The Annals of Probability
1990, Vol. 18, No. 4, 1759–1764

ASSOCIATION OF STABLE RANDOM VARIABLES

By Mei-Ling Ting Lee, Svetlozar T. Rachev[1] and Gennady Samorodnitsky[2]

Boston University, University of California at Santa Barbara and Cornell University

We present necessary and sufficient conditions for association and negative association of jointly α-stable random variables, $0 < \alpha < 2$.

1. Introduction. This research was inspired by Pitt (1982), who proved that jointly normal random variables are associated if and only if their correlations are all nonnegative [see also Joag-dev, Perlman and Pitt (1983)]. Jointly α-stable random variables, $0 < \alpha < 2$, are close "relatives" of jointly normal variables (the latter are "2-stable"), and our purpose is to find necessary and sufficient conditions for α-stable random variables, with $0 < \alpha < 2$, to be associated. Association of α-stable random variables is of a special interest because stable laws have been used to model the distribution of stock price changes [see, e.g., Akgirav and Booth (1988)]. We recall that random variables $X_1, \dots, X_n$ are called *associated* if for any functions $f, g\colon \mathbb{R}^n \to \mathbb{R}$ nondecreasing in each argument, $\operatorname{cov}(f(\mathbf{X}), g(\mathbf{X})) \ge 0$ whenever the covariance exists. Here $\mathbf{X} = (X_1, \dots, X_n)$. Esary, Proschan and Walkup (1967) is our reference for basic properties of associated random variables.

Random variables $X_1, \dots, X_n$ are said to be *jointly α-stable*, $0 < \alpha < 2$, if their joint characteristic function has the form

$$E \exp i(\mathbf{X}, \boldsymbol{\theta}) = \exp\left\{ -\int_{S_n} |(\mathbf{s}, \boldsymbol{\theta})|^\alpha \bigl(1 - i \operatorname{sign}((\mathbf{s}, \boldsymbol{\theta}))\varphi(\alpha; \mathbf{s}, \boldsymbol{\theta})\bigr)\Gamma(d\mathbf{s}) + i(\boldsymbol{\mu}^0, \boldsymbol{\theta})\right\}, \tag{1.1}$$

where $\mathbf{X} = (X_1, \dots, X_n)$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$, S_n is the unit sphere in $\mathbb{R}^n$, $\mathbf{s} = (s_1, \dots, s_n) \in S_n$, Γ is a finite Borel measure on S_n, $\boldsymbol{\mu}^0 = (\mu_1^0, \dots, \mu_n^0) \in \mathbb{R}^n$ and

$$\varphi(\alpha; \boldsymbol{\theta}, \mathbf{s}) = \begin{cases} \tan \dfrac{\pi\alpha}{2} & \text{if } \alpha \ne 1, \\ -\dfrac{2}{\pi} \ln|(\mathbf{s}, \boldsymbol{\theta})| & \text{if } \alpha = 1. \end{cases}$$

Received March 1989; revised April 1989.

[1]Research supported in part by the Army Office of Scientific Research through the Mathematical Sciences Institute during the author's visit to MSI in December 1988.

[2]Research initiated under support by National Science Foundation Grant DMS-88-05627 and completed under support by Air Force Office of Scientific Research Grant 89-0115.

AMS 1980 *subject classifications*. Primary 60E07, 62H20.

Key words and phrases. α-stable, positive dependence, association, negative association.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Probability*. www.jstor.org

There is a one-to-one correspondence between the distribution of jointly α-stable random variables $X_1, \dots, X_n$ and the finite Borel measure Γ in (1.1); the latter is called the *spectral measure* of the α-stable vector $\mathbf{X} = (X_1, \dots, X_n)$. We refer the reader to Weron (1984) for more information on stable distributions.

2. Necessary and sufficient conditions. The following theorem is our main result. A result related to the sufficiency part of our theorem was obtained by Resnick (1988) in terms of a Poisson representation of an infinitely divisible random vector.

THEOREM 2.1. *Let $X_1, \dots, X_n$ be jointly α-stable random variables, $0 < \alpha < 2$, with characteristic function given by* (1.1). *Then $X_1, \dots, X_n$ are associated if and only if the spectral measure Γ satisfies the condition*

$$\Gamma(S_n -) = 0, \tag{2.1}$$

where $S_n - = \{(s_1, \dots, s_n) \in S_n$: *for some* $i, j \in \{1, \dots, n\}$, $s_i > 0$ *and* $s_j < 0\}$.

PROOF. (Necessity) Suppose on the contrary that the $X_1, \dots, X_n$ are associated and still $\Gamma(S_n -) > 0$. Let $(Y_1, \dots, Y_n)$ be an independent copy of $(X_1, \dots, X_n)$, and let $U_i = X_i - Y_i$, $i = 1, \dots, n$. Then the random variables $U_1, \dots, U_n$ are also associated. Moreover, they are also jointly symmetric α-stable, i.e., they have characteristic function of the form (1.1), but with shift vector $\tilde{\boldsymbol{\mu}}^0 = \mathbf{0}$, and spectral measure $\tilde{\Gamma} = \Gamma + \Gamma^*$, where $\Gamma^*(A) = \Gamma(-A)$ for any Borel set A on S_n. Clearly, $\tilde{\Gamma}(S_n -) > 0$. Therefore, there are $i, j \in \{1, \dots, n\}$ such that

$$\tilde{\Gamma}\big(\{(s_1, \dots, s_n) \in S_n : s_i > 0,\ s_j < 0\}\big) > 0. \tag{2.2}$$

Without loss of generality we assume that $i = 1$, $j = 2$. Clearly, the random variables U_1, U_2 are jointly symmetric α-stable in $\mathbb{R}^2$, and a simple change of variables of integration in (1.1) shows together with (2.2) that the spectral measure $\Gamma_{1,2}$ of U_1, U_2 satisfies

$$\Gamma_{1,2}(\{(s_1, s_2) \in S_2 : s_1 > 0,\ s_2 < 0\}) > 0. \tag{2.3}$$

Moreover, U_1, U_2 are associated, being a subset of a set of associated random variables $U_1, \dots, U_n$. For any $\lambda > 0$, the association of U_1 and U_2 implies that

$$P(U_2 \le -\lambda | U_1 > \lambda) \le P(U_2 \le -\lambda). \tag{2.4}$$

Clearly, the right-hand side of (2.4) goes to zero as λ goes to infinity. Moreover, as U_1 and $-U_2$ are also jointly symmetric α-stable with spectral measure $\overline{\Gamma}_{1,2}$ defined by $\overline{\Gamma}_{1,2}(A) = \Gamma_{1,2}(\{(s_1, s_2) \in S_2 : (s_1, -s_2) \in A\})$, we conclude by (2.3) and a result of Samorodnitsky (1988) that

$$\lim_{\lambda\to\infty} P(U_2 \le -\lambda | U_1 > \lambda) = \frac{\int_{\{s_1>0,\, s_2<0\}} (\min(s_1, -s_2))^\alpha \Gamma_{1,2}(d\mathbf{s})}{\int_{S_2} ([s_1]_+)^\alpha \Gamma_{1,2}(d\mathbf{s})} > 0.$$

The contradiction thus obtained proves the necessity part of the theorem.

(Sufficiency) Suppose that (2.1) holds. then the spectral measure Γ is concentrated on $S_n^{\text{pos}} \cup S_n^{\text{neg}}$, where

$$S_n^{\text{pos}} = \{(s_1, \dots, s_n) \in S_n | s_i \ge 0, i = 1, \dots, n\},$$
$$S_n^{\text{neg}} = \{(s_1, \dots, s_n) \in S_n | s_i \le 0, i = 1, \dots, n\}.$$

We may and will assume until the end of the proof that the shift vector $\boldsymbol{\mu}^0$ in the characteristic functions of $X_1, \dots, X_n$ and all other jointly α-stable random variables defined later is zero.

Let $\mathbf{X}^{(+)} = (X_1^{(+)}, \dots, X_n^{(+)})$ and $\mathbf{X}^{(-)} = (X_1^{(-)}, \dots, X_n^{(-)})$ be two independent α-stable random vectors in $\mathbb{R}^n$ with spectral measures $\Gamma^{(+)}$ and $\Gamma^{(-)}$ correspondingly given by $\Gamma^{(+)}(A) = \Gamma(A \cap S_n^{\text{pos}})$, $\Gamma^{(-)}(A) = \Gamma((-A) \cap S_n^{\text{neg}})$ for any Borel set A on S_n. Clearly, $(X_1^{(+)}, \dots, X_n^{(+)}) - (X_1^{(-)}, \dots, X_n^{(-)}) =_d (X_1, \dots, X_n)$, so that it is enough to prove both that $X_1^{(+)}, \dots, X_n^{(+)}$ are associated and that $X_1^{(-)}, \dots, X_n^{(-)}$ are associated. To simplify the following notation, we assume that, to start with, the spectral measure Γ of the α-stable vector $(X_1, \dots, X_n)$ is concentrated on S_n^{pos}, and we will prove that the random variables $X_1, \dots, X_n$ are associated.

Let M be an α-stable random measure on S_n^{pos} with control measure Γ and skewness intensity 1 [see Hardin (1984) and Samorodnitsky (1988) for information on stable random measures and integrals with respect to these measures]. Then $(X_1, \dots, X_n) =_d (Y_1, \dots, Y_n)$, where

$$Y_i = \int_{S_n^{\text{pos}}} s_i M(d\mathbf{s}), \qquad i = 1, \dots, n.$$

By the definition of the integral, for each $i = 1, \dots, n$ there is a sequence of simple Borel functions from S_n^{pos} to $\mathbb{R}$, denoted by $\{f_m^{(i)}, m = 1, 2, \dots\}$, such that $Y_m^{(i)} \to_{m\to\infty} Y_i$ in probability, where $Y_m^{(i)} = \int_{S_n^{\text{pos}}} f_m^{(i)}(\mathbf{s}) M(d\mathbf{s})$, $m = 1, 2, \dots$. Clearly, we may always choose the functions $f_m^{(i)}$ in such a way that $f_m^{(i)}(\mathbf{s}) \ge 0$ for every m, i, s. Suppose that

$$f_m^{(i)}(\mathbf{s}) = \sum_{j=1}^{K_{m,i}} f_j^{(m,i)} 1\big(\mathbf{s} \in A_j^{(m,i)}\big), \qquad i = 1, \dots, n,\ m = 1, 2, \dots,$$

for some $f_j^{(m,i)} \ge 0$, $i = 1, \dots, n$, $m = 1, 2, \dots$, $j = 1, \dots, K_{m,i}$, where $A_1^{(m,i)}, \dots, A_{K_{m,i}}^{(m,i)}$ is a partition of S_n^{pos} into Borel sets. Since the measure M is independently scattered, for each fixed $m = 1, 2, \dots$, the collection

$$\mathscr{C}_m = \Big\{M_{j_1}\big(A_{j_1}^{(m,1)} \cap A_{j_2}^{(m,2)} \cap \cdots \cap A_{j_n}^{(m,n)}\big) | j_1 = 1, \dots, K_{m,1}; \dots; \\ j_n = 1, \dots, K_{m,n}\Big\}$$

is a collection of independent, therefore associated, random variables. Since

$$Y_m^{(i)} = \sum_{j=1}^{K_{m,i}} f_j^{(m,i)} M\big(A_j^{(m,i)}\big), \qquad i = 1,\dots,n,$$

we conclude that each $Y_m^{(i)}$ is a nondecreasing function of random variables from $\mathscr{E}_m$. Therefore, $Y_m^{(1)},\dots,Y_m^{(n)}$ are associated for any $m = 1,2,\dots$. But $(Y_m^{(1)},\dots,Y_m^{(n)}) \to_{m\to\infty} (Y_1,\dots,Y_n)$ in probability, and association is preserved under convergence in distribution. Hence, $Y_1,\dots,Y_n$ are associated, completing the proof of the theorem. □

The next corollary is an immediate consequence of Theorem 2.1. The random variables $X_1,\dots,X_n$ are called *positive upper orthant dependent* (PUOD) if

$$P(X_1 > x_1,\dots,X_n > x_n) \ge P(X_1 > x_1)\cdots P(X_n > x_n)$$

for any $x_1,\dots,x_n$, and we call them *positive lower orthant dependent* (PLOD) if

$$P(X_1 \le x_1,\dots,X_n \le x_n) \ge P(X_1 \le x_1)\cdots P(X_n \le x_n)$$

for any $x_1,\dots,x_n$. It is well known that association implies both PUOD and PLOD, but in general these implications cannot be reversed.

COROLLARY 2.1. *Let $X_1,\dots,X_n$ be jointly α-stable. Then all the notions of positive dependence above are equivalent, and each of them is equivalent to* (2.1).

PROOF. By Theorem 2.1, it is enough to prove that each one of PUOD and PLOD separately implies (2.1). The fact that PUOD implies (2.1) follows directly from the proof of the necessity part of Theorem 2.1, while the same argument applied to the vector $(-X_1,\dots,-X_n)$ shows that PLOD implies (2.1) as well. □

Finally, we give necessary and sufficient conditions for negative association of jointly α-stable random variables. Following Alam and Saxena (1981), we call random variables $X_1,\dots,X_n$ *negatively associated* if for any $1 < k < n$, any $f\colon \mathbb{R}^k \to \mathbb{R}$, $g\colon \mathbb{R}^{n-k} \to \mathbb{R}$, nondecreasing in each argument, $\mathrm{cov}(f(\mathbf{Y}), g(\mathbf{Z})) \le 0$ whenever the covariance exists, where $\mathbf{Y}$ and $\mathbf{Z}$ are any k- and $(n-k)$-dimensional random vectors, respectively, representing a partition of the set $(X_1,\dots,X_n)$ into two subsets of sizes k and $n-k$ accordingly. In the normal case, negative association has been characterized by Joag-dev and Proschan (1983).

THEOREM 2.2. *Let $X_1, \dots, X_n$ be jointly α-stable random variables, $0 < \alpha < 2$, with characteristic function given by* (1.1). *Then $X_1, \dots, X_n$ are negatively associated if and only if the spectral measure Γ satisfies the condition*

$$\Gamma(S_n^+) = 0 \tag{2.5}$$

where $S_n^+ = \{(s_1, \dots, s_n) \in S_n:$ for some $i \neq j$, $s_i \cdot s_j > 0\}$.

PROOF. The proof of the necessity part is identical to that of the necessity part of Theorem 2.1 and is omitted. For the sufficiency part note that (2.5) implies that for any $\{i, j\} \neq \{i', j'\}$

$$\Gamma(E_{i,j} \cap E_{i',j'}) = 0, \tag{2.6}$$

where for any $i \neq j \in \{1, \dots, n\}$, $E_{i,j} = \{(s_1, \dots, s_n) \in S_n: s_i \cdot s_j \neq 0\}$. Denoting for $i = 1, \dots, n$, $D_i = \{(s_1, \dots, s_n) \in S_n$: for each $j \neq i$, $s_i \neq 0$ and $s_j = 0\}$, we conclude that

$$(X_1, \dots, X_n) \stackrel{d}{=} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (X_1(i, j), \dots, X_n(i, j)) + (W_1, \dots, W_n), \tag{2.7}$$

where, in the notation of (the proof of) Theorem 2.1, $X_k(i, j) = \int_{E_{i,j}} s_k M(d\mathbf{s})$, $k = 1, \dots, n$, $i = 1, \dots, n-1$, $j = i+1, \dots, n$, and $W_k = \int_{D_k} s_k M(d\mathbf{s})$, $k = 1, \dots, n$.

Note that the random vectors appearing in the right-hand side of (2.7) are all independent, so that we will be done once we prove that each one of these random vectors consists of negatively associated random variables. The latter is trivially true for the random vector $(W_1, \dots, W_n)$ which consists of independent random variables. Moreover, for each fixed (i, j), (2.6) implies that $X_k(i, j) = 0$ a.s. if $k \notin \{i, j\}$, which reduces our task to showing that $X_i(i, j)$ and $X_j(i, j)$ are negatively associated. But (2.5) implies that $s_i \cdot s_j \leq 0$, Γ-a.e. on $E_{i,j}$, so that by Theorem 2.1, $X_i(i, j)$ and $-X_j(i, j)$ are associated, which implies, of course, the negative association of $X_i(i, j)$ and $X_j(i, j)$. The proof of the theorem is now complete. □

REFERENCES

AKGIRAV, V. and BOOTH, G. G. (1988). The stable-law model of stock returns. *J. Bus. Econ. Statist.* **6** 51–57.

ALAM, K. and SAXENA, K. M. L. (1981). Positive dependence in multivariate distributions. *Comm. Statist. A—Theory Methods* **10** 1183–1196.

BRINDLEY, J. E. C. and THOMPSON, J. W. A. (1972). Dependence and aging aspects of multivariate survival. *J. Amer. Statist. Assoc.* **67** 821–829.

ESARY, J. D., PROSCHAN, F. and WALKUP, D. W. (1967). Association of random variables, with applications. *Ann. Math. Statist.* **38** 1466–1474.

HARDIN, C. D., JR. (1984). Skewed stable variables and processes. Technical Report 79, Center for Stochastic Processes, Univ. North Carolina, Chapel Hill.

JOAG-DEV, K., PERLMAN, M. D. and PITT, L. D. (1983). Association of normal random variables and Slepian's inequality. *Ann. Probab.* **11** 451–455.

JOAG-DEV, K. and PROSCHAN, F. (1983). Negative association of random variables, with applications. *Ann. Statist.* **11** 286–295.

PITT, L. D. (1982). Positively correlated normal variables are associated. *Ann. Probab.* **10** 496–499.

RESNICK, S. I. (1988). Association and multivariate extreme value distributions. In *Gani Festschrift: Studies in Statistical Modeling and Statistical Science* (C. C. Heyde, ed.) 261–271. Statist. Soc. of Australia.

SAMORODNITSKY, G. (1988). Extrema of skewed stable processes. *Stochastic Process. Appl.* **30** 17–39.

WERON. A. (1984). Stable processes and measures: A survey. *Probability Theory on Vector Spaces III. Lecture Notes in Math.* **1080** 306–364. Springer, Berlin.

M.-L. T. LEE
DEPARTMENT OF MATHEMATICS
BOSTON UNIVERSITY
BOSTON, MASSACHUSETTS 02215

S. T. RACHEV
STATISTICS AND APPLIED
PROBABILITY PROGRAM
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA 93106

G. SAMORODNITSKY
SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853