 we recall Dudley's theorem in the form we shall use it; the proof of Theorem 1.4 is in Section 3.

**2. Dudley's theorem.** Let $(T, d)$ be a (compact) pseudometric space, with diameter $\delta = \sup_{s,t \in T} d(s,t)$. For all $u > 0$, define the covering numbers $N(u) = N(T, d, u)$ to be the least number of $d$-balls of radius $u$ needed to cover $T$. Let $Z_t$, $t \in T$, be a stochastic process which satisfies

$$P(|Z_s - Z_t| > \lambda) \le C \exp\left( - \frac{\lambda^2}{d^2(s,t)} \right), \qquad s, t \in T, \tag{2.1}$$

for some fixed $C > 0$. The following is a known extension of Dudley's theorem [Dudley (1967)], which is also related to Borell's inequality [Borell (1975)].

THEOREM 2.2. *If for all* $0 < \varepsilon < \delta$, $m(\varepsilon) = \varepsilon + \int_0^\varepsilon (\log N(u))^{1/2}\, du < +\infty$, *then* $Z$ *admits a version whose sample paths are a.s. d-continuous. Moreover, for all* $\lambda, \varepsilon > 0$,

$$P\Big( \sup_{d(s,t) < \varepsilon} |Z_s - Z_t| > \lambda + m(\varepsilon) \Big) \le K \exp\left( \left( - \frac{\lambda}{K\varepsilon} \right)^2 \right),$$

*where* $K$ *is an absolute constant.*

The important point for us is the exponential squared estimate on the modulus of continuity of $Z$, which is not contained in Dudley's original paper. Nevertheless, this estimate is well known to experts. [A proof is in Lacey (1989).]

**3. Proof of Theorem 1.4.** By (1.3), only the upper half of Theorem 1.4 need be shown. That is,

$$\lim_{\lambda \to +\infty} \lambda^{-\alpha} \log P(L_1^* > \lambda) \le -C_{\alpha h}. \tag{3.1}$$

In the proof below, $D$ will be a constant, possibly depending on $\alpha$ and $h$, which might change from line to line. The principal new step in proving (3.1) is the next lemma, which considers the supremum of $L_1^{\cdot}$ over a fixed finite interval.

LEMMA 3.2. *There are constants $D$ and $\lambda_0 > 0$ so that for all $\lambda > \lambda_0$ and all (nonrandom) intervals $I$, of length* 1,

$$P\Big(\sup_{x \in I} L_1^x > \lambda + \lambda^{2/3}\Big) \le D\lambda^D \exp(-C_{\alpha h}\lambda^\alpha).$$

Before the proof of Lemma 3.2, we show that $\log P(L_1^* > \lambda) \le -D\lambda^\alpha$ follows from this lemma. Let $X'$ be an independent copy of $X$, and $\log M = \frac{1}{2}C_{\alpha h}\lambda^\alpha$. Then for large $\lambda$,

$$\begin{aligned}
P(L_1^* > \lambda) &\le P\Big(\sup_{0 \le t \le 1} |X(t)| > M\Big) + P\Big(\sup_{-M \le x \le M} L_1^x > \lambda\Big) \\
&\le 2P\Big(\sup_{0 \le t \le 1} |X(t) - X'(t)| > \frac{M}{2}\Big) + \sum_{-M \le k \le M} P\Big(\sup_{k \le x \le k+1} L_1^x > \lambda\Big) \\
&\le D\Big(2P\Big(|X(1) - X'(1)| > \frac{M}{2}\Big) + M\lambda^D \exp(-C_{\alpha h}\lambda^\alpha)\Big) \\
&\le 2D\lambda^D \exp\Big(-\frac{1}{2}C_{\alpha h}\lambda^\alpha\Big).
\end{aligned}$$

Here, to control the $X(t)$ term, we have used a symmetrization argument [e.g., Lemma 2.5 of Giné and Zinn (1984)] and Lévy's maximal inequality. To prove the sharp result (3.1) will require a more delicate argument. Note, however, for use later, that a very similar argument shows that

$$P\Big(\sup_{|x| \le \lambda^{2\alpha}} L_1^x > \lambda + \lambda^{2/3}\Big) \le D\lambda^D \exp(-C_{\alpha h}\lambda^\alpha) \tag{3.3}$$

for all sufficiently large $\lambda$.

Lemma 3.2 will be proved by using Dudley's theorem, and the following inequality due to Barlow (1985). (See Lemma 2.8 and the remarks in Section 4, op. cit.)

LEMMA 3.4. *For all $\lambda$, $a > 0$ and all $|x - y| \le 1$,*

$$P(|a \wedge L_1^x - a \wedge L_1^y| > \lambda) \le 2\exp\left(-D\frac{\lambda^2}{a|x - y|^{\alpha - 1}}\right).$$

PROOF OF LEMMA 3.2. Fix $\lambda > 1$, and let $Z_x = 2\lambda \wedge L_1^x$. By (1.3) and the strong Markov property of $X_t$, for $\lambda$ sufficiently large, independent of $x$,

$$P(Z_x > \lambda) \le P(L_1^0 > \lambda) \le \exp(-C_{\alpha h}\lambda^\alpha). \tag{3.5}$$

Fix an interval $I$ of length 1, and consider the metric $d(x, y) = D(\lambda|x - y|^{\alpha - 1})^{1/2}$ on $I$. Then $(I, d)$ has diameter $\delta = D\lambda^{1/2}$, and, by Lemma 3.4, (2.1) is satisfied. Moreover, the covering numbers satisfy

$$N(u) \le 1 + (D^2\lambda u^{-2})^{1/(\alpha - 1)}, \qquad u > 0.$$

Let $\varepsilon = \varepsilon(\lambda) = \lambda^{(1-\alpha)/2} \to 0$ as $\lambda \to +\infty$. Then

$$(3.6) \qquad \begin{aligned} m(\varepsilon) &= \varepsilon + \int_0^{\varepsilon} (\log N(u))^{1/2}\, du \\ &\le D\varepsilon(\log \varepsilon)^{1/2} \to 0 \quad \text{as } \lambda \to +\infty. \end{aligned}$$

With these preliminaries completed, we can now give the desired estimate. Let $I_\varepsilon$ be a minimal $\varepsilon$-net in the metric space $(I, d)$; that is,

$$\sup_{x\in I} \min_{y \in I_\varepsilon} d(x,y) \le \varepsilon,$$

and the cardinality of $I_\varepsilon$ is as small as possible. By the definition of $\varepsilon$, $\operatorname{card}(I_\varepsilon) \le D\lambda^D$. Then by Dudley's theorem, (3.5) and (3.6),

$$\begin{aligned} P\Big(\sup_{x\in I} L_1^x > \lambda + \lambda^{2/3}\Big) &= P\Big(\sup_{x\in I} Z_x > \lambda + \lambda^{2/3}\Big) \\ &\le P\Big(\sup_{x\in I_\varepsilon} Z_x > \lambda + \lambda^{2/3}\Big) + P\Bigg(\sup_{\substack{x,y\in I \\ d(x,y)<\varepsilon}} |Z_x - Z_y| > \lambda^{2/3}\Bigg) \\ &\le \operatorname{card}(I_\varepsilon) \sup_x P(L_1^x > \lambda) + K \exp\left(-\frac{\lambda^{2/3}}{K\varepsilon}\right)^2 \\ &\le D\lambda^D \exp(-C_{\alpha h}\lambda^\alpha) + K\cos(-K^{-1}\lambda^{\alpha+1/3}) \\ &\le 2D\lambda^D \exp(-C_{\alpha h}\lambda^\alpha) \end{aligned}$$

for sufficiently large $\lambda$, which proves the lemma. □

The remainder of the proof will be taken up with showing that the support of $L_1^{\cdot}$ can be covered with a "small" number of intervals with high probability. Fairly precise information about the size of this set is in Section 4 of Griffin (1985). We take the same approach, but our needs are not as great. To begin, decompose $X$ into two processes $X(t) = X_1(t) + X_2(t)$, where

$$X_2(t) = \sum_{x \le t} (X(s) - X(s-))1(|X(s) - X(s-)| > 1),$$

and $X_1(t) = X(t) - X_2(t)$. Then $X_1$ and $X_2$ are independent Lévy processes. Denoting the Lévy measure of $X$ by $\nu(dx)$, $X_1$ has Lévy measure $1(|x| \le 1)\nu(dx)$. This process has bounded jumps; consequently, it has a moment generating function

$$(3.7) \qquad E\exp(uX_1(t)) = \exp(t\phi(u)),$$

where $\phi(u) = \int_{-1}^{1}(e^{ux} - 1)\nu(dx)$ and $\phi(u) \to 0$ as $u \to 0$.

Inductively define stopping times $T(j)$ by setting $T(0) = 0$, and given $T(j)$, letting $T(j+1) = \inf\{s > T(j): |X(s) - X(s-)| > 1\}$. The random variables

$T(1), T(2) - T(1), \dots$ are independent exponential random variables of parameter $\nu(|x| > 1) = b$. Let $J = \max\{j\colon T(j) \le 1\}$. Then $J$ is Poisson with parameter $b$. Finally, let

$$V_j = \sup_{T(j-1) \le s < T(j)} |X(s) - X(T(j-1))|$$

and $V^* = \max\{V_j\colon j \le J\}$. We now collect some facts about these random variables.

LEMMA 3.8. *There is a* $\beta > 0$ *so that for all* $\lambda \ge D$, *the following estimates hold*:

$$\text{(i)} \qquad P(J > \lambda) \le De^{-\beta\lambda},$$

$$\text{(ii)} \qquad P(V_1 > \lambda) \le De^{-\beta\lambda},$$

$$\text{(iii)} \qquad P(V^* > \lambda) \le De^{-\beta\lambda}.$$

PROOF. (i) is a standard Chernoff type bound for Poisson random variables. To see (ii), let $X_1^*(t) = \sup_{0 \le s \le t}|X_1(s)|$, and let $X_1'$ be an independent copy of $X_1$. Then by (3.7), there is a $u > 0$ so that

$$E \exp(u|X_1(t) - X_1'(t)|) \le e^t.$$

Then, again using symmetrization and Lévy's inequality [see the argument which lead to (3.3)], we have for $t > 1$ and $\lambda > 4t/u$,

$$\begin{aligned} P(X_1^*(t) > \lambda) &\le DP\Big(\sup_{0 \le s \le t} |X_1(s) - X_1'(s)| > \lambda/2\Big) \\ &\le 2DP(|X_1(t) - X_1'(t)| > \lambda/2) \\ &\le 2D\exp(-u\lambda/2 + t) \\ &\le 2D\exp(-u\lambda/4). \end{aligned}$$

Therefore, for $\lambda > 8/u$,

$$\begin{aligned} P(V_1 > \lambda) &\le P(T(1) > u\lambda/8) + P(X_1^*(u\lambda/8) > \lambda) \\ &\le De^{-\beta\lambda}, \end{aligned}$$

for some $D$ and $\beta > 0$. This proves (ii).

(iii) is an immediate consequence of the previous two inequalities. □

We can now conclude the proof of (3.1). For $s < t$, let $L(x; s, t) = L_t^x - L_s^x$. Then observe that for $\lambda > 2$, the conditions $J \le \lambda^{2\alpha} - 1$ and $V^* \le \lambda^{2\alpha}$ imply that

$$\operatorname{supp} L_1^{\cdot} \subset \bigcup_{j \le \lambda^{2\alpha}} (X(T(j)) - \lambda^{2\alpha}, X(T(j)) + \lambda^{2\alpha}).$$

Consequently,

$$\begin{aligned}
P(L_1^* > \lambda + \lambda^{2/3}) &\\
&\le P(J > \lambda^{2\alpha} - 1) + P(V > \lambda^{2\alpha})\\
&\quad + D \sum_{0 \le j \le \lambda^{2\alpha}} P\Big( \sup_{|x| \le \lambda^{2\alpha}} L(x + X(T(j)); T(j), T(j) + 1) > \lambda + \lambda^{2/3}\Big)\\
&\le 2D \exp(-\beta \lambda^{2\alpha}) + D\lambda^{2\alpha} P\Big( \sup_{|x| \le \lambda^{2\alpha}} L_1^x > \lambda + \lambda^{2/3}\Big)\\
&\le D\lambda^D \exp(-C_{\alpha h} \lambda^\alpha).
\end{aligned}$$

The penultimate line follows from the strong Markov property and Lemma 3.8, and the last by (3.3). This finishes the proof of (3.1). □

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