The Annals of Probability
1990, Vol. 18, No. 3, 959–977

RANDOM WALKS AND INTERSECTION LOCAL TIME[1]

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With each random walk on $\mathbb{Z}^2$ we associate a functional related to the number of steps which the walk spends in sites occupied at least k times. We show that if our random walk is in the domain of attraction of a stable process of order greater than $2(2k-2)/(2k-1)$, then our functional coverges to the intersection local time of the limiting process.

1. Introduction. We begin with k independent random walks $S(1,\cdot)$, $S(2,\cdot),\dots,S(k,\cdot)$ in $\mathbb{Z}^2$ and study the set of times $(i_1,\dots,i_k)$ such that

$$S(1,i_1)=S(2,i_2)=\cdots=S(k,i_k).$$

Note that we do not require our walks to collide; it is enough that their paths cross. The number of such times, suitably normalized, we define as

$$(1.1)\quad I(n)=\sum_{i_1,\dots,i_k=1}^{n}\delta(S(1,i_1),S(2,i_2))\cdots\delta(S(k-1,i_{k-1}),S(k,i_k))\frac{1}{n}$$

where

$$\delta(x,y)=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{otherwise}\end{cases}$$

is the usual Kroenecker delta.

If our random walks converge in some sense to independent Brownian motions $W(1,), W(2,),\dots,W(k,)$, we would expect $I(n)$ to converge (in some sense) to

$$(1.2)\quad \int_0^1\cdots\int_0^1\delta(W(1,t_1)-W(2,t_2))\cdots \times\delta(W(k-1,t_{k-1})-W(k,t_k))\,dt_1\cdots dt_k,$$

where δ is now the Dirac delta "function." Of course, (1.2) is purely formal, but can be defined rigorously as follows. We replace the δ functions by approximate δ's,

$$p_\varepsilon(x)=\frac{e^{-x^2/2\varepsilon}}{2\pi\varepsilon},$$

Received March 1987; revised August 1988.
[1]Supported in part by NSF Grant DMS-86-02651.
AMS 1980 *subject classifications.* Primary 60G60; secondary 60J55, 60J65.
Key words and phrases. Random walks, intersection local time, multiple points, domain of attraction.

and define

$$\alpha(\varepsilon, B) = \int_B \cdots \int p_\varepsilon(W(1, t_1) - W(2, t_2)) \cdots \times p_\varepsilon(W(k-1, t_{k-1}) - W(k, t_k))\, dt_1 \cdots dt_k. \tag{1.3}$$

It can be shown that $\alpha(B) = \lim_{\varepsilon \to 0} \alpha(\varepsilon, B)$ exists and defines a measure on

$$\{(t_1, \dots, t_k) | W(1, t_1) = \cdots = W(k, t_k)\}.$$

α is called the intersection local time [see Geman, Horowitz and Rosen (1984)]. Formally, (1.2) corresponds to $\alpha([0,1]^k)$, and our first theorem says that under suitable conditions $I(n) \to \alpha([0,1]^k)$. Let us spell out the conditions on our random walks. If

$$S(n) = X_1 + \cdots + X_n,$$

where the X_i are i.i.d., we require that

1. X_i have mean zero,
2. $\mathrm{Var}(X_i) = 1$, and
3. $S(n)$ be strongly aperiodic [see Spitzer (1964)].

THEOREM 1. *$I(n) \to \alpha([0,1]^k)$ in distribution.*

REMARK. This theorem is not a simple consequence of Donsker's theorem, since $\alpha(\cdot)$ is not a continuous functional of the Brownian paths.

Our theorem was motivated by the work of Le Gall (1986), who studied a related functional of k random walks. He was able to show that for his functionals, all moments converge to those of $\alpha(\cdot)$, but except for $k = 2, 3$, the standard criteria do not allow us to conclude that the distribution of $\alpha([0,1]^k)$ is determined by its moments. The particular form of $I(n)$ was proposed by Le Gall.

Let us now turn to the study of self-intersections of a single random walk $S(n)$. If, in analogy with (1.1), we set

$$\tilde{I}(n) = \sum_{1 \le i_1 \le \cdots < i_k \le n} \delta(S(i_1), S(i_2)) \cdots \delta(S(i_{k-1}), S(i_k)) \frac{1}{n},$$

we cannot expect a nice limit as $n \to \infty$. This is due to the fact that the analogue of (1.3),

$$\alpha(\varepsilon, B) = \int_B \cdots \int p_\varepsilon(W(t_1) - W(t_2)) \cdots p_\varepsilon(W(t_{k-1}) - W(t_k))\, dt, \tag{1.4}$$

will, in general, diverge as $\varepsilon \to 0$. Actually, if we restrict our attention to sets $B \subseteq R^k_\le = \{(t_1, \dots, t_k) | 0 \le t_1 \le \cdots \le t_k\}$ which lie away from the diagonals

$t_i = t_{i+1}$, then $\alpha(\varepsilon, B)$ will converge to a measure $\alpha(B)$ supported on

$$\{(t_1, \dots, t_k) \in R^k_{\leq} | W(t_1) = \cdots = W(t_k)\}$$

[see Rosen (1984)], but if we do not wish to limit ourselves to B away from the diagonals, we must find a remedy for the divergence of $\alpha(\varepsilon, B)$.

In Rosen (1986a) we showed that, with the notation $\{Y\} = Y - E(Y)$,

$$\gamma(\varepsilon, B) = \int_B \cdots \int \{p_\varepsilon(W(t_1) - W(t_2))\} \cdots \times \{p_\varepsilon(W(t_{k-1}) - W(t_k))\}\, dt \tag{1.5}$$

converges (in L_2) as $\varepsilon \to 0$, for all bounded Borel sets $B \subseteq R^k_{\leq}$, to a random variable $\gamma(B)$, called the renormalized intersection local time. The analogue of (1.5) for random walks is

$$R(n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \{\delta(S(i_1), S(i_2))\} \cdots \times \{\delta(S(i_{k-1}), S(i_k))\} \frac{1}{n}. \tag{1.6}$$

With the notation $[0,1]^k_{\leq} = [0,1]^k \cap R^k_{\leq}$, we have the following theorem.

THEOREM 2. *$R(n) \to \gamma([0,1]^k_{\leq})$ in distribution.*

Dynkin (1988a) has proposed a different renormalization for $\alpha(\varepsilon, B)$, and proven (1988b) an analogue of our Theorem 2 for his renormalization. Our methods are very different, as are those of Stoll (1986), who considers the case $k = 2$ and $S(1)$ finite range.

We now remove the requirement that our random walks have finite variance (condition 2).

We consider the case of random walks in the domain of attraction of a nondegenerate strictly stable process $X(t)$ of order $\beta > 1$, i.e.,

$$\frac{S(n)}{b(n)} \to X(1), \tag{1.7}$$

with $b(x)$ a function of regular variation of index $1/\beta$.

THEOREM 3. *If $\beta > 2 - 2/k$, then*

$$\frac{b(n)^{2(k-1)}}{n^{k-1}} I(n) \to \alpha_x([0,1]^k). \tag{1.8}$$

Here $\alpha_X(\cdot)$ denotes the analogue of (1.2) and (1.3), where we substitute k independent copies of $X(t)$ for the $W(i,t)$'s [see, e.g., Rosen (1984)].

THEOREM 4. *If $\beta > 2 - 2/(2k - 1)$, then*

$$\frac{b(n)^{2(k-1)}}{n^{k-1}} R(n) \to \gamma_X([0,1])^k_{\leq}.$$

Once more, $\gamma_X(\cdot)$ is obtained from the analogue of (1.5) with $X(t)$ substituted for $W(t)$ [see Rosen (1986b)].

Theorems 1 and 2 are subsumed under the general Theorems 3 and 4, but we intend to spell out the details of the proof in the simpler case (Sections 2–4) so that the interested reader will be able to follow more easily. In Section 5 we will show how to modify the proof for the more general case.

A forthcoming article with J.-F. Le Gall [Le Gall and Rosen (1988)] will apply the results of this paper to develop the asymptotics of the range of a random walk in the plane.

Let $S(n)$ be as in Theorem 4, thus, in the domain of attraction of a stable process X_t. $|S(0,n)|$ will denote the number of distinct sites visited by our walk during its first n steps. Then if $\frac{4}{3} < \beta < 2$,

$$\frac{b^2(n)}{n^2}\left(|S(0,n)| - \mathbb{E}(|S(0,n)|)\right) \to -q^2\gamma_X\left([0,1]^2_{\leq}\right),$$

where q is the probability of never returning to the origin. If $\beta \leq \frac{4}{3}$, a suitably normalized version of $|S(0,n)|$ converges to a normal law.

The case of a random walk with finite covariance was considered in Le Gall (1986). We note that the range of a random walk in dimensions ≥ 3 was considered in a series of papers by Jain, Pruitt and Orey [see references in Le Gall (1986)]. However, it is only now with the concept of intersection local times γ_X that random walks in the plane can be successfully analyzed.

2. Proof of Theorem 1. Although we cannot apply Donsker's theorem directly to the discontinuous functional α, we will apply it to $\alpha(\varepsilon,\cdot)$—and then show how to remove the cutoff.

We begin by rewriting

$$I(n) = \sum_{i_1,\dots,i_k=1}^{n} \delta(S(1,i_1),S(2,i_2)) \cdots \delta(S(k-1,i_{k-1}),S(k,i_k))\frac{1}{n}$$

$$(2.1) \qquad = \sum_{i_1,\dots,i_k=1}^{n} \prod_{j=2}^{k} \frac{1}{(2\pi)^2}\int_{|p_j|_0\leq\pi} \exp\left[ip_j\left(S(j,i_j) - S(j-1,i_{j-1})\right)\right] dp_j \frac{1}{n}$$

$$= \sum_{i_1,\dots,i_k=1}^{n} \frac{1}{n^k}\prod_{j=2}^{k} \frac{1}{(2\pi)^2}\int_{|p_j|_0\leq\pi\sqrt{n}} \exp\left[ip_j\frac{\left(S(j,i_j) - S(j-1,i_{j-1})\right)}{\sqrt{n}}\right] dp_j,$$

where $|(x,y)|_0 = \max(|x|,|y|)$.

We then define a functional which will interpolate between $I(n)$ and $\alpha(\varepsilon,\cdot)$, a "link" (a term coined by Dynkin):

$$(2.2) \qquad I(\varepsilon,n) = \sum_{i_1,\dots,i_k=1}^{n} \frac{1}{n^k}\prod_{j=2}^{k} \frac{1}{(2\pi)^2} \times\int_{|p_j|_0\leq\sqrt{n}\pi} \exp\left[ip_j\frac{\left(S(j,i_j) - S(j-1,i_{j-1})\right)}{\sqrt{n}}\right]\exp\left[-\varepsilon p_j^2\right] dp_j.$$

LEMMA 1.

$$\|I(n) - I(\varepsilon, n)\|_2 \le c\varepsilon^{\delta}, \tag{2.3}$$

for some $c < \infty$, $\delta > 0$ *independent of* n.

Lemma 1 implies that

$$\left|E(e^{i\lambda I(n)}) - E(e^{i\lambda I(\varepsilon, n)})\right| \le c\varepsilon^{\delta} \tag{2.4}$$

independent of n. Choose $\varepsilon_0 > 0$ so that both $c\varepsilon_0^{\delta} \le \gamma$ for small, given γ, and also

$$\left|E\left(\exp\left\{i\lambda\alpha\left(\varepsilon_0, [0,1]^k\right)\right\}\right) - E\left(\exp\left\{i\lambda\alpha\left([0,1]^k\right)\right\}\right)\right| \le \gamma \tag{2.5}$$

[see Geman, Horowitz and Rosen (1984) for (2.5)].

From (2.2) we see that

$$\begin{aligned} I(\varepsilon_0, n) &= \sum_{i_1, \dots, i_k = 1}^{n} \frac{1}{n^k} \prod_{j=2}^{k} p_{\varepsilon_0}\left(\frac{S(j, i_j) - S(j-1, i_{j-1})}{\sqrt{n}}\right) + O(e^{-\varepsilon_0 n}) \\ &= \int \cdots \int_{[0,1]^k} \prod_{j=2}^{k} p_{\varepsilon_0}\left(W^{(n)}(j, t_j) - W^{(n)}(j-1, t_{j-1})\right) dt + O(e^{-\varepsilon_0 n}), \end{aligned} \tag{2.6}$$

where $W^{(n)}(j, t) = S(j, [nt])/\sqrt{n}$, and $O(e^{-\varepsilon_0 n})$ is generic. By Donsker's theorem [see Skorohod (1957)] we can find N_0, such that for $n \ge N_0$,

$$\left|E(\exp\{i\lambda I(\varepsilon_0, n)\}) - E\left(\exp\left\{i\lambda\alpha\left(\varepsilon_0, [0,1]^k\right)\right\}\right)\right| \le \gamma. \tag{2.7}$$

Combining (2.4), (2.5) and (2.7), we have that for all $n \ge N_0$

$$\left|E(e^{i\lambda I(n)}) - E\left(\exp\left\{i\lambda\alpha\left([0,1]^k\right)\right\}\right)\right| \le 3\gamma$$

which completes the proof of Theorem 1, subject to the following.

PROOF OF LEMMA 1.

$$\begin{aligned} &\mathbb{E}(I(n) - I(\varepsilon, n))^2 \\ &\quad = \frac{1}{n^{2k}} \sum_{\substack{i_1, \dots, i_k = 1 \\ j_1, \dots, j_k}}^{n} \frac{1}{(2\pi)^{4(k-1)}} \\ &\qquad \times \int_{|p|_0, |q|_0 \le \pi\sqrt{n}} \mathbb{E}\left(\prod_{l=2}^{k} \exp\left[ip_l \frac{(S(l, i_l) - S(l-1, i_{l-1}))}{\sqrt{n}} \right.\right. \\ &\qquad\qquad \left.\left. + iq_l \frac{(S(l, j_l) - S(l-1, j_{l-1}))}{\sqrt{n}}\right]\right) F(\varepsilon, p, q)\, dp\, dq, \end{aligned} \tag{2.8}$$

where

$$(2.9)\qquad F(\varepsilon,p,q)=\left(1-\exp\left[-\varepsilon\sum_{l=2}^{k}\frac{p_l^2}{2}\right]\right)\left(1-\exp\left[-\varepsilon\sum_{l=2}^{k}\frac{q_l^2}{2}\right]\right),$$

for $|p_l|_0,\ |q_l|_0\le\pi\sqrt{n}$, and for ease in our proof we consider $F(\varepsilon,p,q)$ to be extended periodically, with period $2\pi\sqrt{n}$.

The expectation in (2.8) is

$$(2.10)\qquad \begin{aligned}&\mathbb{E}\left(\prod_{l=1}^{k}\exp\left[i(p_l-p_{l+1})\frac{S(l,i_l)}{\sqrt{n}}\right]\exp\left[i(q_l-q_{l+1})\frac{S(l,j_l)}{\sqrt{n}}\right]\right)\\&\quad=\prod_{l=1}^{k}\mathbb{E}\left(\exp\left[i\tilde{u}_l\frac{S(l,i_l)}{\sqrt{n}}\right]\exp\left[iu_l\frac{S(l,j_l)}{\sqrt{n}}\right]\right),\end{aligned}$$

where $p_1=q_1=0$ and $\tilde{u}_l=p_l-p_{l+1}$, $u_l=q_l-q_{l+1}$.

If $i_l\le j_l$, this expectation is

$$\phi_l^{i_l}\left(\frac{v_l}{\sqrt{n}}\right)\phi_l^{j_l-i_l}\left(\frac{u_l}{\sqrt{n}}\right),$$

where $v_l=\tilde{u}_l+u_l$ and $\phi_l(u)=E(e^{iuS(l,1)})$. If $j_l\le i_l$, we have an analogous expression. For simplicity we concentrate on the case $i_l\le j_l$, $\forall\ l$.

We then bound the contribution to (2.8) by

$$(2.11)\qquad \int_{|p|_0,\ |q|_0\le\pi\sqrt{n}}\prod_{l=1}^{k}G(l,n,u_l)G(l,n,v_l)|F(\varepsilon,p,q)|\,dp\,dq,$$

where

$$G(l,n,u)=\frac{1}{n}\sum_{k=0}^{n}\left|\phi_l^k\left(\frac{u}{\sqrt{n}}\right)\right|.$$

Since our assumptions imply

$$(2.12)\qquad \left|\phi_l\left(\frac{u}{\sqrt{n}}\right)\right|\le e^{-b((u))^2/n},$$

where $((u))$ is the representative of $u \bmod 2\pi\sqrt{n}$ of smallest absolute value, we have

$$(2.13)\qquad \begin{aligned}G(l,n,u)&\le\sum_{k=0}^{n}e^{-b(k/n)((u))^2}\frac{1}{n}\\&\le\left(\frac{1}{n}+\int_0^1 e^{-bt((u))^2}\,dt\right)\\&\le c\left(\frac{1}{n}+\frac{1}{1+((u))^2}\right)\\&\le\frac{c}{1+((u))^2},\end{aligned}$$

since $|((u))|\le 4\pi\sqrt{n}$.

Thus, (2.11) is bounded by

$$c\int_{|p|_0,\,|q|_0\le\pi\sqrt{n}}\prod_{l=1}^{k}\left(1+((u_l))^2\right)^{-1}\left(1+((v_l))^2\right)^{-1}|F(\varepsilon,p,q)|\,dp\,dq$$

$$(2.14)\qquad \begin{aligned}&= c\int_{|p|_0,\,|q|_0\le\pi\sqrt{n}}\prod_{m=1}^{k}\left(\prod_{l\ne m}\left(1+((u_l))^2\right)^{-1}\left(1+((v_l))^2\right)^{-1}\right)^{1/k-1}|F(\varepsilon,p,q)|\,dp\,dq\\ &\le c\prod_{m=1}^{k}\Bigg[\int_{|p|_0,\,|q|_0\le\pi\sqrt{n}}\prod_{l\ne m}\left(1+((u_l))^2\right)^{-k/k-1}\\ &\qquad\qquad\times\left(1+((v_l))^2\right)^{-k/k-1}|F(\varepsilon,p,q)|\,dp\,dq\Bigg]^{1/k}.\end{aligned}$$

Since each p_l, q_l is a linear combination of elements of the set $\{u_i, v_i, i\ne m\}$ for any fixed m, we have

$$(2.15)\qquad \begin{aligned}|F(\varepsilon,p,q)| &\le c\varepsilon^{\delta}\sum|p_l|^{\delta}\sum|q_l|^{\delta}\\ &\le c\varepsilon^{\delta}\left(1+\sum_{l\ne m}((u_l))^{2\delta}+((v_l))^{2l}\right).\end{aligned}$$

Here we used the fact that

$$|((x+y))|\le|((x))|+|((y))|.$$

From (2.15) we see that (2.14) is bounded by $c\varepsilon^{\delta}$, once we prove the finiteness of

$$(2.16)\qquad \int_{|u|_0\le 4\pi\sqrt{n}}\left(1+((u))^{2\delta}\right)\frac{1}{\left(1+((u))^2\right)^{k/k-1}}\,d^2u,$$

where we have expanded the region of integration, since, e.g., $|v_l| = |p_l - p_{l+1} + q_l - q_{l+1}| \le 4\pi\sqrt{n}$. (2.16) is clearly finite if $\delta < 1/(k-1)$. □

3. Proof of Theorem 2. As in the proof of Theorem 1, we define

$$(3.1)\qquad \begin{aligned}R(\varepsilon,n) = \sum_{1\le i_1<\cdots<i_k\le n}&\frac{1}{n^k}\prod_{j=2}^{k}\frac{1}{(2\pi)^2}\\ &\times\int_{|p_j|_0\le\pi\sqrt{n}}\left\{\exp\left[ip_j\frac{(S(i_j)-S(i_{j-1}))}{\sqrt{n}}\right]\right\}e^{-\varepsilon p_j^2/2}\,dp_j\end{aligned}$$

and once again it suffices to prove the following.

LEMMA 2.

$$(3.2)\qquad \|R(\varepsilon,n)-R(n)\|_2\le c\varepsilon^{\delta},$$

for some $c<\infty$, $\delta>0$ independent of n.

PROOF. We have

$$E(R(n)-R(\varepsilon,n))^2$$

$$(3.3)\quad = \frac{1}{n^{2k}} \sum_{\substack{1\le i_1<\cdots<i_k\le n\\ 1\le j_1<\cdots<j_k\le n}} \int_{|p|_0,\,|q|_0\le \pi\sqrt{n}} \mathbb{E}\Bigg(\prod_{l=2}^{k}\left\{\exp\left[ip_l\frac{(S(i_l)-S(i_{l-1}))}{\sqrt{n}}\right]\right\} \times\left\{\exp\left[iq_l\frac{(S(j_l)-S(j_{l-1}))}{\sqrt{n}}\right]\right\}\Bigg)F(\varepsilon,p.q),$$

where $F(\varepsilon,p,q)$ is given by (2.9).

The expectation will depend on the relative ordering of the $2k$ integers $i_1,\dots,i_k,\ j_1,\dots,j_k$. Fix such an ordering. We assume for now that our $2k$ integers are distinct and we rename them

$$r_1<r_2<\cdots<r_{2k}.$$

Later we deal with the case when some of the integers coincide. We will refer to $i_1,\dots,i_k$ as blue integers and to $j_1,\dots,j_k$ as white.

We will say that $[r_{m-1},r_m]$ is an isolated interval if r_{m-1},r_m are of the same color. If, e.g., $[r_{m-1},r_m]=[i_l,i_{l-1}]$, then we say that p_l is an isolated variable and write $v_m=p_l$. Let $\mathscr{I}=\{m|[r_{m-1},r_m]\text{ is isolated}\}$.

We can write

$$(3.4)\quad \begin{aligned}&\sum p_l(S(i_l)-S(i_{l-1}))+\sum q_l(S(j_l)-S(j_{l-1}))\\ &\quad=\sum u_m(S(r_m)-S(r_{m-1})),\end{aligned}$$

where u_m is necessarily the sum of one p and one q—more precisely, if

$$[i_{l-1},i_l]\cap[j_{h-1},j_h]\supseteq[r_{m-1},r_m],$$

then $u_m=p_l+q_h$.

If $[r_{m-1},r_m]$ is an isolated interval, one of these (p_l or q_h) will be the isolated variable v_m, and we denote the other by w_m and refer to it as the coisolated variable for m.

If we expand those brackets in the expectation of (3.3) which refer to nonisolated intervals, then we will obtain a sum of many terms, the simplest of which can be written, with $\mathscr{I}^c$ denoting the complement of $\mathscr{I}$, as

$$(3.5)\quad \begin{aligned}&\mathbb{E}\Bigg(\prod_{m\in\mathscr{I}^c}\exp\left[iu_m\frac{S(r_m)-S(r_{m-1})}{\sqrt{n}}\right]\prod_{m\in\mathscr{I}}\left\{\exp\left[iv_m\frac{S(r_m)-S(r_{m-1})}{\sqrt{n}}\right]\right\}\\ &\qquad\qquad\times\exp\left[iw_m\frac{S(r_m)-S(r_{m-1})}{\sqrt{n}}\right]\Bigg)\\ &\quad=\prod_{m\in\mathscr{I}^c}\phi^{\tilde r_m}\left(\frac{u_m}{\sqrt{n}}\right)\prod_{m\in\mathscr{I}}\phi^{\tilde r_m}\left(\frac{v_m+w_m}{\sqrt{n}}\right)-\phi^{\tilde r_m}\left(\frac{v_m}{\sqrt{n}}\right)\phi^{\tilde r_m}\left(\frac{w_m}{\sqrt{n}}\right),\\ &\qquad\qquad\tilde r_m=r_m-r_{m-1}.\end{aligned}$$

We first control this term—and then come back to the other terms which arise from expanding the brackets.

As in the proof of Lemma 1, it suffices to bound

$$(3.6)\qquad \int_{|p|_0,\ |q|_0 \le \pi\sqrt{n}} \prod_{m \in \mathscr{I}^c} \left(1 + ((u_m))^{2\delta}\right) G_1(n, u_m) \times \prod_{m \in \mathscr{I}} \left(1 + ((u_m))^{2\delta}\right) H(n, v_m, w_m)\, dp\, dq,$$

where

$$(3.7)\qquad G_1(n,u) = \frac{1}{n} \sum_{k=1}^{n} \left| \phi^k\left(\frac{u}{\sqrt{n}}\right) \right|$$

and

$$(3.8)\qquad H(n,v,w) = \frac{1}{n} \sum_{k=1}^{n} \left| \phi^k\left(\frac{v+w}{\sqrt{n}}\right) - \phi^k\left(\frac{v}{\sqrt{n}}\right) \phi^k\left(\frac{w}{\sqrt{n}}\right) \right|.$$

In the next section we prove the following lemma.

LEMMA 3. *For any $\delta > 0$ small, we can find $\bar{\delta} > 0$ such that*

$$(3.9)\qquad \int_{|v|_0 \le \pi\sqrt{n}} [1 + v^{2\bar{\delta}}] H(n,v,w)\, d^2 v \le c(1 + w^{2\delta}).$$

Since all coisolated variables w_m are linear combinations of u_i, $i \in \mathscr{I}^c$, and recalling (2.13), we see that it suffices to bound

$$(3.10)\qquad \int_{|p|_0,\ |q|_0 \le \pi\sqrt{n}} \prod_{m \in \mathscr{I}^c} \left(1 + ((u_m))^{2\delta}\right) \frac{1}{1 + ((u_m))^2} \widehat{dp\, dq}$$

where $\widehat{dp\, dq}$ means that integration is restricted to nonisolated variables.

It is easy to check that, even after omitting any one element from the set u_m, $m \in \mathscr{I}^c$, we can still generate all nonisolated variables. As in the proof of Lemma 1, this shows (3.10) bounded for δ small.

We now consider the other terms generated from expanding the brackets in (3.3) for nonisolated intervals. The effect is to replace some of the factors $\phi^{\bar{r}_m}(u_m/\sqrt{n})$ in (3.5) by $\phi^{\bar{r}_m}(p_{i(m)}/\sqrt{n})\phi^{\bar{r}_m}(q_{j(m)}/\sqrt{n})$ where $u_m = p_{i(m)} + q_{j(m)}$. Then in (2.13) we replace $\exp[-bk/n((u_m))^2]$ by

$$\exp\left[-bk/n\left((p_{i(m)})\right)^2 + \left((q_{j_m})\right)^2\right] \le \exp\left[-\bar{b}k/n((u_m))^2\right],$$

since $|((u_m))| = |((p_{i(m)} + q_{j(m)}))| \le |((p_{i(m)}))| + |((q_{j(m)}))|$. This leads once again to (3.10).

Finally, we consider the effect of allowing some of our r_i's to coincide. Since integers of the same color cannot coincide, we see that the effect is to allow some of our nonisolated intervals to collapse—thus, in the definition of $G_1(n,u)$, (3.7), we must also allow $k = 0$. However, this has already been dealt with in (2.13); hence, once again we are led to the bound (3.10).

This completes the proof of Lemma 2, hence of Theorem 2, subject to the proof of Lemma 3, which is the subject of the next section. □

4. Proof of Lemma 3. We must show that for any $\delta > 0$ small we can find $\bar{\delta} > 0$ such that

$$\sum_{k=1}^{n} \frac{1}{n} \int_{|q|_0 \le \pi\sqrt{n}} (1 + q^{2\bar{\delta}}) \left| \phi^k\left(\frac{p+q}{\sqrt{n}}\right) - \phi^k\left(\frac{p}{\sqrt{n}}\right)\phi^k\left(\frac{q}{\sqrt{n}}\right) \right| dq \le c(1 + p^{2\delta}). \tag{4.1}$$

We begin by bounding the left-hand side by the sum of two terms,

$$\sum_{k=1}^{n} \frac{1}{n} \int_{|q|_0 \le \pi\sqrt{n}} (1 + q^{2\bar{\delta}}) \left| \phi^k\left(\frac{q}{\sqrt{n}}\right) \right| \left| 1 - \phi^k\left(\frac{p}{\sqrt{n}}\right) \right| dq \tag{4.2}$$

and

$$\sum_{k=1}^{n} \frac{1}{n} \int_{|q|_0 \le \pi\sqrt{n}} (1 + q^{2\bar{\delta}}) \left| \phi^k\left(\frac{q+p}{\sqrt{n}}\right) - \phi^k\left(\frac{q}{\sqrt{n}}\right) \right| dq. \tag{4.3}$$

For (4.2) we use the bound

$$\begin{aligned}
\left| 1 - \phi^k\left(\frac{p}{\sqrt{n}}\right) \right| &= |\mathbb{E}(1 - e^{ipS(k)/\sqrt{n}})| \\
&\le c\mathbb{E}\left(\frac{p^{2\gamma}}{n^{\gamma}} |S(k)|^{2\gamma} \right) \\
&\le c\frac{p^{2\gamma}}{n^{\gamma}} \mathbb{E}(|S(k)|^2)^{\gamma} \\
&\le cp^{2\gamma}\left(\frac{k}{n}\right)^{\gamma} \quad \text{for } 0 \le \gamma \le \tfrac{1}{2}
\end{aligned}$$

to see that (4.2) is bounded by

$$c\frac{p^{2\gamma}}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{\gamma} \int_{|q|_0 \le \pi\sqrt{n}} (1 + (q)^{2\bar{\delta}}) e^{-ck/nq^2}\, dq \le c\frac{p^{2\gamma}}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{\gamma} \frac{1}{(k/n)^{1+\bar{\delta}}}.$$

If we take $\gamma = 2\bar{\delta}$, this is less than

$$\begin{aligned}
c\frac{p^{4\bar{\delta}}}{n} \sum_{k=1}^{n} \frac{1}{(k/n)^{1-\bar{\delta}}} &= cp^{4\bar{\delta}} \frac{1}{n^{\delta}} \sum_{k=1}^{n} \frac{1}{k^{1-\bar{\delta}}} \\
&\le cp^{4\bar{\delta}}.
\end{aligned}$$

Thus, (4.2) is bounded as required by the lemma.

Let now $J(k)$ denote the integral in (4.3). We will derive two distinct bounds for $J(k)$ and then interpolate between them.

For the first bound, by periodicity,

$$\begin{aligned}
&\int_{|q|_0 \le \pi\sqrt{n}} (1 + q^{2\bar{\delta}}) \left| \phi^k\left(\frac{p+q}{\sqrt{n}} \right) \right| dq \\
&\quad = \int_{|q|_0 \le \pi\sqrt{n}} \left(1 + ((q-p))^{2\bar{\delta}}\right) \left| \phi^k\left(\frac{q}{\sqrt{n}} \right) \right| dq \\
(4.4)\quad &\quad \le \int_{|q|_0 \le \pi\sqrt{n}} (1 + p^{2\bar{\delta}} + q^{2\bar{\delta}}) \exp\left[-c\frac{k}{n}((q))^2 \right] dq \\
&\quad \le c\frac{1 + p^{2\bar{\delta}}}{(k/n)^{1+\bar{\delta}}}.
\end{aligned}$$

A similar estimate with $\phi^k((p+q)/\sqrt{n})$ replaced by $\phi^k(q/\sqrt{n})$ gives our first bound,

$$J(k) \le c\frac{1 + p^{2\bar{\delta}}}{(k/n)^{1+\bar{\delta}}}. \tag{4.5}$$

For our second bound we use the mean value theorem together with the fact that

$$\left| \nabla\phi\left(\frac{x}{\sqrt{n}} \right) \right| \le c\frac{|((x))|}{\sqrt{n}}$$

to find that

$$\begin{aligned}
J(k) &\le \frac{|p|}{\sqrt{n}} \int_{|q|_0 \le \pi\sqrt{n}} k\frac{|((q^*))|}{\sqrt{n}} \left| \phi^{k-1}\left(\frac{q^*}{\sqrt{n}} \right) \right| \left(1 + ((q))^{2\bar{\delta}}\right) dq \\
(4.6)\quad &\le c|p| \left(\frac{k}{n} \right)^{1/2} \int \left[\frac{k}{n}((q^*))^2 \right]^{1/2} \\
&\qquad \times \exp\left[-c\frac{(k-1)}{n}((q^*))^2 \right] \left(1 + ((q))^{2\bar{\delta}}\right) dq,
\end{aligned}$$

where $q^* = q + \theta p$, $|\theta| \le 1$.

If $k \ne 1$, we see from this that

$$J(k) \le c|p| \left(\frac{k}{n} \right)^{1/2} \int_{|q|_0 \le \pi\sqrt{n}} \exp\left[-c\frac{k}{n}((q^*))^2 \right] \left(1 + ((q))^{2\bar{\delta}}\right) dq. \tag{4.7}$$

If $|p| \le |q|/4$, then, since $|p|_0 \le \pi\sqrt{n}$, $|q|_0 \le \pi\sqrt{n}$, we must have $|((q^*))| \ge |q|/4$; hence, if $J_1(k)$ denotes the integral in $J(k)$ over $|q| \ge 4p$,

$$\begin{aligned}
J_1(k) &\le c|p| \left(\frac{k}{n} \right)^{1/2} \int_{|q|_0 \le \pi\sqrt{n}} \exp\left[-c\frac{k}{n}|q|^2 \right] (1 + |q|^{2\bar{\delta}})\, dq \\
(4.8)\quad &\le c\frac{|p|}{(k/n)^{1/2+\bar{\delta}}}, \qquad |p| \le |q|/4,\ k \ne 1.
\end{aligned}$$

On the other hand, if $|q| \le 4|p|$, then by (4.7) if $J_2(k)$ is the integral over $|q| \le 4|p|$,

$$J_2(k) \le c\left(1 + |((p))|^{3+3\bar{\delta}}\right)\left(\frac{k}{n}\right)^{1/2} \int \frac{1 + |q|^{2\bar{\delta}}}{1 + |q|^{2+3\bar{\delta}}}\, dq \tag{4.9}$$

$$\le c\left(1 + |p|^{3+3\bar{\delta}}\right)\left(\frac{k}{n}\right)^{1/2}, \qquad |p| \ge \frac{|q|}{4, k \ne 1}.$$

Putting together (4.8) and (4.9),

$$J(k) \le c\frac{\left(1 + |p|^{3+3\bar{\delta}}\right)}{(k/n)^{1/2+\bar{\delta}}}, \qquad k \ne 1. \tag{4.10}$$

For $k = 1$, we return to (4.6), which gives

$$J(1) \le \frac{|p|}{n} \int_{|q|_0 \le \pi\sqrt{n}} |((q^*))|\left(1 + |q|^{2\bar{\delta}}\right) dq$$

$$\le c\left(1 + |p|^2\right)\frac{1}{n} n^{3/2+\bar{\delta}}$$

$$\le \frac{c\left(1 + |p|^2\right)}{(1/n)^{1/2+\bar{\delta}}}.$$

Thus, for all k we have

$$J(k) \le \frac{c\left(1 + |p|^{3+3\bar{\delta}}\right)}{(k/n)^{1/2+\bar{\delta}}}. \tag{4.11}$$

We now interpolate between (4.5) and (4.11). For any $0 \le t \le 1$,

$$J(k) \le c\frac{1 + |p|^{(3+3\bar{\delta})t+2\bar{\delta}(1-t)}}{(k/n)^{(1/2+\bar{\delta})t+(1+\bar{\delta})(1-t)}}. \tag{4.12}$$

We need to choose t so that

$$\beta = \left(\tfrac{1}{2} + \bar{\delta}\right)t + (1 + \bar{\delta})(1 - t) < 1,$$

i.e., $2\bar{\delta} < t$.

By choosing $\bar{\delta}$ sufficiently small, we can arrange for

$$(3 + 3\bar{\delta})t + 2\bar{\delta}(1 - t) \le 2\delta,$$

for any $\delta > 0$ small. Thus,

$$J(k) \le c\left(1 + |p|^{2\delta}\right)\frac{1}{(k/n)^{\beta}}, \qquad \beta < 1.$$

As before, this bounds (4.3):

$$\sum_{k=1}^{n} \frac{1}{n} J(k) \le \sum_{k=1}^{n} \frac{1}{n} c(1+|p|^{2\delta}) \frac{1}{(k/n)^{\beta}}$$
$$\le c(1+|p|^{2\delta}) \frac{1}{n^{1-\beta}} \sum_{k=1}^{n} \frac{1}{k^{\beta}}$$
$$\le c(1+|p|^{2\delta}).$$

5. Proof of Theorems 3 and 4. Let S_n be a random walk in $\mathbb{Z}^2$, lying in the domain of attraction of a stable process X_t of order β. Thus,

$$\frac{S_n}{b(n)} \to X_1, \tag{5.1}$$

with $b(x)$ a function of regular variation of index $1/\beta$. For the facts that we use concerning functions of regular variation we refer to Bingham, Goldie and Teugels (1987). We may assume that $b(x)$ is continuous and strictly monotone increasing and then its inverse, $l(y)$ will be of regular variation of index β.

We note that for any $\varepsilon > 0$.

$$cn^{1/\beta(1+\varepsilon)} \le b(n) \le dn^{1/\beta(1-\varepsilon)}. \tag{5.2}$$

We can write

$$b(x) = x^{1/\beta} s(x), \tag{5.3}$$

for $s(x)$ a function of slow variation. Then, setting

$$h(x) \doteq s^{\beta}(l(x)),$$

we find that $h(x)$ is of slow variation and satisfies

$$h(b(x)) = s^{\beta}(l(b(x))) = s^{\beta}(x)$$

so that by (5.3),

$$n = \frac{b^{\beta}(n)}{h(b(n))}. \tag{5.4}$$

For small p we can write $\phi(p) = \mathbb{E}e^{(ipS_1)} = e^{-\psi(p)}$, and (5.1) implies that

$$\left| n\psi\left(\frac{p}{b(n)}\right) - p^{\beta} S\left(\frac{p}{|p|}\right) \right| \le \gamma, \tag{5.5}$$

for all $|p| \le 1$ and $n \ge n_0$. Therefore,

$$\left| n \frac{\psi(p/b(n))}{p^{\beta}} - S\left(\frac{p}{|p|}\right) \right| \le 2^{\beta}\gamma, \tag{5.6}$$

for all $\frac{1}{2} \le |p| \le 1$ and $n \ge n_0$.

It is a basic property of functions of slow variation that

$$H(p,n) = \frac{h(b(n))}{h(b(n)/|p|)} \to 1 \quad \text{as } n \to \infty,$$

uniformly in $\frac{1}{2} \le |p| \le 1$.

Together with (5.4), we now have

$$\begin{aligned} &\left| \frac{\psi(p/b(n))}{(|p|/b(n))^{\beta} h(b(n)/|p|)} - S\left(\frac{p}{|p|}\right) \right| \\ &\quad \le \left| \frac{\psi(p/b(n))}{(|p|/b(n))^{\beta} h(b(n))} - S\left(\frac{p}{|p|}\right) \right| H(p,n) \\ &\qquad + \left| S\left(\frac{p}{|p|}\right) \right| |1 - H(p,n)| \\ &\quad \le \bar{\gamma}, \quad \text{which can be taken arbitrarily small.} \end{aligned} \tag{5.7}$$

Thus,

$$\left| \frac{\psi(q)}{q^{\beta} h(1/q)} - S\left(\frac{q}{|q|}\right) \right| \le \bar{\gamma}, \tag{5.8}$$

for all $1/[2b(n)] \le |q| \le 1/b(n)$ and $n \ge n_0$.

Since $b(n)$ is of regular variation of index $1/\beta$, we have

$$\frac{2b(n)}{b(2n)} \to \frac{2}{2^{1/\beta}} > 1$$

(we, of course, assume $\beta > 1$), hence, possibly increasing n_0, the intervals $1/[2b(n)] \le q \le 1/b(n)$, $n = n_0, 2n_0, 4n_0, \ldots$ overlap; thus, (5.8) holds for all $0 \ne |q| \le \delta \doteq 1/b(n_0)$.

From this we have

$$\operatorname{Re} \psi(q) \ge c q^{\beta} h(1/|q|), \qquad |q| \le \delta, \tag{5.9}$$

$$|\psi(q)| \le c q^{\beta} h(1/|q|), \qquad |q| \le \delta. \tag{5.10}$$

By (5.9) for all $|p| \le \delta b(n)$,

$$\begin{aligned} \left| \phi\left(\frac{p}{b(n)}\right) \right| &\le \exp\left[- \operatorname{Re} \psi\left(\frac{p}{b(n)}\right) \right] \\ &\le \exp\left[-c p^{\beta} \frac{h(b(n)/|p|)}{b^{\beta}(n)} \right] \\ &= \exp\left[-c \frac{p^{\beta}}{n} \frac{h(b(n)/|p|)}{h(b(n))} \right]. \end{aligned} \tag{5.11}$$

By Bingham, Goldie and Teugels [(1987), Theorem 1.5.6], for any $\bar{\varepsilon} > 0$ we can choose $\delta > 0$, and constants A, $\bar{A}$ such that

$$Ap^{-\bar{\varepsilon}} \le \frac{h(b(n)/|p|)}{h(b(n))} \le \bar{A}p^{\varepsilon}, \qquad 1 \le |p| \le \delta b(n), \tag{5.12}$$

first for $n \ge n_0$ large, and then for all n by adjusting the constants. Similarly, an analogous inequality holds for $|p| \le 1$.

In particular, returning to (5.11), we have

$$\left|\phi\left(\frac{p}{b(n)}\right)\right| \le \exp\left[-\frac{c}{n}p^{\beta(1-\varepsilon)}\right], \qquad 1 \le |p| \le \delta b(n). \tag{5.13}$$

On the other hand, for $|p| \le \pi b(n)$ we automatically have, by (5.2), that

$$|p| \le \bar{c}n^{1/\beta(1-\varepsilon)},$$

hence,

$$\frac{1}{n}p^{\beta(1-\varepsilon)} \le \bar{\bar{c}}.$$

Using strong aperiodicity, and possibly readjusting the constant c, we find that (5.13) also holds for $\delta \le |p|/b(n) \le \pi$. Summarizing,

$$\left|\phi\left(\frac{p}{b(n)}\right)\right| \le \exp\left[-\frac{c}{n}p^{\beta(1-\varepsilon)}\right], \qquad 1 \le |p| \le \pi b(n). \tag{5.14}$$

Therefore,

$$\begin{aligned} G(n,u) &\doteq \frac{1}{n}\sum_{k=0}^{n}\left|\phi^k\left(\frac{u}{b(n)}\right)\right| \\ &\le c\frac{1}{n} + \int_0^1 \exp\left[-ct((u))^{\beta(1-\varepsilon)}\right]dt \\ &\le \frac{c}{1+((u))^{\beta(1-\varepsilon)}}, \end{aligned} \tag{5.15}$$

first for $1 \le |((u))| \le \pi b(n)$ [we have used (5.2) once more], but of course (5.15) will also hold for $|((u))| \le 1$.

Proof of Theorem 3. This is quite similar to the proof of Theorem 1. Skorohod [(1957), Section 3] provides an analogue of Donsker's theorem which is sufficient for our purposes. As before, we change variables to find

$$\begin{aligned} L(n) &= \frac{b^{2(k-1)}(n)}{n^{k-1}}I(n) \\ &= \sum_{i_1,\dots,i_k=1}^{n}\frac{1}{n^k}\prod_{j=2}^{k}\frac{1}{(2\pi)^2} \\ &\qquad \times \int_{|p_j|_0 \le \pi b(n)} \exp\left[ip_j\frac{(S(j,i_j)-S(j-1,i_{j-1}))}{b(n)}\right]dp_j \end{aligned}$$

and we define our link,

$$L(\varepsilon,n) = \sum_{i_l,\dots,i_k=1}^{n} \frac{1}{n^k} \prod_{j=2}^{k} \frac{1}{(2\pi)^2}$$

$$\times \int_{|p_j|_0 \le \pi b(n)} \exp\left[ip_j \frac{\left(S(j,i_j) - S(j-1), i_{j-1}\right)}{b(n)}\right] \exp\left[-\varepsilon p_j^{\beta}\right] dp_j$$

and it suffices to prove the analogue of Lemma 1.

By (5.15) we are reduced to bounding

$$\int (1 + u^{\beta\delta}) \frac{1}{1 + u^{\beta(1-\bar{\varepsilon})(k/k-1)}} \, du,$$

which is finite for $\delta, \bar{\varepsilon}$ small if

$$\beta\left(\frac{k}{k-1}\right) > 2,$$

which is our hypothesis: $\beta > 2 - 2/k$. □

PROOF OF THEOREM 4. We proceed as in the proof of Theorem 2, arriving at the analogue of (3.6), where G is now given by (5.15), and

$$H(n,v,w) = \frac{1}{n} \sum_{k=1}^{n} \left| \phi^k\left(\frac{v+w}{b(n)}\right) - \phi^k\left(\frac{v}{b(n)}\right)\phi^k\left(\frac{w}{b(n)}\right)\right|.$$

We will soon establish that, for any $\delta > 0$ small, we can find $\bar{\delta} > 0$ such that

$$\int_{|q|_0 \le \pi b(n)} (1 + q^{\beta\bar{\delta}}) H(n,q,p) \, d^2q \le c(1 + p^{2-\beta+\delta}). \tag{5.16}$$

This will complete the proof of Theorem 4, as before, once we know the finiteness of

$$\int \prod_{m \in \mathscr{I}^c} \frac{1}{1 + u_m^{\beta-\varepsilon}} \prod_{m \in \mathscr{I}} w_m^{2-\beta+\delta} \widehat{dp\, dq},$$

where we integrate over nonisolated variables. This is essentially proven in Rosen (1986b), under our hypothesis $\beta > 2(2k-2)/(2k-1)$.

It remains to prove (5.16)—the analogue of Lemma 3. As in the proof of that lemma, we bound (5.16) by the sum of two terms:

$$\sum_{k=1}^{n} \frac{1}{n} \int_{|q| \le \pi b(n)} (1 + q^{\beta\bar{\delta}}) \left|\phi^k\left(\frac{q}{b(n)}\right)\right| \left|1 - \phi^k\left(\frac{p}{b(n)}\right)\right| dq, \tag{5.17}$$

$$\sum_{k=1}^{n} \frac{1}{n} \int_{|q|_0 \le \pi b(n)} (1 + q^{\beta\bar{\delta}}) \left|\phi^k\left(\frac{q+p}{b(n)}\right) - \phi^k\left(\frac{q}{b(n)}\right)\right| dq. \tag{5.18}$$

By (5.10) and (5.12) we have that, for any $0 < \gamma < 1$.

$$
\begin{aligned}
\left|1 - \phi^k\left(\frac{p}{b(n)}\right)\right| &= \left|1 - e^{-k\psi(p/b(n))}\right| \\
&\le ck^\gamma\left|\psi\left(\frac{p}{b(n)}\right)\right|^\gamma \\
&\le ck^\gamma\frac{p^{\beta\gamma}}{b^{\beta\gamma}(n)}h^\gamma\left(\frac{b(n)}{|p|}\right) \qquad (5.19)\\
&= c\left(\frac{k}{n}\right)^\gamma p^{\beta\gamma}\left(\frac{h(b(n)/|p|)}{h(b(n))}\right)^\gamma \\
&\le c\left(\frac{k}{n}\right)^\gamma\left(p^{\gamma\beta(1+\varepsilon)} + 1\right), \quad \text{for } \frac{|p|}{b(n)} \le \delta,
\end{aligned}
$$

while if $|p| \ge \delta b(n) \ge cn^{1/\beta(1+\varepsilon)}$ [see (5.2)], we have $p^{\beta(1+\varepsilon)}/n \ge \bar{c} > 0$, so the above bound holds for all p. Thus, (5.17) is bounded by

$$
\begin{aligned}
&c\left[1+p^{(2-\beta+2\beta\bar{\delta})(1+\varepsilon)}\right]\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{2/\beta-1+2\bar{\delta}}\left[1+\int_{|q|0\le\pi b(n)}\left(1+q^{\beta\bar{\delta}}\right)e^{-(ck/n)q^{\beta(1-\bar{\varepsilon})}}d_q\right] \\
&\le c\left(1 + p^{2-\beta+\delta}\right)\frac{1}{n}\sum_{k=1}^{n}\frac{1}{(k/n)^{1-\bar{\delta}}} \qquad (\text{for } \bar{\delta}, \bar{\varepsilon} \text{ small}) \\
&\le c\left(1 + p^{2-\beta+\delta}\right) \quad \text{as required.}
\end{aligned}
$$

Let $M(k)$ denote the integral in (5.18). Our first bound is quite similar to (4.5):

$$
M(k) \le c\left(1 + p^{\beta\bar{\bar{\delta}}}\right)\frac{1}{(k/n)^{2/\beta+\bar{\bar{\delta}}}}. \qquad (5.20)
$$

Now let $M_1(k)$ and $M_2(k)$ denote, respectively, the integral in (5.18) over $|q| \ge 4|p|$ and $|q| \le 4|p|$. The following bound for $M_2(k)$ is elementary:

$$
\begin{aligned}
M_2(k) &\le c\left(1 + p^{2+2\beta\bar{\delta}}\right)\int\frac{1 + q^{\beta\bar{\delta}}}{1 + q^{2+2\beta\bar{\delta}}}\,dq \qquad (5.21)\\
&\le c\left(1 + p^{2+2\beta\bar{\delta}}\right).
\end{aligned}
$$

In $M_1(k)$ we can assume $|q| \ge 4$. We use the mean value theorem,

$$
M_1(k) \le \frac{c|p|}{b(n)}\int_{\substack{2\le|q|_0\le\pi b(n)\\ |p|\le|q|/4}} k\left|\nabla\phi\left(\frac{q^*}{b(n)}\right)\right|\left|\phi^{k-1}\left(\frac{q^*}{b(n)}\right)\right|\left(1+q^{\beta\bar{\delta}}\right)dq. \qquad (5.22)
$$

In the region $|q| \leq \delta b(n)$, and with $k \neq 1$, we make use of the bounds (5.9), (5.13) and

$$|\nabla\psi(p)| \leq \frac{c}{|p|} \operatorname{Re}\psi(p), \qquad |p| \leq \delta$$

[Proposition 5.4 of Le Gall and Rosen (1988)] to obtain the bound

$$\begin{aligned} &c|p|\int \frac{k}{|q^*|} \operatorname{Re}\psi(q^*/b(n))\exp\left[-ck \operatorname{Re}\psi\left(\frac{q^*}{b(n)}\right)\right](1+q^{\beta\bar{\delta}})\,dq \\ (5.23)\quad &\leq c|p|\int \frac{1}{q} \exp\left[-c\frac{k}{n}q^{\beta(1-\varepsilon)}\right](1+q^{\beta\bar{\delta}})\,dq \leq \frac{c|p|}{(k/n)^{1/\beta(1-\varepsilon)+\bar{\delta}/(1-\varepsilon)}}. \end{aligned}$$

On the other hand, we always have

$$|\nabla\phi(q^*/b(n))| \leq \mathbb{E}(|S(1)|) \leq c$$

and if $|q| \geq \delta b(n)$ and $|p| \leq |q|/4$,

$$|\phi(q^*/b(n))| \leq z < 1,$$

so that the expression in (5.22) will be bounded by

$$c|p|b^{1+\beta\bar{\delta}}(n)kz^k.$$

Since $kz^k \leq d/k^{1/\beta(1-\varepsilon)+\bar{\delta}/(1-\varepsilon)}$ for d large (think first of k large), we are led to the bound appearing in (5.23). Note that this approach also works for the case $k = 1$, for all $|q|_0 \leq \pi b(n)$. Thus,

$$M_1(k) \leq c\frac{(1+|p|)}{(k/n)^{1/\beta+\bar{\bar{\delta}}}}. \tag{5.24}$$

We first interpolate between (5.21) and (5.20),

$$M_2(k) \leq c\left(1 + \frac{p^{(2+2\beta\bar{\delta})s+(1-s)\beta\bar{\delta})}}{(k/n)^{(2/\beta+\bar{\delta})(1-s)}}\right), \qquad 0 \leq s \leq 1. \tag{5.25}$$

Taking $s = 1 - \beta/2 + \varepsilon$, we find that $(2/\beta + \bar{\delta})(1-s) < 1$ for $\bar{\delta}$ small, and this leads to the bound required in (5.16).

We then interpolate between (5.20) and (5.24),

$$M_1(k) \leq c\frac{1+p^{s+(1-s)\beta\bar{\delta}}}{(k/n)^{(1/\beta+\bar{\delta})s+(2/\beta+\bar{\delta})(1-s)}}, \qquad 0 \leq s \leq 1. \tag{5.26}$$

We now take $s = 2 - \beta + \varepsilon$, and once more obtain the bound required in (5.16). □

Acknowledgment. I wish to thank E. B. Dynkin for several stimulating conversations.

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