The Annals of Probability
1990, Vol. 18, No. 2, 754–789

SOME APPLICATIONS OF ISOPERIMETRIC METHODS TO STRONG LIMIT THEOREMS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

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We develop several applications to almost sure limit theorems for sums of independent vector valued random variables of an isoperimetric inequality due to Talagrand. A general treatment of the classical laws of large numbers of Kolmogorov and Prokorov and laws of the iterated logarithm of Kolmogorov and Hartman and Wintner is described. New results as well as simpler new proofs of known ones illustrate the usefulness of isoperimetric methods in this context. We show further how this approach can be used in the study of limit theorems for trimmed sums of independent and identically distributed random variables.

1. Introduction and theoretical statement. Isoperimetric methods already appeared in the study of strong limit theorems for sums of independent vector valued random variables through concentration inequalities, mainly of Gaussian type. These inequalities are commonly used with the tool of randomization and applied conditionally. In particular, this was the basic idea of the main result of [24], which characterizes Banach space valued random variables satisfying the classical Hartman–Wintner–Strassen law of the iterated logarithm.

In a recent work [32], a new isoperimetric inequality for product measures was obtained and applied to the integrability of sums of independent vector valued random variables, providing at the same time new perspectives in the study of strong limit theorems. This inequality was indeed further applied in [25] to get extensions of Kolmogorov's law of the iterated logarithm and Prokorov's law of large numbers in Banach space. The purpose of this article is to demonstrate further the usefulness of this approach. We study a general statement (Section 2) on sums of independent Banach space valued random variables, actually already of interest in the real case, to be applied to almost sure limit theorems like the strong law of large numbers (Section 3) and the law of the iterated logarithm (Section 4). In this way we obtain several extensions of classical limit theorems as well as new and simpler proofs of known results like the recent law of the iterated logarithm in Banach spaces of [24]. This contribution, in particular, unifies the treatment of the classical laws of large numbers of Kolmogorov and Prokorov and laws of the iterated logarithm of Kolmogorov and Hartman, Wintner and Strassen. Moreover, the

Received March 1988; revised March 1989.
[1]Work partially supported by an NSF grant.
AMS 1980 *subject classifications*. Primary 60B12, 60F15.
Key words and phrases. Isoperimetric inequality, law of large numbers, law of the iterated logarithm, trimming.

isoperimetric inequality seems to be also of some interest in the context of trimming, when extreme values are excluded from sums, and we establish by this method some new results on the law of the iterated logarithm in Banach space in this case (Section 4). Similarly, extensions and new proofs of Mori's strong laws of large numbers for trimmed sums can be obtained (Section 4). In Section 5 we discuss the identification of the limits in the vector valued Hartman–Wintner law of the iterated logarithm.

The isoperimetric inequality of [32] is proved in the abstract setting of product measures. One natural framework for applications to limit theorems for sums of independent random variables concerns random variables taking their values in a Banach space. B will therefore always denote below a real separable Banach space with norm $\|\cdot\|$ and topological dual space B^*; duality is denoted $f(x)$ for f in B^* and x in B. Let us then describe the isoperimetric inequality of [32] in this setting. Let N be a fixed, but arbitrary, integer and let $\mathbb{X} = (X_i)_{i \le N}$ be a sample of independent random variables with values in B. Let $A \subset B^N = \text{s}\{x = (x_i)_{i \le N},\ x_i \in B\}$. For q, k integers, set

$$H(A,q,k) = \{x \in B^N \colon \exists\, x^1,\dots,x^q \in A,$$
$$\operatorname{card}\{i \le N \colon x_i \notin \{x_i^1,\dots,x_i^q\}\} \le k\}.$$

Then, if $\mathbb{P}\{\mathbb{X} \in A\} \ge \frac{1}{2}$ and $k \ge q$, for some *universal* constant K_0,

$$(*) \qquad \mathbb{P}_*\{\mathbb{X} \in H(A,q,k)\} \ge 1 - \left(\frac{K_0}{q}\right)^k,$$

where $\mathbb{P}_*$ denotes inner probability. For simplicity, we will assume in the applications K_0 to be an *integer*. The isoperimetric character of this inequality proceeds from both its proof which uses delicate symmetrization techniques and the conclusion itself which estimates some neighborhood of A (actually for any value of $\mathbb{P}\{\mathbb{X} \in A\}$ in [32]). On the set $H(A,q,k)$, the sample $\mathbb{X} = (X_i)_{i \le N}$ is controlled by a finite number q of points in the set A provided k elements of the sample are neglected. The isoperimetric inequality $(*)$ precisely estimates the size of this set which increases when more terms are neglected (and/or q is bigger). In the applications, in particular to the study of sums of independent random variables, these discarded terms are intuitively the largest elements of the sample and these large values have therefore to be estimated. Moreover, this observation strongly suggests possible consequences for the study of sums of independent random variables when extreme values are excluded. Indeed, if the largest elements are deleted, it is likely that, in some sense, the integer k can be diminished and various conditions can be weakened.

This whole approach, which we call isoperimetric, is already of great interest on the line as will be clear from the results we will describe. It will be one of the purposes of the next sections to find appropriate bounds on large values in various situations which are identical in the scalar and vector valued cases. What we would like to do first, in the remainder of this section, is to show how the sums of independent random variables we study are controlled when the sample, on $H(A, q, k)$, falls in the finite set of points in A. The basic idea is quite simple and makes essential use of the tool of randomization; its application, however, rests on some powerful observations on Rademacher averages, like concentration and comparison properties. But these are much simpler on the line, reducing essentially to the classical subgaussian estimate, and the interested reader is perhaps invited to study first this case. It will lead to some quite optimal estimates, as will be clear in the next sections. We describe the theoretical result in the next proposition which we introduce with some notation.

Let $\mathbb{X} = (X_i)_{i \le N}$ be a family of independent and symmetric random variables with values in B. By the symmetry assumption, (X_i) has the same distribution as $(\varepsilon_i X_i)$, where (ε_i) denotes a (Bernoulli or Rademacher) sequence of independent random variables taking the values ± 1 with probability $\frac{1}{2}$ and independent from the X_i's. Accordingly we will sometimes denote, by Fubini's theorem, by $\mathbb{P}_\varepsilon, \mathbb{E}_\varepsilon$ (resp., $\mathbb{P}_X, \mathbb{E}_X$) conditional probability and expectation with respect to (X_i) [resp., (ε_i)]. We denote by $(\|X_i\|^*)_{i \le N}$ the nonincreasing rearrangement of the sequence $(\|X_i\|)_{i \le N}$.

Proposition 1.1. *Let $(X_i)_{i \le N}$ be independent and symmetric random variables with values in B. For integers $k \ge q$ and positive numbers s, t, we have*

$$\begin{aligned} &\mathbb{P}\left\{\left\|\sum_{i=1}^{N} X_i\right\| > t + 2s + 8qM\right\} \\ (1.1)\qquad &\le \left(\frac{K_0}{q}\right)^k + \mathbb{P}\left\{\sum_{i=1}^{k} \|X_i\|^* > s\right\} + 4\exp\left(-\frac{t^2}{64q\sigma^2}\right) + 4\exp\left(-\frac{kt^2}{768qMs}\right) \end{aligned}$$

where

$$M = \mathbb{E}\left\|\sum_{i=1}^{N} u_i\right\|, \qquad \sigma = \sup_{\|f\| \le 1}\left(\sum_{i=1}^{N} \mathbb{E} f^2(u_i)\right)^{1/2}$$

and $u_i = X_i I_{\{\|X_i\| \le s/k\}}$, $i \le N$.

We decided in this proposition to keep track, for clarity, of the numerical constants (not best possible!); K_0 is the constant of the isoperimetric inequality $(*)$ and the first term in the right-hand side of (1.1) comes from $(*)$. The second corresponds to the largest values and the others to the estimates we have now to prove. Let us mention that M and σ defined from truncated random variables are of course majorized (using the contraction principle for M) as

$$M \le \mathbb{E}\left\|\sum_{i=1}^{N} X_i\right\|, \qquad \sigma \le \sup_{\|f\| \le 1}\left(\sum_{i=1}^{N} \mathbb{E} f^2(X_i)\right)^{1/2}$$

and can often be used in these forms in applications.

PROOF. We prepare ourselves to apply the isoperimetric inequality. Recall the symmetry assumption on the X_i's. Let $A \subset B^N$ be defined as

$$A = \left\{ x \in B^N \colon \mathbb{E}_\varepsilon \left\| \sum_{i=1}^N \varepsilon_i x_i I_{\{\|x_i\| \le s/k\}} \right\| \le 4M, \right.$$

$$\left. \sup_{\|f\| \le 1} \sum_{i=1}^N f^2(x_i) I_{\{\|x_i\| \le s/k\}} \le 4\sigma^2 + \frac{48Ms}{k} \right\}.$$

Let us first show that $\mathbb{P}\{\mathbb{X} \in A\} \ge \frac{1}{2}$. Clearly,

$$\mathbb{P}_X \left\{ \mathbb{E}_\varepsilon \left\| \sum_{i=1}^N \varepsilon_i u_i \right\| > 4M \right\} \le \tfrac{1}{4}.$$

On the other hand, by centering,

$$\mathbb{E}\left(\sup_{\|f\| \le 1} \sum_{i=1}^N f^2(u_i) \right) \le \sigma^2 + \mathbb{E}\left(\sup_{\|f\| \le 1} \left| \sum_{i=1}^N f^2(u_i) - \mathbb{E} f^2(u_i) \right| \right)$$

$$\le \sigma^2 + 2\mathbb{E}\left(\sup_{\|f\| \le 1} \left| \sum_{i=1}^N \varepsilon_i f^2(u_i) \right| \right).$$

At this stage, we need to recall the comparison theorem for Rademacher averages of [25], Theorem 5.

LEMMA 1.2. *Let $\varphi_i \colon \mathbb{R} \to \mathbb{R}$ be contractions such that $\varphi_i(0) = 0$, $i = 1, \dots, N$, and let $\mathscr{F}$ be a class of functions on $\{1, \dots, N\}$. Then, for $p \ge 1$,*

$$\mathbb{E}\left(\sup_{h \in \mathscr{F}} \left| \sum_{i=1}^N \varepsilon_i \varphi_i(h(i)) \right|^p \right) \le 3^p \mathbb{E}\left(\sup_{h \in \mathscr{F}} \left| \sum_{i=1}^N \varepsilon_i h(i) \right|^p \right).$$

We apply this lemma to $\mathbb{E}_\varepsilon(\sup_{\|f\| \le 1} |\sum_{i=1}^N \varepsilon_i f^2(u_i)|)$, conditionally therefore on the sequence (X_i); we can choose $\mathscr{F} = \{h_f;\ f \in U\}$, where $h_f(i) = \|u_i\| f(u_i)$ and U is the unit ball of B^* and $\varphi_i(t) = \min(t^2/2\|u_i\|^2, \|u_i\|^2/2)$. We then get

$$\mathbb{E}_\varepsilon\left(\sup_{\|f\| \le 1} \left| \sum_{i=1}^N \varepsilon_i f^2(u_i) \right| \right) \le 6\mathbb{E}_\varepsilon \left\| \sum_{i=1}^N \varepsilon_i u_i \|u_i\| \right\|.$$

If we recall that $\|u_i\| \le s/k$, we finally obtain from the contraction principle ([18], page 18, or [16]) that

$$\mathbb{E}\left(\sup_{\|f\| \le 1} \sum_{i=1}^N f^2(u_i) \right) \le \sigma^2 + \frac{12Ms}{k}.$$

From the very definition of A, together with our first observation, it follows indeed that $\mathbb{P}\{\mathbb{X} \in A\} \ge \frac{1}{2}$.

We are now in a position to apply the isoperimetric inequality $(*)$. By definition of $H = H(A, q, k)$, if $\mathbb{X} \in H$, there exist points $x^1, \dots, x^q$ in A and integers $\{i_1, \dots, i_j\}$ with $j \le k$ such that

$$\{1, \dots, N\} = \{i_1, \dots, i_j\} \cup I,$$

where $I = \bigcup_{l=1}^{q} \{i \le N: X_i = x_i^l\}$. Thus, in this case,

$$\left\| \sum_{i=1}^{N} \varepsilon_i X_i \right\| \le \sum_{i=1}^{k} \|X_i\|^* + \left\| \sum_{i \in I} \varepsilon_i X_i \right\|.$$

Now assume, moreover, that $\sum_{i=1}^{k} \|X_i\|^* \le s$; then, certainly, at most $k - 1$ X_i's satisfy $\|X_i\| > s/k$. Hence

$$\left\| \sum_{i=1}^{N} \varepsilon_i X_i \right\| \le s + \left\| \sum_{i \in I} \varepsilon_i X_i \right\| \le 2s + \left\| \sum_{i \in I} \varepsilon_i u_i \right\|.$$

Summarizing,

$$\begin{aligned} &\mathbb{P}\left\{ \left\| \sum_{i=1}^{N} \varepsilon_i X_i \right\| > t + 2s + 8qM \right\} \\ &\quad \le \mathbb{P}^*\{\mathbb{X} \notin H\} + \mathbb{P}\left\{ \sum_{i=1}^{k} \|X_i\|^* > s \right\} \\ &\qquad + \sup \int_F \mathbb{P}_\varepsilon \left\{ \left\| \sum_{i \in I} \varepsilon_i u_i \right\| > t + 8qM \right\} d\mathbb{P}_X, \end{aligned} \tag{1.2}$$

where the sup runs over all measurable sets F such that $F \subset \{\mathbb{X} \in H\}$. The last step in this proof is therefore the (conditional) estimate of

$$\mathbb{P}_\varepsilon \left\{ \left\| \sum_{i \in I} \varepsilon_i u_i \right\| > t + 8qM \right\}.$$

To this aim, we will make use of a concentration inequality for Bernoulli averages, also of isoperimetric type, recently obtained in [31]. Inequalities of the same type were used for Gaussian averages in [24] and for averages by uniform random variables in [25], and the latter ones can also be used here (with some modifications). The more natural Bernoulli symmetrization, however, distinguishes the following result.

LEMMA 1.3. *Let $x_1, \dots, x_N$ be points in B. Let μ be a median of $\|\sum_{i=1}^{N} \varepsilon_i x_i\|$ and set $\sigma = \sup_{\|f\| \le 1} (\sum_{i=1}^{N} f^2(x_i))^{1/2}$. Then, for every $t > 0$,*

$$\mathbb{P}\left\{ \left| \left\| \sum_{i=1}^{N} \varepsilon_i x_i \right\| - \mu \right| > t \right\} \le 4 \exp\left(-\frac{t^2}{8\sigma^2} \right).$$

In particular (but this is weaker than the previous inequality)

$$\mathbb{P}\left\{ \left\| \sum_{i=1}^{N} \varepsilon_i x_i \right\| > 2\mathbb{E}\left\| \sum_{i=1}^{N} \varepsilon_i x_i \right\| + t \right\} \le 4 \exp\left(-\frac{t^2}{8\sigma^2} \right). \tag{1.3}$$

In order to apply this lemma, we make the following observation. For each i in I, fix $1 \le l(i) \le q$ with $X_i = x_i^{l(i)}$. Let $I_l = \{i: l(i) = l\}$, $1 \le l \le q$. Then

$$\mathbb{E}_\varepsilon \Big\| \sum_{i \in I} \varepsilon_i u_i \Big\| = \mathbb{E}_\varepsilon \Big\| \sum_{l=1}^{q} \sum_{i \in I_l} \varepsilon_i u_i \Big\| \le \sum_{l=1}^{q} \mathbb{E}_\varepsilon \Big\| \sum_{i \in I_l} \varepsilon_i x_i^l I_{\{\|x_i^l\| \le s/k\}} \Big\|.$$

Now, by the *monotonicity of Rademacher averages* and the definition of A, it follows that

$$\mathbb{E}_\varepsilon \Big\| \sum_{i \in I} \varepsilon_i u_i \Big\| \le \sum_{l=1}^{q} \mathbb{E}_\varepsilon \Big\| \sum_{i=1}^{N} \varepsilon_i x_i^l I_{\{\|x_i^l\| \le s/k\}} \Big\| \le 4qM.$$

In the *same way*,

$$\sup_{\|f\| \le 1} \sum_{i \in I} f^2(u_i) \le 4q\sigma^2 + \frac{48qMs}{k}.$$

Hence, by (1.3),

$$\mathbb{P}_\varepsilon \Big\{ \Big\| \sum_{i \in I} \varepsilon_i u_i \Big\| > t + 8qM \Big\} \le 4 \exp\left(- \frac{t^2}{8(4q\sigma^2 + 48qMs/k)} \right),$$

from which, together with (1.2), (1.1) readily follows. The proof of Proposition 1.1 is complete. □

Since we are also interested in some applications of the isoperimetric inequality to extreme values, we mention, to conclude this introduction, the analog of Proposition 1.1 in the context of sums of independent random variables when extremes are excluded. However, we need to introduce first some more convenient notation concerning large values which will also be useful in the sequel. If (X_i) is a sequence of random variables, we set, for $1 \le r \le N$, $X_N^{(r)} = X_i$ whenever $\|X_i\|$ is the rth maximum of the sample $(\|X_1\|, \dots, \|X_N\|)$ (breaking ties by priority of index). We agree, moreover, that $X_N^{(r)} = 0$ when $r > N$.

PROPOSITION 1.4. *Let $(X_i)_{i \le N}$ be independent and symmetric random variables with values in B. Let also $(r_j)_{j_0 \le j \le N}$ be integers and set $r = \min_{j_0 \le j \le N} r_j$. Then, for $k \ge q$ and $k > r$, and $s, t > 0$,*

$$\mathbb{P}\Big\{ \max_{j_0 \le j \le N} \Big\| \sum_{i=1}^{j} X_i - \sum_{i=1}^{r_j} X_j^{(i)} \Big\| > t + 2s + 8qM \Big\}$$

$$\le \left(\frac{K_0}{q} \right)^k + \mathbb{P}\Big\{ \sum_{i=r+1}^{k} \|X_N^{(i)}\| > s \Big\} + 8 \exp\left(- \frac{t^2}{64q\sigma^2} \right) + 8 \exp\left(- \frac{(k-r)t^2}{768qMs} \right),$$

where M and σ are as in Proposition 1.1 *but with $u_i = X_i I_{\{\|X_i\| \le s/(k-r)\}}$, $i \le N$.*

PROOF. It is completely similar to the proof of Proposition 1.1 which has simply to be followed with the same notation. Take A as before with $s/(k-r)$ instead of s/k in the truncations; we have $\mathbb{P}\{\mathbb{X} \in A\} \geq \frac{1}{2}$. Suppose now that $\mathbb{X} \in H$ and $\sum_{i=r+1}^{k} \|X_N^{(i)}\| \leq s$; then, if J denotes, for each $j \geq j_0$, the set of indices corresponding to the r_j maxima of the sample $(X_1, \dots, X_j)$,

$$\left\| \sum_{i=1}^{j} \varepsilon_i X_i - \sum_{i \in J} \varepsilon_i X_i \right\| \leq \sum_{i=r+1}^{k} \|X_N^{(i)}\| + \left\| \sum_{\substack{i \in I \setminus J \\ i \leq j}} \varepsilon_i X_i \right\|$$

$$\leq s + \left\| \sum_{\substack{i \in I \setminus J \\ i \leq j}} \varepsilon_i X_i \right\|.$$

Now, since $\sum_{i=r+1}^{k} \|X_N^{(i)}\| \leq s$, there are at most $k-r-1$ indices $i \leq j$ in $I \setminus J$ such that $\|X_i\| > s/(k-r)$. Hence

$$\left\| \sum_{\substack{i \in I \setminus J \\ i \leq j}} \varepsilon_i X_i \right\| \leq s + \left\| \sum_{\substack{i \in I \setminus J \\ i \leq j}} \varepsilon_i u_i \right\|.$$

Therefore

$$\max_{j_0 \leq j \leq N} \left\| \sum_{i=1}^{j} \varepsilon_i X_i - \sum_{i \in J} \varepsilon_i X_i \right\| \leq 2s + \max_{j_0 \leq j \leq N} \left\| \sum_{\substack{i \in I \setminus J \\ i \leq j}} \varepsilon_i u_i \right\|.$$

If we now recall Lévy's maximal inequality (see, e.g., [18]),

$$\mathbb{P}_\varepsilon\left\{ \max_{j_0 \leq j \leq N} \left\| \sum_{\substack{i \in I \setminus J \\ i \leq j}} \varepsilon_i u_i \right\| > a \right\} \leq 2\mathbb{P}_\varepsilon\left\{ \left\| \sum_{i \in I} \varepsilon_i u_i \right\| > a \right\}$$

(where $a > 0$) we see that the conclusion to Proposition 1.4 is then obtained exactly as in Proposition 1.1. □

2. A general statement for strong limit theorems. In this section we draw from the preceding formulation of the isoperimetric inequality a general statement for almost sure limit theorems for sums of independent random variables. This general form will then be used in applications to the law of large numbers and the law of the iterated logarithm in the next sections. It follows rather easily from what we have already obtained. For the sake of completeness, we start with some classical facts on symmetrization and blocking which will be useful later on.

We deal with a sequence $(X_i)_{i \in \mathbb{N}}$ of independent random variables with values in a Banach space B. As usual, we set $S_n = X_1 + \cdots + X_n$ for each n. Let also $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers increasing to ∞. We study the *almost sure* behavior of the sequence (S_n/a_n). As is classical in probability in Banach spaces, such a study can be developed reasonably only if one assumes some (*necessary*) boundedness or convergence in *probability*. This corresponds to the control of M in Proposition 1.1 (see also Lemma 1.3) and the difficulty in the vector valued setting is to find "good" conditions, in terms of the individual summands only if possible, for such a property to hold; typical in this regard are the law of large numbers (e.g., [23] and [6]) and the law of the iterated logarithm (e.g., [24]). This basic hypothesis we shall keep along all limit theorems in the sequel allows in particular a simple symmetrization procedure summarized in the next trivial lemma.

LEMMA 2.1. *Let $(Z_n), (Z'_n)$ be independent sequences of random variables such that the sequence $(Z_n - Z'_n)$ is almost surely bounded (resp., convergent to 0) and (Z_n) is bounded (resp., convergent to 0) in probability. Then (Z_n) is almost surely bounded (resp., convergent to 0). More quantitatively (if necessary), if, for some numbers M and A,*

$$\limsup_{n\to\infty} \|Z_n - Z'_n\| \le M \quad \textit{almost surely}$$

and

$$\limsup_{n\to\infty} \mathbb{P}\{\|Z_n\| > A\} < 1,$$

then

$$\limsup_{n\to\infty} \|Z_n\| \le 2M + A \quad \textit{almost surely}.$$

In the context of sums of independent random variables, let (X'_i) be an independent copy of the sequence (X_i) and set, for each i, $\tilde{X}_i = X_i - X'_i$ defining thus independent and symmetric random variables. Lemma 2.1 thus tells us that under appropriate assumptions in probability on (S_n/a_n), it is enough to study $(\sum_{i=1}^n \tilde{X}_i/a_n)$, reducing ourselves to symmetric random variables. Therefore we only detail below our general statement in the symmetrical case. Let us mention that if this symmetry assumption is of basic use at various places, essentially through Proposition 1.1, it is, however, completely superfluous at some others, like for example the control of the largest values.

As is well known also, we can study, in quite general situations, the sums S_n through blocks. Assume there exists a subsequence (a_{m_n}) of (a_n) such that for each n,

$$1 < c \le \frac{a_{m_{n+1}}}{a_{m_n}} \le C < \infty$$

and let $I(n)$ be the set of integers $\{m_{n-1} + 1, \dots, m_n\}$. The next lemma may be found in the book of Stout [29], page 158, to which we actually refer for a general introduction on almost sure stability (for real random variables).

LEMMA 2.2. *Let (X_i) be independent and symmetric random variables. Then (S_n/a_n) is almost surely bounded (resp., convergent to 0) if and only if the same holds for $(\sum_{i\in I(n)} X_i/a_{m_n})$.*

Since the blocks $I(n)$ are disjoint, by independence and the Borel–Cantelli lemma, what we have thus to study is the convergence of the series

$$\sum_n \mathbb{P}\left\{\left\|\sum_{i \in I(n)} X_i\right\| > \varepsilon a_{m_n}\right\},$$

for some, or all, $\varepsilon > 0$. Our general result that we describe now provides sufficient conditions for this to hold. The somewhat technical formulation of the theorem is justified by the sharpness of the various conditions.

THEOREM 2.3. *Let (X_i) be a sequence of independent and symmetric random variables with values in B. Assume there exist an integer $q \geq 2K_0$ and a sequence of integers (k_n) such that*

$$\sum_n \left(\frac{K_0}{q}\right)^{k_n} < \infty, \tag{2.1}$$

$$\sum_n \mathbb{P}\left\{\sum_{i=1}^{k_n} \|X_{I(n)}^{(i)}\| > \varepsilon a_{m_n}\right\} < \infty, \tag{2.2}$$

for some $\varepsilon > 0$, where $X_{I(n)}^{(r)}$ denotes the rth maximum of $(X_i)_{i \in I(n)}$. Set then, for each n,

$$M_n = \mathbb{E}\left\|\sum_{i \in I(n)} X_i I_{\{\|X_i\| \leq \varepsilon a_{m_n}/k_n\}}\right\|,$$

$$\sigma_n = \sup_{\|f\| \leq 1}\left(\sum_{i \in I(n)} \mathbb{E}\left(f^2(X_i) I_{\{\|X_i\| \leq \varepsilon a_{m_n}/k_n\}}\right)\right)^{1/2}.$$

Then, if $L = \limsup_{n \to \infty} M_n / a_{m_n} < \infty$ and, for some $\delta > 0$,

$$\sum_n \exp\left(-\frac{\delta^2 a_{m_n}^2}{\sigma_n^2}\right) < \infty, \tag{2.3}$$

we have

$$\sum_n \mathbb{P}\left\{\left\|\sum_{i \in I(n)} X_i\right\| > 10^2 \alpha(\varepsilon, \delta, q, L) a_{m_n}\right\} < \infty, \tag{2.4}$$

where

$$\alpha(\varepsilon, \delta, q, L) = \varepsilon + (\varepsilon L + \delta^2)^{1/2}\left(q \log \frac{q}{K_0}\right)^{1/2} \leq \varepsilon + qL + q(\varepsilon L + \delta^2)^{1/2}.$$

There is actually nothing to prove concerning this theorem that readily follows from the inequality of Proposition 1.1 applied to the sample $(X_i)_{i \in I(n)}$

with $k = k_n$ ($\geq q$ for n large enough), $s = \varepsilon a_{m_n}$ and

$$t = 10^2(\varepsilon L + \delta^2)^{1/2}\left(q \log \frac{q}{K_0}\right)^{1/2} a_{m_n}.$$

(Of course, the numerical constant 10^2 is not the best one, just a convenient number!)

In order to apply Theorem 2.3 to concrete situations, several comments are in order. Conditions (2.1) and (2.2) are rather technical, but we will see later that they follow from a handy (but weaker) condition (Lemma 2.5 below). Theorem 2.3 can often be applied by replacing the quantities M_n and σ_n by the corresponding quantities without truncations. As stated, however, the conditions on M_n and σ_n are kind of optimal. More precisely, if the sequence (S_n/a_n) is almost surely bounded, i.e., (2.4) holds for some $\alpha > 0$, and if there exist q and (k_n) such that (2.1) [and (2.2)] is satisfied, then L is finite and (2.3) holds for some $\delta > 0$. For L, since it is necessary that the sequence (S_n/a_n) is bounded in probability, we have simply to translate this property in terms of expectation. This is, however, classical and is expressed in the next lemma that is proved using Hoffmann-Jørgensen's inequality (cf. [23] and [6]); it applies directly to L using a contraction principle (cf. [18] and [16]) [or (2.2)].

LEMMA 2.4. *Let (X_i) be independent and symmetric random variables. Then, if the sequence (S_n/a_n) is bounded (resp., convergent to 0) in probability, for any $p > 0$ and any $\varepsilon > 0$, the sequence $(\mathbb{E}\|\sum_{i=1}^n X_i I_{\{\|X_i\| \leq \varepsilon a_n\}}/a_n\|^p)$ is bounded (resp., convergent to 0).*

The necessity of (2.3) is obtained from Kolmogorov's lower bound inequality ([29], page 262) and we sketch the proof following [3]. We assume thus that for some $\alpha > 0$,

$$\sum_n \mathbb{P}\left\{\left\| \sum_{i \in I(n)} X_i \right\| > \alpha a_{m_n}\right\} < \infty$$

and set, with $\varepsilon = 1$ for simplicity,

$$X_i^n = X_i I_{\{\|X_i\| \leq a_{m_n}/k_n\}}, \qquad i \in I(n).$$

For each n, choose f_n in B^*, $\|f_n\| \leq 1$, such that

$$\sigma_n^2 \leq 2 \sum_{i \in I(n)} \mathbb{E} f_n^2(X_i^n) \qquad (\leq 2\sigma_n^2).$$

We also have

$$\sum_n \mathbb{P}\left\{ \sum_{i \in I(n)} f_n(X_i^n) > \alpha a_{m_n}\right\} < \infty,$$

where we have used either a contraction principle or (2.2). By Lemma 2.4, $a_{m_n}/\sigma_n > c > 0$ for each n. In order to apply Kolmogorov's inequality, note that it is sufficient, by (2.1), to consider the integers n such that $a_{m_n}^2 \leq \eta\sigma_n^2 k_n$,

$\eta > 0$. But then, it is easily seen that for some $C(\geq \alpha)$, some small $\eta > 0$ and all n large enough, we are under the hypotheses described on page 262 of [29] and thus

$$\mathbb{P}\left\{ \sum_{i \in I(n)} f_n(X_i^n) > C a_{m_n} \right\} \geq \exp\left(-\frac{2C^2 a_{m_n}^2}{\sigma_n^2} \right),$$

which proves our claim. Let us note further that if $S_n/a_n \to 0$ almost surely, then, in the prescribed setting, it is possible to show similarly that $L = 0$ and that (2.3) holds for every $\delta > 0$.

Concerning conditions (2.1) and (2.2) in Theorem 2.3, it would be desirable to find, if possible, simple, or at least easy to be handled, hypotheses on (X_i) in order for these conditions to be fulfilled. There could be many ways to do this. We suggest a possible one in the next lemma in terms of the probabilities $\mathbb{P}\{\max_{i \in I(n)} \|X_i\| > t\}$ (or $\sum_{i \in I(n)} \mathbb{P}\{\|X_i\| > t\}$).

Lemma 2.5. *In the notation of Theorem* 2.3, *assume that, for some* $u > 0$,

$$\sum_n \mathbb{P}\left\{ \max_{i \in I(n)} \|X_i\| > u a_{m_n} \right\} < \infty, \tag{2.5}$$

and that, for some $v > 0$, *all* n *and* t, $0 < t \leq 1$,

$$\mathbb{P}\left\{ \max_{i \in I(n)} \|X_i\| > t v a_{m_n} \right\} \leq \delta_n \exp\left(\frac{1}{t} \right), \tag{2.6}$$

where $\sum_n \delta_n^s < \infty$ *for some integer* s. *Then, for each* $q > K_0$, *there exists a sequence of integers* (k_n) *satisfying* (2.1) *and such that*

$$\sum_n \mathbb{P}\left\{ \sum_{i=1}^{k_n} \|X_{I(n)}^{(i)}\| > 2s\left(u + v\left(\log \frac{q}{K_0} \right)^{-1} \right) a_{m_n} \right\} < \infty. \tag{2.7}$$

Proof. The idea is simply that if the largest element of $(X_i)_{i \in I(n)}$ is exactly estimated by (2.5), the $2s$th one is already small enough so that quite a large number of values after it are under control. Write indeed (for any k_n)

$$\sum_{i=1}^{k_n} \|X_{I(n)}^{(i)}\| \leq 2s \|X_{I(n)}^{(1)}\| + k_n \|X_{I(n)}^{(2s)}\|.$$

We have, $0 < t \leq 1$,

$$\mathbb{P}\left\{ \|X_{I(n)}^{(2s)}\| > t v a_{m_n} \right\} \leq \left(\mathbb{P}\left\{ \max_{i \in I(n)} \|X_i\| > t v a_{m_n} \right\} \right)^{2s}$$

(which is easily checked by induction from

$$\mathbb{P}\left\{ \|X_{I(n)}^{(r)}\| > a \right\} \leq \mathbb{P}\left\{ \max_{i \in I(n)} \|X_i\| > a \right\} \mathbb{P}\left\{ \|X_{I(n)}^{(r-1)}\| > a \right\}).$$

(We thank A. de Acosta for pointing out to us the preceding inequality.) Therefore, by (2.6),

$$\mathbb{P}\{\|X_{I(n)}^{(2s)}\| > tva_{m_n}\} \le \left(\delta_n \exp\left(\frac{1}{t}\right)\right)^{2s}.$$

We choose next, for each n large enough, t to be $(\log 1/\sqrt{\delta_n})^{-1}$ and set k_n to be the integer part of

$$2s\left(\log \frac{q}{K_0}\right)^{-1} \log \frac{1}{\sqrt{\delta_n}}.$$

It is plain that

$$\sum_n \left(\frac{K_0}{q}\right)^{k_n} < \infty.$$

On the other hand,

$$\mathbb{P}\left\{\sum_{i=1}^{k_n} \|X_{I(n)}^{(i)}\| > 2s\left(u + v\left(\log \frac{q}{K_0}\right)^{-1}\right) a_{m_n}\right\}$$

$$\le \mathbb{P}\{\|X_{I(n)}^{(1)}\| > ua_{m_n}\} + \mathbb{P}\left\{\|X_{I(n)}^{(2s)}\| > \left(\log \frac{1}{\sqrt{\delta_n}}\right)^{-1} va_{m_n}\right\},$$

so that (2.7) holds by our preceding observation. □

REMARK 2.6. Assume that (2.6) of Lemma 2.5 is strengthened into

$$\mathbb{P}\left\{\max_{i \in I(n)} \|X_i\| > tva_{m_n}\right\} \le \delta_n \frac{1}{t^p},$$

for some $p > 0$. It is easily seen then that the preceding proof can be improved to yield the existence of a sequence (k_n) of integers such that (2.7) is satisfied *and*

$$\sum_n \frac{1}{k_n^{2ps}} < \infty$$

(or even $\sum_n k_n^{-p's} < \infty$ for $p' > p$). This observation is sometimes useful as will be the case in the next sections.

REMARK 2.7. Theorem 2.3 and Lemma 2.5 have trivial analogs in case some extreme values are deleted from sums. More precisely, if, in Theorem 2.3, (2.2) is replaced by

$$\sum_n \mathbb{P}\left\{\sum_{i=r+1}^{k_n} \|X_{I(n)}^{(i)}\| > \frac{\varepsilon}{2} a_{m_n}\right\} < \infty,$$

for some fixed integer $r \ge 0$, and the other hypotheses are unchanged, then, using Proposition 1.4 instead of Proposition 1.1, we get in the same way that

$$\sum_n \mathbb{P}\left\{ \max_{j\in I(n)} \left\| \sum_{i=m_{n-1}+1}^{j} X_i - \sum_{i=1}^{r} X^{(i)}_{(m_{n-1},j]} \right\| > 10^2 \alpha(\varepsilon,\delta,q,L) a_{m_n} \right\} < \infty,$$

where $X^{(i)}_{(m_{n-1},j]}$ denotes the ith maximum of $(X_{m_{n-1}+1},\dots,X_j)$.

Similarly, in Lemma 2.5, if (2.5) is weakened into

$$\sum_n \left(\mathbb{P}\left\{ \max_{i\in I(n)} \|X_i\| > u a_{m_n} \right\} \right)^{r+1} < \infty,$$

the proof leads to the same conclusion for $\sum_{i=r+1}^{k_n} \|X^{(i)}_{I(n)}\|$, i.e., the existence, for each $q > K_0$, of a sequence (k_n) of integers satisfying $\sum_n (K_0/q)^{k_n} < \infty$ and

$$\sum_n \mathbb{P}\left\{ \sum_{i=r+1}^{k_n} \|X^{(i)}_{I(n)}\| > 2s\left(u + v\left(\log\frac{q}{K_0}\right)^{-1}\right) a_{m_n} \right\} < \infty.$$

Further, Remark 2.6 clearly also applies in this context. We refer below to this remark as the trimmed version of Theorem 2.3 and Lemma 2.5

3. The law of large numbers. This section is devoted to extensions of the classical strong laws of large numbers of Kolmogorov and Prokorov (cf. [29]) for Banach space valued random variables. What the isoperimetric approach provides is actually already of interest on the line, in the results themselves like Theorem 3.1 and, perhaps more importantly, in the conceptual idea of the proofs. We apply in a rather trivial manner the general statement of the preceding paragraph. For clarity, we do not seek the greatest generality in normalizing sequences and only deal with the natural and classical case $a_n = n$. The same approach leads, however, to similar statements in general (see, e.g., [3] in case of Prokorov's law of large numbers). We thus also simply take $m_n = 2^n$ as the blocking subsequence.

The classical law of large numbers of Kolmogorov states that if (X_i) are real independent random variables with mean 0, then $S_n/n \to 0$ almost surely as soon as

$$\sum_i \frac{\mathbb{E}X_i^2}{i^2} < \infty.$$

In Banach spaces ([17] and [23]), if

$$\sum_i \frac{\mathbb{E}\|X_i\|^2}{i^2} < \infty,$$

then $S_n/n \to 0$ almost surely if and only if $S_n/n \to 0$ in probability. Several recent works in smooth norm spaces (starting with the law of the iterated logarithm), however, showed that assumptions on the norms of the X_i's only are in general too heavy and should be weakened and complemented with hypotheses on weak moments ([13] and [15]). The isoperimetric inequality confirms this confirms this intuition in general Banach space and leads to some general extension of Kolmogorov's law of large numbers which appears to be sharp even on the line. The result is the following. Recall that, for each n, $I(n) = \{2^{n-1}+1,\dots,2^n\}$.

THEOREM 3.1. *Let (X_i) be a sequence of independent random variables with values in a Banach space B. Assume that*

$$\frac{X_i}{i} \to 0 \quad \textit{almost surely} \tag{3.1}$$

and

$$\frac{S_n}{n} \to 0 \quad \textit{in probability}. \tag{3.2}$$

Assume, moreover, that for some $v > 0$, all n and t, $0 < t \le 1$,

$$\mathbb{P}\Big\{\max_{i\in I(n)} \|X_i\| > tv2^n\Big\} \le \delta_n \exp\left(\frac{1}{t}\right), \tag{3.3}$$

where $\sum_n \delta_n^s < \infty$ for some $s > 0$, and that, for each $\delta > 0$,

$$\sum_n \exp\left(-\delta 2^{2n}\Big/ \sup_{\|f\|\le 1} \sum_{i\in I(n)} \mathbb{E}\big(f^2(X_i) I_{\{\|X_i\|\le 2^n\}}\big)\right) < \infty. \tag{3.4}$$

Then the strong law of large numbers holds, i.e.,

$$\frac{S_n}{n} \to 0 \quad \textit{almost surely}.$$

PROOF. We simply apply Theorem 2.3 and Lemma 2.5. We can first assume, by Lemma 2.1, the X_i's to be symmetric and, by (3.2) and Lemma 2.4, take L (in Theorem 2.3) to be 0. Since $X_i/i \to 0$ almost surely, for every $u > 0$,

$$\sum_n \mathbb{P}\Big\{\max_{i\in I(n)} \|X_i\| > u2^n\Big\} < \infty.$$

Summarizing the conclusions of Lemma 2.5 and Theorem 2.3, for all $u, \delta > 0$ and $q \ge 2K_0$, and for s assumed to be an integer,

$$\sum_n \mathbb{P}\left\{\left\|\sum_{i\in I(n)} X_i\right\| > 10^2\left[2s\left(u + v\left(\log\frac{q}{K_0}\right)^{-1}\right) + q\delta\right]2^n\right\} < \infty.$$

It obviously follows that

$$\sum_n \mathbb{P}\left\{\left\|\sum_{i\in I(n)} X_i\right\| > \varepsilon 2^n\right\} < \infty$$

for every $\varepsilon > 0$ and the theorem is proved by Lemma 2.2. □

COROLLARY 3.2. *Under the hypotheses of Theorem* 3.1, *but with* (3.3) *replaced by*

$$\sum_n \left(\frac{1}{2^{np}} \sum_{i \in I(n)} \mathbb{E}\|X_i\|^p \right)^s < \infty, \tag{3.3$'$}$$

for some $p > 0$ *and some* $s > 0$, *the strong law of large numbers is satisfied*:

$$\frac{S_n}{n} \to 0 \quad \textit{almost surely}.$$

PROOF. Simply note that

$$\sum_{i \in I(n)} \mathbb{P}\{\|X_i\| > tv2^n\} \le \frac{1}{(tv)^p} \frac{1}{2^{np}} \sum_{i \in I(n)} \mathbb{E}\|X_i\|^p$$

$$\le C(p)\exp\left(\frac{1}{t}\right) \frac{1}{v^p} \frac{1}{2^{np}} \sum_{i \in I(n)} \mathbb{E}\|X_i\|^p,$$

from which (3.3) of Theorem 3.1 follows. Note that the sums $\sum_{i \in I(n)} \mathbb{E}\|X_i\|^p$ can also be replaced, if one wishes it, by expressions of the type

$$\sup_{t>0} t^p \sum_{i \in I(n)} \mathbb{P}\{\|X_i\| > t\}.$$

□

Conditions (3.1) and (3.2) in Theorem 3.1 are of course necessary, (3.2) describing the classical assumption in probability on the sequence (S_n) (usually trivially satisfied in the real valued statements). Under (3.1), it is legitimate and sometimes convenient for comparison to assume that $\|X_i\| \le i$; for example, (3.3) [via (3.3′)] holds under the stronger hypothesis

$$\sum_i \frac{\mathbb{E}\|X_i\|^p}{i^p} < \infty \quad \text{for some } p > 0,$$

which is then seen to be weaker and weaker as p increases. This condition is easily comparable to the classical statements as described before. In this way, it is possible to recover from Theorem 3.1 almost all known vector valued laws of large numbers of Kolmogorov's type, e.g., [23] and [15]. It is also possible to include laws of Brunk's type ([4], [33], [6] and [15]); Brunk's theorem states that if (X_i) is a sequence of independent mean zero real random variables satisfying

$$\sum_i \frac{\mathbb{E}|X_i|^p}{i^{p/2+1}} < \infty \quad \text{for some } p \ge 2,$$

then the law of large numbers holds. If we observe that, for $p \ge 2$,

$$\frac{1}{2^{2n}} \sum_{i \in I(n)} \mathbb{E} f^2(X_i) \le \left(\frac{1}{2^{n(p/2+1)}} \sum_{i \in I(n)} \mathbb{E}|f(X_i)|^p \right)^{2/p},$$

it is easily seen how Theorem 3.1 (or better Corollary 3.2) contains Brunk's theorem as well as extensions to the vector valued setting.

One feature of the isoperimetric approach is a common treatment of Kolmogorov's and Prokorov's laws of large numbers. As easily as we obtained Theorem 3.1 from Section 2, we get an extension of Prokorov's result to the vector valued case: We still work under conditions (3.2) and (3.4), but reinforce (3.1) into

$$\|X_i\| \le \frac{i}{LLi} \quad \text{almost surely for each } i,$$

where we denote $Lt = \max(1, \log t)$ and $LLt = L(Lt)$, $t \in \mathbb{R}_+$. This boundedness assumption provides the exact bound on large values and actually fits (3.3) of Theorem 3.1. Indeed, for each n and t, $0 < t \le 1$,

$$\mathbb{P}\Big\{ \max_{i \in I(n)} \|X_i\| > 2t2^n \Big\} \le \delta_n \exp\left(\frac{1}{t}\right),$$

with $\delta_n = \exp(-2LL2^n)$. We thus obtain as a next corollary the following result which already appeared in [25]; it improves upon previous results of [14] and [2]. Note that the boundedness assumption, as is well known on the line (cf. [29]), is optimal under (3.4) and that (3.4) becomes necessary under this hypothesis, following the necessity of (2.3) in Theorem 2.3 (see [2]). Further, the convergence $S_n/n \to 0$ in probability for real mean zero random variables always holds under (3.4).

Corollary 3.3. *Let (X_i) be a sequence of independent random variables with values in a Banach space B. Assume that*

$$\|X_i\| \le \frac{i}{LLi} \quad \textit{almost surely for each } i.$$

Then, if (and only if)

$$\frac{S_n}{n} \to 0 \quad \textit{in probability}$$

and

$$\sum_n \exp\left(-\delta 2^{2n} \Big/ \sup_{\|f\| \le 1} \sum_{i \in I(n)} \mathbb{E} f^2(X_i) \right) < \infty,$$

for each $\delta > 0$, we have

$$\frac{S_n}{n} \to 0 \quad \textit{almost surely}.$$

4. The law of the iterated logarithm and extreme values. As for the law of large numbers, the framework provided by the isoperimetric inequality allows one to study quite easily the law of the iterated logarithm. The two classical and main theorems are here the laws of the iterated logarithm of Kolmogorov [19] and Hartman, Wintner and Strassen ([12] and [30]) (cf. [29]

for a general reference). An extension to Banach space of the first one using the isoperimetric approach has already been obtained in [25]. The general idea of the proof is the same as the one used for Prokorov's law of large numbers so that we need only briefly describe the result.

THEOREM 4.1. *Let* (X_i) *be a sequence of independent random variables with values in a Banach space* B *such that* $\mathbb{E} f(X_i) = 0$ *and* $\mathbb{E} f^2(X_i) < \infty$ *for each* f *in* B^* *and each* i *in* $\mathbb{N}$. *Let, for each* n, $s_n = \sup_{\|f\| \le 1}(\sum_{i=1}^n \mathbb{E} f^2(X_i))^{1/2}$, *assumed to increase to* ∞. *We suppose that*

$$\|X_i\| \le \frac{\eta_i s_i}{\left(LLs_i^2\right)^{1/2}} \quad \textit{almost surely for each } i,$$

for some sequence (η_i) *of positive numbers tending to* 0. *Then, if (and only if) the sequence* $(S_n/(2s_n^2 LLs_n^2)^{1/2})$ *is bounded in probability, with probability* 1,

$$1 \le \limsup_{n \to \infty} \frac{\|S_n\|}{\left(2s_n^2 LLs_n^2\right)^{1/2}} \le M,$$

for some finite number M.

We do not know whether it is possible to take $M = 1$ as it is the case on the line even if the sequence $(S_n/(2s_n^2 LLs_n^2)^{1/2})$ converges to 0 in probability.

PROOF. The upper bound with M unspecified only requires the sequence (η_i) be bounded, for example by 1. We need only consider the case of symmetric random variables. For each n, set $a_n = (2s_n^2 LLs_n^2)^{1/2}$ and define m_n as the smallest m such that $s_m > 2^n$. It is easily seen that

$$\frac{s_{n+1}}{s_n} \sim 1, \qquad s_{m_n} \sim 2^n, \qquad \frac{s_{m_{n+1}}}{s_{m_n}} \sim 2,$$

so that we can make use of Lemma 2.2. To apply Theorem 2.3, note that (2.3) will be trivially satisfied by the very definition of s_n and that L will be finite from Lemma 2.4. Concerning (2.1) and (2.2), we use Lemma 2.5 and observe, as for Prokorov's law of large numbers that, for each n and t, $0 < t \le 1$,

$$\mathbb{P}\Big\{ \max_{i \in I(n)} \|X_i\| > t a_{m_n} \Big\} \le \delta_n \exp\left(\frac{1}{t}\right),$$

with $\delta_n = \exp(-\sqrt{2}\, LLs_{m_n}^2)$. It follows that for some M,

$$\sum_n \mathbb{P}\left\{ \left\| \sum_{i \in I(n)} X_i \right\| > M a_{m_n} \right\} < \infty,$$

hence the upper bound. The lower bound (which uses $\eta_i \to 0$) is detailed in [25] and rests, as the necessity of (2.3) in Theorem 2.3, on Kolmogorov's real exponential minoration inequality. □

Hartman and Wintner deduced their law of the iterated logarithm for independent and identically distributed random variables from Kolmogorov's result and a (clever) truncation argument. (Along this line, cf. [7] for a simpler proof of the Hartman–Wintner theorem.) In the framework of the isoperimetric inequality, both, however, can be obtained essentially by the same procedure: The identical distribution of the random variables and the integrability condition we will require play the role of the truncation in Theorem 4.1. We thus obtain, with the tools of Sections 1 and 2, a new and simpler proof of the characterization of [24] of Banach space valued random variables satisfying the law of the iterated logarithm in the Hartman–Wintner–Strassen form along the same lines used for Kolmogorov's law of the iterated logarithm (Theorem 4.1).

The proof we give applies similarly when extreme values are deleted from sums and we deduce, in the second part of this section, several consequences on the law of the iterated logarithm for trimmed sums. Extensions and new proofs of Mori's laws of large numbers when extreme values are deleted from sums ([27] and [28]) complete in the same spirit this section.

Let X be a random variable with values in a Banach space B. From now on, (X_i) will denote a sequence of independent random variables with the same distribution as X. For convenience, we also set $a_n = (2nLLn)^{1/2}$ for each n, where we recall that $Lt = \max(1, \log t)$, $LLt = L(Lt)$, $t \in \mathbb{R}_+$. With this notation, the random variable X is said to satisfy the (compact) law of the iterated logarithm if there is a compact set K in B such that, with probability 1,

$$\lim_{n \to \infty} d\left(\frac{S_n}{a_n}, K\right) = 0 \quad \text{and} \quad C\left(\frac{S_n}{a_n}\right) = K, \tag{4.1}$$

where $d(x, K)$ denotes the distance from the point x to the set K and $C(x_n)$ the set of limit points of the sequence (x_n). It is known ([20]) that if such a set K exists, it is necessarily the unit ball of the reproducing kernel Hilbert space associated to the covariance structure of X and that (4.1) holds if and only if the sequence (S_n/a_n) is *almost surely relatively compact in* B. We refer to [24] for further details, information and references concerning these definitions and equivalent formulations. In [24] the following characterization of random variables satisfying the law of the iterated logarithm was obtained.

THEOREM 4.2. *Let X be a random variable with values in B. In order that X satisfy the law of the iterated logarithm, it is necessary and sufficient that the following conditions be fulfilled*:

$$\mathbb{E}\left(\frac{\|X\|^2}{LL\|X\|}\right) < \infty; \tag{4.2}$$

(4.3) *the family of real random variables* $\{f^2(X);\ f \in B^*,\ \|f\| \le 1\}$ *is uniformly integrable*;

$$\frac{S_n}{a_n} \to 0 \quad \textit{in probability}. \tag{4.4}$$

Proof. We refer to [24] for the easy necessity part (see also the end of the proof of Theorem 4.3). Turning to sufficiency, we first assume the random variable X to be symmetric. We would like to apply Theorem 2.3 and the results of Section 2. Set there, for each n, $m_n = 2^n$. Note that, by Lemma 2.4 and (4.4), whatever the choice of ε and (k_n) will be, we can take $L = 0$ in Theorem 2.3. Further, if we let

$$\sigma^2 = \sup_{\|f\|\le 1} \mathbb{E} f^2(X) \qquad (< \infty),$$

(2.3) will hold with $\delta = \sigma$. We are therefore left with conditions (2.1) and (2.2) that we check using the integrability hypothesis (4.2) applying Lemma 2.5. Recall first that $\mathbb{E}(\|X\|^2/LL\|X\|) < \infty$ if and only if

$$\sum_n 2^n \mathbb{P}\{\|X\| > \varepsilon a_{2^n}\} < \infty,$$

for each (or some) $\varepsilon > 0$. Set $\gamma_n = \gamma_n(\varepsilon) = 2^n \mathbb{P}\{\|X\| > \varepsilon a_{2^n}\}$. It is easily seen that there exists a sequence (β_n) such that $\beta_n \ge \gamma_n$ for each n, $\sum_n \beta_n < \infty$ and satisfying the regularity condition $\beta_n \le 2\beta_{n+1}$ for each n. In order to verify (2.6) of Lemma 2.5, let $t \le 1$ be such that $2^{-k} < t^2 \le 2^{-k+1}$. When $k < n$,

$$2^n \mathbb{P}\{\|X\| > t\varepsilon a_{2^n}\} \le 2^k \gamma_{n-k} \le 2^k \beta_{n-k} \le 2^{2k} \beta_n \le \frac{4}{t^4} \beta_n.$$

If $k \ge n$,

$$2^n \mathbb{P}\{\|X\| > t\varepsilon a_{2^n}\} \le 2^n \le \frac{4}{t^4} 2^{-n},$$

so that, in any case, (2.6) holds with

$$\delta_n = 96 \max(\beta_n, 2^{-n}),$$

which satisfies $\sum_n \delta_n < \infty$. Lemma 2.5 thus tells us that, taking for example $q = 2K_0$, there exists a sequence (k_n) of integers such that $\sum_n 2^{-k_n} < \infty$ and

$$\sum_n \mathbb{P}\left\{ \sum_{i=1}^{k_n} \|X_{2^n-1}^{(i)}\| > 5\varepsilon a_{2^n} \right\} < \infty.$$

It follows from Theorem 2.3 that for some numerical constant C and all $\varepsilon > 0$,

$$\sum_n \mathbb{P}\left\{ \left\| \sum_{i=1}^{2^n} X_i \right\| > C(\varepsilon + \sigma) a_{2^{n-1}} \right\} < \infty.$$

Hence, by Lévy's maximal inequality for symmetric random variables and the Borel–Cantelli lemma, with probability 1,

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} \le C\sigma.$$

By (4.4) and the symmetrization Lemma 2.1, the same inequality holds when X is not symmetric, with perhaps $2\sqrt{2}\,C$ instead of C. The proof is almost complete: If we replace in the argument the norm of B by quotient norms by finite-dimensional subspaces of B, the uniform integrability (4.3) implies that the corresponding σ's can be made arbitrarily small for large enough subspaces. By the preceding inequality, the same holds for the $\limsup \|S_n/a_n\|$ and it immediately follows that the sequence (S_n/a_n) is almost surely relatively compact in B. Theorem 3.1 of Kuelbs [20], as mentioned before, leads then to the existence of K satisfying (4.1). □

Theorem 4.2 and its proof can be extended to the case where extreme values are deleted from the sums S_n. The result we obtain below has its origin in the observation (see [21]) that the integrability condition $\mathbb{E}(\|X\|^2/LL\|X\|) < \infty$ in Theorem 4.2 exactly expresses that the maximal term of the sample $(\|X_1\|, \dots, \|X_n\|)$ is asymptotically negligible with respect to the normalization a_n; thus deleting large values from sums should be reflected in this integrability condition.

Before stating this result for trimmed sums, recall that for $n \geq r \geq 1$, $X_n^{(r)} = X_i$ whenever $\|X_i\|$ is the rth maximum of $(\|X_1\|, \dots, \|X_n\|)$ (0 if $r > n$). We also let ${}^{(r)}S_n = S_n - X_n^{(1)} - \cdots - X_n^{(r)}$. ${}^{(0)}S_n$ is just S_n. We denote also by $L_{p,q}(0 < p, q < \infty)$ the space of all real random variables ζ such that

$$\int_0^\infty \left(t^p \mathbb{P}\{|\zeta| > t\}\right)^{q/p} \frac{dt}{t} < \infty.$$

$L_{p,p}$ is just L_p by the usual integration by parts formula and $L_{p,q_1} \subset L_{p,q_2}$ if $q_1 \leq q_2$. Moreover, as is easy to see, $\lim_{t\to\infty} t^p \mathbb{P}\{|\zeta| > t\} = 0$ when $\zeta \in L_{p,q}$.

THEOREM 4.3. *Let r be an integer and let X be a random variable with values in B. In order that there exist a compact set K in B such that, with probability* 1,

$$\lim_{n\to\infty} d\left(\frac{{}^{(r)}S_n}{a_n}, K\right) = 0 \quad \text{and} \quad C\left(\frac{{}^{(r)}S_n}{a_n}\right) = K,$$

it is necessary and sufficient that the following conditions be fulfilled:

$$\frac{\|X\|^2}{LL\|X\|} \in L_{1,r+1}; \tag{4.5}$$

$$\{f^2(X);\ f \in B^*, \|f\| \leq 1\} \quad \text{is uniformly integrable}; \tag{4.6}$$

$$\frac{S_n}{a_n} \left(\text{or } \frac{{}^{(r)}S_n}{a_n}\right) \to 0 \quad \text{in probability}. \tag{4.7}$$

If such a set K exists, it is the same as the one which appears for $r = 0$, i.e., the usual cluster set of the law of the iterated logarithm. The proof of this result is actually only a small variation on the proof of Theorem 4.2 with the modifications we mentioned about trimming at the ends of Sections 1 and 2. Theorem 4.3 improves upon a result in [21] (to which we refer for a short history of these questions) under the assumption of the central limit theorem.

Moreover, Theorem 4.3 points out the certainly well-known fact that trimming a finite number of values does not affect the stability of the real law of the iterated logarithm, i.e., for any r,

$$\limsup_{n\to\infty} \frac{{}^{(r)}S_n}{a_n} = (\mathbb{E}X^2)^{1/2} \quad \text{almost surely},$$

if and only if the real valued random variable X satisfies $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$. This is established in the course of the proof of Theorem 4.3 to which we turn now. □

PROOF. It is completely similar to the proof of Theorem 4.2 (corresponding to $r = 0$) but a few points like the symmetrization procedure and the necessity part require some more details. We start, however, with the portion reproducing Theorem 4.2. Note first, for further use, that the integrability condition $\|X\|^2/LL\|X\| \in L_{1,r+1}$ is equivalent to saying that

$$\lim_{n\to\infty} \frac{\|X_n^{(r+1)}\|}{a_n} = 0 \quad \text{almost surely} \tag{4.8}$$

(cf. [27], Lemma 3). Moreover, but this is trivial; it holds if and only if

$$\sum_n \left(2^n \mathbb{P}\{\|X\| > \varepsilon a_{2^n}\}\right)^{r+1} < \infty,$$

for some (or every) $\varepsilon > 0$. Hence, by the comments at the end of Remark 2.7 and the exact same argument for Theorem 4.2, we see that for $q = 2K_0$ and every $\varepsilon > 0$, there exists a sequence of integers (k_n) such that $\sum_n 2^{-k_n} < \infty$ and satisfying

$$\sum_n \mathbb{P}\left\{\sum_{i=r+1}^{k_n} \|X_{2^{n-1}}^{(i)}\| > 5(r+1)\varepsilon a_{2^n}\right\} < \infty.$$

Let X be symmetric. Theorem 2.3 for trimmed sums (Remark 2.7) implies now that for each $\varepsilon > 0$,

$$\sum_n \mathbb{P}\left\{\max_{j\le 2^n}\|{}^{(r)}S_j\| > C(r)(\varepsilon+\sigma)a_{2^{n-1}}\right\} < \infty,$$

where $C(r)$ only depends on r and $\sigma = \sup_{\|f\|\le 1}(\mathbb{E}f^2(X))^{1/2}$. Hence

$$\limsup_{n\to\infty} \frac{\|{}^{(r)}S_n\|}{a_n} \le C(r)\sigma \quad \text{almost surely}. \tag{4.9}$$

This already concludes the proof in the symmetrical case. To deal with the general case, we make use of (4.8) and of the following elementary symmetrization lemma for trimmed sums. Let (X_i') denote an independent copy of the sequence (X_i) and set, for each i, $\tilde{X}_i = X_i - X_i'$, defining thus symmetrical random variables. The notation $X_n'^{(r)}$, S_n', ${}^{(r)}S_n'$, $\tilde{X}_n^{(r)}$, $\tilde{S}_n$, ${}^{(r)}\tilde{S}_n$ is consistent with that introduced above.

LEMMA 4.4. *For each* $n \geq r + 1$,

$$\left\| {}^{(r)}\tilde{S}_n - \left({}^{(r)}S_n - {}^{(r)}S'_n \right) \right\| \leq 6r \max\left\{ \| X_n^{(r+1)} \|, \| X_n'^{(r+1)} \|, \| \tilde{X}_n^{(r+1)} \| \right\}.$$

PROOF. Denote by I (resp., J, K) the set of indices corresponding to the r maxima of $(X_1, \dots, X_n)$ [resp., $(X'_1, \dots, X'_n), (\tilde{X}_1, \dots, \tilde{X}_n)$]. Then

$${}^{(r)}\tilde{S}_n - \left({}^{(r)}S_n - {}^{(r)}S'_n \right) = - \sum_{k \in K} \tilde{X}_k + \sum_{i \in I} X_i - \sum_{j \in J} \tilde{X}'_j.$$

Let A (resp., B, C) denote $I \cap J \cap K$ (resp., the indices belonging to 2 exactly, 1 exactly, of the sets I, J, K). Then, letting a denote the maximum in the right-hand side of the inequality of the lemma,

$$\begin{aligned} &\text{if } i \in A, \quad && -\tilde{X}_i + X_i - X'_i = 0, \\ &\text{if } i \in B, && \| -\tilde{X}_i I_K(i) + X_i I_I(i) - X_i I_J(i) \| \leq a, \\ &\text{if } i \in C, && \| -\tilde{X}_i I_K(i) + X_i I_I(i) - X_i I_J(i) \| \leq 2a. \end{aligned}$$

Hence

$$\left\| {}^{(r)}\tilde{S}_n - \left({}^{(r)}S_n - {}^{(r)}S'_n \right) \right\| \leq a(\operatorname{card} B + 2 \operatorname{card} C).$$

Since $\operatorname{card} B + \operatorname{card} C \leq 3r$, the conclusion follows. □

Using the integrability condition $\|X\|^2/LL\|X\| \in L_{1,r+1}$ (that also holds for $\tilde{X}$ and X'), we easily deduce from (4.8), Lemmas 4.4 and 2.1 that (4.9) will hold for all random variables X satisfying the hypotheses of Theorem 4.3, and not only symmetric ones. As in Theorem 4.2, the same theorem of Kuelbs [20] concludes the sufficiency part of this proof since it is plain, by the one-dimensional law of the iterated logarithm, that

$$\limsup_{n \to \infty} \frac{f\left({}^{(r)}S_n \right)}{a_n} = \left(\mathbb{E} f^2(X) \right)^{1/2} \quad \text{almost surely},$$

for every $r \geq 0$ and f in B^* [note that $\mathbb{E} f(X) = 0$ under (4.7)].

The necessity of (4.5) is well known and the argument is due to Mori (see [21]). In order to prove the necessity of (4.7), note that the almost sure relative compactness of the sequence $({}^{(r)}S_n/a_n)$ implies that the sequence of the laws of the ${}^{(r)}S_n/a_n$ is tight. Since under $\|X\|^2/LL\|X\| \in L_{1,r+1}$, $\lim_{n \to \infty} n \mathbb{P}\{\|X\| > \varepsilon a_n\} = 0$ for every $\varepsilon > 0$, the sequence of the laws of the S_n/a_n is also tight. Now, if we assume that $\mathbb{E} f^2(X) < \infty$ for every f, clearly $f(S_n)/a_n \to 0$ in probability and (4.7) follows by tightness. We are thus left with the necessity of (4.6). This will follow from the certainly known next lemma.

LEMMA 4.5. *If, with probability* 1,

$$\limsup_{n \to \infty} \frac{\| {}^{(r)}S_n \|}{a_n} < \infty, \tag{4.10}$$

then $\mathbb{E} f^2(X) < \infty$ *for every* f *in* B^*.

PROOF. Lemma 4.4 and the necessity of (4.5) ensure that it is enough to consider the case of a symmetric X, something we assume henceforth. Let $c > 0$ arbitrary, but fixed, and define

$$\bar{X} = XI_{\{\|X\| \le c\}} - XI_{\{\|X\| > c\}}.$$

By symmetry, $\bar{X}$ has the same distribution as X. We can suppose that $\mathbb{P}\{\|X\| \le c\} < 1$ (if not there is nothing to prove). In this case, if we let, for each n,

$$C_n = \{\text{there exist at least } r \text{ indices } i \le n \text{ such that } \|X_i\| > c\},$$

it is plain that $\mathbb{P}\{\liminf_{n\to\infty} C_n\} = 1$ [since $\sum_n \mathbb{P}(C_n^c) < \infty$]. Therefore, for almost every ω, there exists an integer n_0 such that for every $n \ge n_0$, $\omega \in C_n$; if $\omega \in C_n$, it is easy to see that

$$2S_{n,c}(\omega) = {}^{(r)}S_n(\omega) + {}^{(r)}\bar{S}_n(\omega),$$

where $S_{n,c} = \sum_{i=1}^n X_i I_{\{\|X_i\| \le c\}}$ and ${}^{(r)}\bar{S}_n$ has the same meaning for $\bar{X}$ as ${}^{(r)}S_n$ for X. Since by the 0–1 law, (4.10) has really to be meant as

$$\limsup_{n\to\infty} \frac{\|{}^{(r)}S_n\|}{a_n} = M \quad \text{almost surely},$$

for some finite number M, it follows that, for every $c > 0$, with probability 1,

$$\limsup_{n\to\infty} \frac{\|S_{n,c}\|}{a_n} \le M.$$

By symmetry and the classical law of the iterated logarithm, we then get, for each f in B^*,

$$\left(\mathbb{E} f^2(X) I_{\{\|X\| \le c\}}\right)^{1/2} = \limsup_{n\to\infty} \frac{f(S_{n,c})}{a_n} \le M\|f\|,$$

from which the lemma follows when $c \to \infty$. □

To prove that the family $\{f^2(X);\ f \in B^*,\ \|f\| \le 1\}$ is uniformly integrable is now exactly as in the proof of Theorem 1.2 in [24]. This concludes the proof of Theorem 4.3.

When r increases, the integrability conditions $\|X\|^2/LL\|X\| \in L_{1,r+1}$ in Theorem 4.3 are less and less constraining. If one decides then to subtract from the sums S_n a number $r = r_n$ depending on the size n of the sample, increasing to ∞ but still chosen to be reasonably small (for applications), it is possible to replace the preceding integrability conditions by some tail behavior, namely $\lim_{t\to\infty} t^2\mathbb{P}\{\|X\| > t\} = 0$. The following theorem extends further a result obtained in [21] under the stronger assumption of the central limit property.

THEOREM 4.6. *Let X be a random variable with values in B satisfying*

$$\lim_{t\to\infty} t^2\mathbb{P}\{\|X\| > t\} = 0. \tag{4.11}$$

Then, there exists a sequence of positive numbers (ζ_n) decreasing to 0 such *that if r_n denotes the integer part of $\zeta_n LLn$ and if*

$$\{f^2(X);\ f \in B^*, \|f\| \le 1\} \quad \textit{is uniformly integrable}, \tag{4.12}$$

with probability 1,

$$\lim_{n\to\infty} d\left(\frac{{}^{(r_n)}S_n}{a_n}, K\right) = 0 \quad \textit{and} \quad C\left(\frac{{}^{(r_n)}S_n}{a_n}\right) = K$$

if and only if

$$\frac{{}^{(r_n)}S_n}{a_n} \to 0 \quad \textit{in probability}; \tag{4.13}$$

K is here again the usual cluster set of the law of the iterated logarithm.

PROOF. The sequence (ζ_n) is chosen as in [21]. More precisely, by $\lim_{t\to\infty} t^2\mathbb{P}\{\|X\| > t\} = 0$, let (ζ_n) be any sequence of positive numbers decreasing to 0 such that, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \zeta_n \log\left(\frac{\zeta_n}{\Lambda(\varepsilon\beta(n))}\right) = \infty, \tag{4.14}$$

where $\beta(n) = (n/LLn)^{1/2}$ and $\Lambda(t) = \sup_{s\ge t} s^2\mathbb{P}\{\|X\| > s\}$. We may also assume that $r_n = [\zeta_n LLn] \to \infty$ with n, where $[\cdot]$ is the integer part function. Set, for each n, $s_n = [\zeta_{2^n} LL2^{n-1}] + 1$. We first note that, for each $\varepsilon > 0$,

$$\sum_n \mathbb{P}\{LL2^n \|X_{2^n}^{(s_n)}\| > \varepsilon a_{2^n}\} < \infty. \tag{4.15}$$

Indeed, the classical binomial estimates (see, e.g., [5]) give

$$\mathbb{P}\{LL2^n\|X_{2^n}^{(s_n)}\| > \varepsilon a_{2^n}\} \le \left(\frac{e2^n\mathbb{P}\{\|X\| > \varepsilon\beta(2^n)\}}{s_n}\right)^{s_n},$$

from which (4.15) easily follows by (4.14). As a consequence of (4.15) we note in particular that

$$\lim_{n\to\infty} \frac{r_n}{a_n}\|X_n^{(r_n+1)}\| = 0 \quad \text{almost surely;}$$

hence the symmetrization Lemma 4.4 allows us to reduce to the case of symmetric random variables.

We now make use of the framework developed in Sections 1 and 2. We find it convenient to go back to Proposition 1.4, recalling that in our setting: For $k_n \ge q, s_n$, and $s, t > 0$,

$$\begin{aligned}
&\mathbb{P}\left\{\max_{2^{n-1}<j\le 2^n} \|{}^{(r_j)}S_j\| > t + 2s + 8qM_n\right\} \\
&\quad \le \left(\frac{K_0}{q}\right)^{k_n} + \mathbb{P}\left\{\sum_{i=s_n}^{k_n} \|X_{2^n}^{(i)}\| > s\right\} + 8\exp\left(-\frac{t^2}{64q\sigma_n^2}\right) \\
&\qquad + 8\exp\left(-\frac{(k_n - s_n)t^2}{768qM_n s}\right),
\end{aligned} \tag{4.16}$$

where

$$\sigma_n^2 = \sup_{\|f\| \le 1} \sum_{i=1}^{2^n} \mathbb{E}\left(f^2(X_i) I_{\{\|X_i\| \le s/(k_n - s_n)\}} \right)$$

and

$$M_n = \mathbb{E} \left\| \sum_{i=1}^{2^n} X_i I_{\{\|X_i\| \le s/(k_n - s_n)\}} \right\|.$$

We will use this inequality for each n (large enough), with $q = 2K_0$, $k_n = [2LL2^n]$ and $s = \varepsilon a_{2^n}$, $\varepsilon > 0$. Clearly, $\sigma_n^2 \le 2^n \sigma^2$, where $\sigma^2 = \sup_{\|f\| \le 1} \mathbb{E} f^2(X)$. More interesting is to notice that under hypotheses (4.11) and (4.13) of the theorem, we have $\lim_{n \to \infty} M_n / a_{2^n} = 0$. Indeed, we know from (4.15) that almost surely for all n large enough, the number of X_i, $i \le 2^n$, satisfying $\|X_i\| > 2\varepsilon\beta(2^n)$ is less than r_{2^n}. Hence, combining with (4.13), we see that

$$\frac{1}{a_{2^n}} \sum_{i=1}^{2^n} X_i I_{\{\|X_i\| \le 2\varepsilon\beta(2^n)\}} \to 0 \quad \text{in probability.}$$

Since $k_n - s_n \ge LL2^n$ for n large enough, we deduce from Lemma 2.4 that $M_n / a_{2^n} \to 0$ as announced. As yet another remark, note that, obviously,

$$\sum_{i=s_n}^{k_n} \|X_{2^n}^{(i)}\| \le k_n \|X_{2^n}^{(s_n)}\|.$$

We can now conclude the proof of the theorem. By our preceding choices and observations, if we take t to be of the order of σa_{2^n} in (4.16), we easily obtain from this inequality that for some numerical constant C and all $\varepsilon > 0$,

$$\sum_n \mathbb{P}\left\{ \max_{2^{n-1} < j \le 2^n} \|{}^{(r_j)}S_j\| > C(\varepsilon + \sigma) a_{2^{n-1}} \right\} < \infty.$$

Hence

$$\limsup_{n \to \infty} \frac{\|{}^{(r_n)}S_n\|}{a_n} \le C\sigma \quad \text{almost surely.}$$

As in Theorems 4.2 and 4.3, condition (4.12) implies further the relative compactness in B of the sequence $({}^{(r_n)}S_n / a_n)$. The theorem of Kuelbs [20] will complete our proof whenever we know that

$$\limsup_{n \to \infty} \frac{f\left({}^{(r_n)}S_n\right)}{a_n} = \left(\mathbb{E} f^2(X)\right)^{1/2} \quad \text{almost surely,}$$

for each f in B^*. By the classical law of the iterated logarithm, it is of course enough to be convinced that, with probability 1,

$$\lim_{n \to \infty} \frac{1}{a_n} f\left(\sum_{i=1}^{r_n} X_n^{(i)} \right) = 0.$$

To this aim, simply observe that for each n,

$$\left|\frac{1}{a_n}\sum_{i=1}^{r_n} f(X_n^{(i)})\right| \le \zeta_n + \left|\frac{1}{a_n}\sum_{i=1}^{r_n} f(X_n^{(i)}) I_{\{|f(X_n^{(i)})|>(2n/LLn)^{1/2}\}}\right|$$

$$\le \zeta_n + \left[\left(\frac{1}{n}\sum_{i=1}^{n} f^2(X_i)\right)\left(\frac{1}{2LLn}\sum_{i=1}^{n} I_{\{|f(X_i)|>(2n/LLn)^{1/2}\}}\right)\right]^{1/2}.$$

By the strong law of large numbers and Kronecker's lemma, the hypothesis $\mathbb{E} f^2(X) < \infty$ implies that the second term in the right-hand side of the previous inequality converges almost surely to 0; our claim is therefore satisfied. As in Theorem 4.3, it proves by the same way the necessity of (4.13) and the proof is complete. □

It is interesting to point out that the general method based on the isoperimetric inequality ($*$) also provides new and simple proofs of Mori's strong laws of large numbers for independent identically distributed random variables when extreme values are deleted from sums ([27] and [28]). Extensions to the vector valued case already appeared in a first version of the article [22]. For $r = 0$, we recover the Marcinkiewicz–Zygmund laws of large numbers ([6]). Statements analogous to Theorem 4.4 with $r = r_n$ can also be obtained and some are described in [22].

THEOREM 4.7. *Let r be an integer and $0 < p < 2$. Let also X be a random variable with values in B. In order that there exist a sequence (c_n) in B such that*

$$\frac{{}^{(r)}S_n}{n^{1/p}} - c_n \to 0 \quad \textit{almost surely},$$

it is necessary and sufficient that

$$\frac{{}^{(r)}S_n}{n^{1/p}} - c_n \to 0 \quad \textit{in probability} \tag{4.17}$$

and

$$X \textit{ is in } L_{p,(r+1)p}. \tag{4.18}$$

The sequence (c_n) in this theorem can be chosen to be identically 0 in the case of symmetric random variables (see the proof). Note, also, that under the integrability condition (4.18), all the properties (4.17) when r varies are equivalent, in particular $r = 0$. Furthermore, (4.18) always implies (4.17) (with $c_n \equiv 0$) whenever $0 < p < 1$; as is well known this is no more the case however for $1 \le p < 2$ and various conditions (type) have to be imposed on the Banach space in order that such an implication holds (cf., e.g., [26] and [6]). These conditions are of course always satisfied on the real line in which case one might take $c_n = n\mathbb{E}X$ when $1 < p < 2$ and $c_n = \mathbb{E}(XI_{\{|X| \le \varepsilon n\}})$, $\varepsilon > 0$, in the remaining case $p = 1$ (see [28]).

PROOF. The necessity of (4.18) follows as in [28]. Turning to sufficiency, since, following again [28], (4.18) implies that (is equivalent to)

$$\lim_{n\to\infty} \frac{\|X_n^{(r+1)}\|}{n^{1/p}} = 0 \quad \text{almost surely,}$$

it is easily seen, by Lemma 4.4, that we can restrict ourselves to symmetric random variables and prove the theorem with $c_n \equiv 0$. We would like to apply the trimmed versions of Theorem 2.3 and Lemma 2.5 (Remark 2.7). To this aim, recall that (4.1) is equivalent to saying that

$$\sum_n \left(2^n \mathbb{P}\{\|X\| > \varepsilon 2^{n/p}\}\right)^{r+1} < \infty$$

for some (all) $\varepsilon > 0$. Let $\varepsilon > 0$ be fixed and set $\gamma_n = \gamma_n(\varepsilon) = (2^n \mathbb{P}\{\|X\| > \varepsilon 2^{n/p}\})^{r+1}$. There exists a sequence of positive numbers (β_n) such that $\beta_n \geq \gamma_n$ for every n, $\sum_n \beta_n < \infty$ and satisfying the regularity property $\beta_n \leq 2\beta_{n+1}$ for all n. Let now $0 < t \leq 1$ be such that $2^{-k} < t^p \leq 2^{-k+1}$, $k \geq 1$. If $k < n$,

$$\begin{aligned} 2^n \mathbb{P}\{\|X\| > t\varepsilon 2^{n/p}\} &\leq 2^n \mathbb{P}\{\|X\| > \varepsilon 2^{(n-k)/p}\} \\ &\leq 2^k (\gamma_{n-k})^{1/(r+1)} \\ &\leq 2^{2k} \beta_n^{1/(r+1)} \leq \frac{4}{t^{2p}} \beta_n^{1/(r+1)}. \end{aligned}$$

When $k \geq n$,

$$2^n \mathbb{P}\{\|X\| > t\varepsilon 2^{n/p}\} \leq 2^n \leq \frac{4}{t^{2p}} 2^{-n}.$$

Therefore, in any case, for every n and $0 < t \leq 1$,

$$2^n \mathbb{P}\{\|X\| > t\varepsilon 2^{n/p}\} \leq \delta_n \frac{1}{t^{2p}},$$

where $\sum_n \delta_n^{r+1} < \infty$. It follows from Lemma 2.5 completed with Remarks 2.6 and 2.7 that, for $q = 2K_0$ for example, there exists a sequence (k_n) of integers satisfying $\sum_n 2^{-k_n} < \infty$, and even

$$\sum_n \exp\left(-\delta k_n^{2-p}\right) < \infty \quad \text{for every } \delta > 0, \tag{4.19}$$

such that

$$\sum_n \mathbb{P}\left\{ \sum_{i=r+1}^{k_n} \|X_{2^{n-1}}^{(i)}\| > 5(r+1)\varepsilon 2^{n/p} \right\} < \infty.$$

We are now in a position to apply Theorem 2.3. Since $S_n/n^{1/p} \to 0$ in probability, by Lemma 2.4, $L = 0$. In order to check (2.3), note that, for each n,

$$\sigma_n^2 \leq 2^n \mathbb{E}\left(\|X\|^2 I_{\{\|X\| \leq 5(r+1)\varepsilon 2^{n/p}/k_n\}}\right);$$

since $\mathbb{P}\{\|X\| > t\} \leq C/t^p$ for all $t > 0$ under (4.18), a simple integration by parts shows that

$$\sigma_n^2 \le C(p,r)2^{2n/p}k_n^{p-2}.$$

By (4.19), (2.3) holds for every $\delta > 0$. The trimmed version of Theorem 2.3 (Remark 2.7) then tells us that, for some constant $C(r)$, and every $\varepsilon, \delta > 0$,

$$\sum_n \mathbb{P}\Big\{ \max_{j \le 2^n} \|^{(r)}S_j\| > C(r)(\varepsilon + \delta)2^{n/p} \Big\} < \infty.$$

By the Borel–Cantelli lemma the proof of Theorem 4.7 is complete. □

5. On the identification of the limits in the law of the iterated logarithm. We keep the notation of the preceding section on the law of the iterated logarithm. When a random variable X satisfies the law of the iterated logarithm, (4.1), i.e.,

$$\lim d\left(\frac{S_n}{a_n}, K\right) = 0 \quad \text{and} \quad C\left(\frac{S_n}{a_n}\right) = K \quad \text{almost surely},$$

completely describes the limits of the sequence (S_n/a_n). However, it is also of interest to try to have some idea of what (4.1) could be if X does not satisfy the law of the iterated logarithm. In particular, one might want to estimate the nonrandom (0–1 law) limit

$$\Lambda = \limsup_{n \to \infty} \frac{\|S_n\|}{a_n}, \tag{5.1}$$

when the sequence (S_n/a_n) is only bounded almost surely and not relatively compact (*bounded* law of the iterated logarithm).

In his remarkable recent work, Alexander [1] obtains a complete description of the cluster set $C(S_n/a_n)$. Assume that $\mathbb{E} f(X) = 0$ and $\mathbb{E} f^2(X) < \infty$ for every f in B^* so that the reproducing kernel Hilbert space associated with the covariance structure of X, and its unit ball K, are well defined. Alexander shows that $C(S_n/a_n)$ can only be a multiple αK of K with $0 \le \alpha \le 1$, or empty, and examples are given to show that all cases actually occur. Moreover, a series condition involving the laws of the partial sums S_n determines the value of α. We retain in particular that $C(S_n/a_n) = K$ almost surely when $S_n/a_n \to 0$ in probability, a result actually obtained earlier by de Acosta, Kuelbs and Ledoux [9].

Concerning the limit (5.1), less is known. Identification in some smooth normed spaces (including Hilbert spaces) was obtained by de Acosta and Kuelbs [8]. In the notation of Theorem 4.2, we are interested in the case where (4.2) and (4.4) hold, i.e., $\mathbb{E}(\|X\|^2/LL\|X\|) < \infty$ and $S_n/a_n \to 0$ in probability, and where (4.3) is replaced by

$$\sigma = \sup_{\|f\| \le 1} \left(\mathbb{E} f^2(X)\right)^{1/2} < \infty.$$

What our proof of Theorem 4.2 actually shows in this case is that $\Lambda \le C\sigma$ for some numerical constant C (> 1!). When $\{f^2(X);\ f \in B^*,\ \|f\| \le 1\}$ is uniformly integrable, which is equivalent to the compactness of K, it could be shown, by an approximation argument, that the sequence (S_n/a_n) is almost surely relatively compact, which implies (4.1). Besides the crucial symmetrization procedure itself, one reason for the poor constant C is that Lemma 1.3 does not provide the right constants. When dealing with Gaussian averages, it was the observation of [10] that one does have the best constants and it could be proved that $\Lambda = \sigma$ for random variables of the form λX, where λ denotes a standard Gaussian variable independent of X. In the last part of this work, we show that the equality $\Lambda = \sigma$ holds for *all* random variables X satisfying (4.2) and (4.4). The general idea of the proof avoids symmetrization and tries to find a substitute for the approximation argument alluded to above. To this aim, it combines the isoperimetric approach with real variable methods and exponential inequalities (instead of Lemma 1.3) that yield the best limit in the classical case.

Before turning to the statement of our result, let us recall that we always have $\Lambda \geq \sigma$ by the one-dimensional law of the iterated logarithm [X has mean 0 under (4.4)]. Further, when (4.4) is replaced by some stochastic boundedness of the sequence (S_n/a_n), the (still finite) limit Λ has to take this fact into account (see [9] and [10]).

THEOREM 5.1. *Let X be a random variable with values in a Banach space B. If*

$$\mathbb{E}\left(\frac{\|X\|^2}{LL\|X\|}\right) < \infty, \tag{5.2}$$

$$\sigma = \sup_{\|f\| \leq 1} \left(\mathbb{E} f^2(X)\right)^{1/2} < \infty \tag{5.3}$$

and

$$\frac{S_n}{a_n} \to 0 \quad \textit{in probability}, \tag{5.4}$$

we have

$$\limsup_{n \to \infty} \frac{\|S_n\|}{a_n} = \sigma \quad \textit{almost surely}.$$

Moreover,

$$\lim_{n \to \infty} d\left(\frac{S_n}{a_n}, K\right) = 0 \quad \textit{and} \quad C\left(\frac{S_n}{a_n}\right) = K$$

almost surely, where K is the unit ball of the reproducing kernel Hilbert space associated with X.

PROOF. It is enough to prove the first assertion of the theorem; indeed, replacing the norm of B by the gauge of $K + \varepsilon B_1$, where B_1 is the unit ball of

B, it is easily seen that $d(S_n/a_n, K) \to 0$; identification of the cluster set follows from [8] and [1]. By homogeneity and the real law of the iterated logarithm, we need only show that

$$\limsup_{n\to\infty} \frac{\|S_n\|}{a_n} \le 1 \quad \text{almost surely},$$

for $\sigma = 1$. As is well known, by the Borel–Cantelli lemma and Ottaviani's inequality, it suffices to prove that for all $\varepsilon > 0$ and $\rho > 1$,

$$\sum_n \mathbb{P}\left\{\left\|\sum_{i=1}^{m_n} X_i\right\| > (1+\varepsilon)a_{m_n}\right\} < \infty,$$

where $m_n = [\rho^n]$, $n \ge 1$.

Let $0 < \varepsilon \le 1$ and $\rho > 1$ be fixed. Recall (see [24]) that under (5.4), X has mean 0, i.e., $\mathbb{E} f(X) = 0$ for all f in B^*. To begin with, we need first recall a randomization property. Let $(\lambda_i)_{i\in\mathbb{N}}$ be an orthogaussian sequence independent of (X_i). Then (5.4) implies (see, e.g., [24]) that

$$\lim_{n\to\infty} \frac{1}{a_n}\mathbb{E}\left\|\sum_{i=1}^{n} \varepsilon_i X_i\right\| = \lim_{n\to\infty} \frac{1}{a_n}\mathbb{E}\left\|\sum_{i=1}^{n} \lambda_i X_i\right\| = 0. \tag{5.5}$$

The first step in the proof, very much in the spirit of Section 4 of [25], uses (5.5) and Gaussian properties to estimate some entropy numbers related to the geometry of the unit ball of B^*. Denote precisely by U this unit ball and set, for every integer n and f, g in U,

$$d_2^n(f,g) = \left(\mathbb{E}(f-g)^2(X) I_{\{\|X\| \le a_{m_n}\}}\right)^{1/2}.$$

Let further $N(U, d_2^n; \varepsilon)$ denote the minimal number of points g in U such that for every f in U there exists such a g with $d_2^n(f,g) \le \varepsilon$.

LEMMA 5.2. *There exists a sequence (α_n) of positive numbers tending to 0 such that, for every n large enough,*

$$N(U, d_2^n; \varepsilon) \le \exp(\alpha_n LLm_n). \tag{5.6}$$

PROOF. Suppose this is not the case. Then, for every sequence $\alpha_n \to 0$, infinitely often in n, there exists $U_n \subset U$ such that for any $f \ne g$ in U_n, $d_2^n(f,g) > \varepsilon$, and

$$\operatorname{card} U_n = [\exp(\alpha_n LLm_n)] + 1.$$

Set $N(n) = m_n^2$ (for example). By classical exponential estimates, in the form for example of Lemma 5.3 below, for $h = f - g$, $f \neq g$ in U_n,

$$\mathbb{P}\left\{\frac{1}{N(n)}\sum_{i=1}^{N(n)} h^2(X_i) I_{\{\|X_i\| \le a_{m_n}\}} \le \frac{\varepsilon^2}{2}\right\}$$

$$\le \mathbb{P}\left\{\frac{1}{N(n)}\sum_{i=1}^{N(n)} \left(-h^2(X_i) I_{\{\|X_i\| \le a_{m_n}\}} + \mathbb{E}h^2(X_i) I_{\{\|X_i\| \le a_{m_n}\}}\right) > \frac{\varepsilon^2}{2}\right\}$$

$$\le \exp\left(-\frac{\varepsilon^4 N(n)}{256 a_{m_n}^2}\right).$$

By the choice of $N(n) = m_n^2$, for n large enough,

$$\operatorname{card} U_n \exp\left(-\frac{\varepsilon^4 N(n)}{256 a_{m_n}^2}\right) < \frac{1}{2}.$$

It follows that, infinitely often in n,

$$\mathbb{P}\left\{\forall\, f \neq g \text{ in } U_n, \frac{1}{N(n)}\sum_{i=1}^{N(n)} (f-g)^2(X_i) I_{\{\|X_i\| \le a_{m_n}\}} > \frac{\varepsilon^2}{2}\right\} \ge \frac{1}{2}.$$

We are then in a position to apply, conditionally on this set of probability larger than $\frac{1}{2}$, Sudakov's minoration inequality for Gaussian processes (cf., e.g., [11], Theorem 2.3.1): For some numerical constant $K > 0$,

$$\frac{1}{\sqrt{N(n)}}\mathbb{E}\left\|\sum_{i=1}^{N(n)} \lambda_i X_i I_{\{\|X_i\| \le a_{m_n}\}}\right\| \ge \frac{\varepsilon}{4K}(\log \operatorname{card} U_n)^{1/2}$$

$$\ge \frac{\varepsilon}{4K}(\alpha_n LLm_n)^{1/2},$$

which holds therefore infinitely often in n. Since by (5.5) and the contraction principle

$$\lim_{n\to\infty} \frac{1}{a_{N(n)}}\mathbb{E}\left\|\sum_{i=1}^{N(n)} \lambda_i X_i I_{\{\|X_i\| \le a_{m_n}\}}\right\| = 0,$$

this clearly leads to a contradiction when (α_n) does not converge quickly enough to 0 since $LLN(n) \sim LLm_n$. The proof of Lemma 5.2 is complete. □

According to Lemma 5.2, we denote, for each n and f in U, by $g_n(f)$ an element of U such that $d_2^n(f, g_n(f)) \le \varepsilon$ in such a way that the set U_n of all $g_n(f)$ has a cardinality less than $\exp(\alpha_n LLm_n)$. We write that

$$\left\|\sum_{i=1}^{m_n} X_i\right\| \le \sup_{g\in U_n}\left|\sum_{i=1}^{m_n} g(X_i)\right| + \sup_{h\in V_n}\left|\sum_{i=1}^{m_n} h(X_i)\right|,$$

where $V_n = \{f - g_n(f),\ f \in U\} \subset 2U$. The main observation concerning V_n is that

$$\mathbb{E}\Big(h^2(X) I_{\{\|X\| \le a_{m_n}\}}\Big) \le \varepsilon^2,$$

for all h in V_n and all n. Although the proofs of Theorems 4.2 and 2.3 are described in the setting of a single true norm of a Banach space, it is clear that they also apply to more general seminorms which might, moreover, eventually depend on n on the blocks $I(n)$. In this way, using the preceding observation, it is just a mere exercise to see how the proofs of Theorems 4.2 and 2.3 yield similarly that

$$\sum_n \mathbb{P}\left\{ \sup_{h \in V_n} \left| \sum_{i=1}^{m_n} h(X_i) \right| > C\varepsilon a_{m_n} \right\} < \infty,$$

for some numerical constant $C > 0$. Taking this fact into account, we have now basically to prove that

$$\sum_n \mathbb{P}\left\{ \sup_{f \in U_n} \left| \sum_{i=1}^{m_n} f(X_i) \right| > (1 + C\varepsilon) a_{m_n} \right\} < \infty. \tag{5.7}$$

To this aim, we will use the real exponential inequality of Kolmogorov in the form put forward in [7], Lemma 2.2 (cf. also [29], page 262).

LEMMA 5.3. *Let $(Y_i)_{i \le N}$ be independent identically distributed real random variables such that $\mathbb{E}Y_i = 0$, $\mathbb{E}Y_i^2 \le \sigma^2$ and $|Y_i| \le c$ almost surely, $i \le N$. Then, for every $t > 0$,*

$$\mathbb{P}\left\{ \left| \sum_{i=1}^{N} Y_i \right| > t \right\} \le 2 \exp\left[-\frac{t^2}{2N\sigma^2} \left(2 - \exp\left(\frac{ct}{N\sigma^2} \right) \right) \right].$$

To apply this lemma, let $\delta = \delta(\varepsilon) > 0$ be specified in a moment and set, for each n, $c_n = \delta m_n / a_{m_n}$. Define further, for each n, $i \le m_n$ and f in U,

$$Y_i(f, n) = \max(-c_n, \min(f(X_i), c_n)) - \mathbb{E}(\max(-c_n, \min(f(X_i), c_n))).$$

Note that $|Y_i(f, n)| \le 2c_n$ and $\mathbb{E}(Y_i(f, n)^2) \le 1$. By Lemma 5.3 it follows that

$$\mathbb{P}\left\{ \sup_{f \in U_n} \left| \sum_{i=1}^{m_n} Y_i(f, n) \right| > (1 + \varepsilon) a_{m_n} \right\} \le 2 \operatorname{card} U_n \exp(-(1 + \varepsilon) LLm_n),$$

provided $\delta = \delta(\varepsilon) > 0$ is small enough in order that $2 - \exp(2(1 + \varepsilon)\delta) \ge (1 + \varepsilon)^{-1}$. By (5.6), it thus already follows that

$$\sum_n \mathbb{P}\left\{ \sup_{f \in U_n} \left| \sum_{i=1}^{m_n} Y_i(f, n) \right| > (1 + \varepsilon) a_{m_n} \right\} < \infty. \tag{5.8}$$

Consider now $Z_i(f,n) = f(X_i) - Y_i(f,n)$, $i \le m_n$, $f \in U$. Note that, by centering of $f(X_i)$,

$$\mathbb{E}|Z_i(f,n)| \le 2\mathbb{E}\big(\|X\| I_{\{\|X\| > c_n\}}\big).$$

The integrability condition $\mathbb{E}(\|X\|^2/LL\|X\|) < \infty$ is equivalent to saying that $\sum_n \gamma_n < \infty$, where

$$\gamma_n = \frac{m_n}{(LLm_n)^2}\mathbb{P}\{\|X\| > c_n\}.$$

There exists a sequence (β_n) such that $\beta_n \ge \gamma_n$, $\sum_n \beta_n < \infty$ and satisfying the regularity property $\beta_{n+1} \le \rho^{1/3}\beta_n$ for every n (recall that $\rho > 1$). It is then easily seen that, for all n,

$$\begin{aligned}
\mathbb{E}\big(\|X\| I_{\{\|X\| > c_n\}}\big) &\le \sum_{l \ge n} c_{l+1}\mathbb{P}\{\|X\| > c_l\} \\
&\le \sum_{l \ge n} \frac{c_{l+1}(LLm_l)^2}{m_l}\beta_l \\
&\le C_1(\rho,\delta)\beta_n \sum_{l \ge n} \frac{(LL\rho^l)^{3/2}}{\rho^{l/2}}\rho^{(l-n)/3} \\
&\le C_2(\rho,\delta)\beta_n \frac{(LLm_n)^{3/2}}{\sqrt{m_n}},
\end{aligned}$$

for some constants $C_1(\rho,\delta), C_2(\rho,\delta) > 0$. Consider the set of integers

$$L = \{n \colon 2C_2(\rho,\delta)\beta_n LLm_n \le \varepsilon\}.$$

The preceding estimate indicates that for all $n \in L$, $f \in U$, and $i \le m_n$,

$$\mathbb{E}|Z_i(f,n)| \le \frac{\varepsilon a_{m_n}}{m_n}.$$

We now use this property to show that if $n \in L$ is large enough

$$\mathbb{E}\left(\sup_{f \in U} \sum_{i=1}^{m_n} |Z_i(f,n)|\right) \le 2\varepsilon a_{m_n}. \tag{5.9}$$

Indeed, note that from Lemma 1.2, (5.5) implies that

$$\lim_{n \to \infty} \frac{1}{a_{m_n}}\mathbb{E}\left(\sup_{f \in U}\left|\sum_{i=1}^{m_n} \varepsilon_i |Z_i(f,n)|\right|\right) = 0;$$

hence

$$\lim_{n \to \infty} \frac{1}{a_{m_n}}\mathbb{E}\left(\sup_{f \in U}\left|\sum_{i=1}^{m_n} |Z_i(f,n)| - \mathbb{E}|Z_i(f,n)|\right|\right) = 0,$$

from which the announced property (5.9) follows.

The main interest in the introduction of the absolute values in (5.9) is that it allows a simple use of the isoperimetric inequality; it provides us indeed with the crucial monotonicity property (which was before at our disposal by symmetrization and Rademacher averages). More precisely, let $n \in L$ and set

$$A = \left\{\omega \in \Omega \colon \sup_{f \in U} \sum_{i=1}^{m_n} |Z_i(f, n)(\omega)| \le 4\varepsilon a_{m_n}\right\}.$$

Then $\mathbb{P}(A) \ge \frac{1}{2}$ by (5.9). Now, if for $\omega \in \Omega$, there exist $\omega^1, \dots, \omega^q$ in A such that

$$X_i(\omega) \in \{X_i(\omega^1), \dots, X_i(\omega^q)\}$$

except perhaps for at most k values of $i \le m_n$, then

$$\begin{aligned}\sup_{f \in U} \sum_{i=1}^{m_n} |Z_i(f, n)(\omega)| &\le \sum_{i=1}^{k} \|Z_i(\omega)\|^* + \sum_{l=1}^{q} \sup_{f \in U} \sum_{i=1}^{m_n} |Z_i(f, n)(\omega^l)| \\ &\le \sum_{i=1}^{k} \|Z_i(\omega)\|^* + 4q\varepsilon a_{m_n},\end{aligned}$$

where $(\|Z_i\|^*)$ denotes the nonincreasing rearrangement of $(\|X_i\| + \mathbb{E}\|X_i\|)_{i \le m_n}$. Hence the isoperimetric inequality $(*)$ ensures that for $k \ge q$,

$$\mathbb{P}\left\{\sup_{f \in U} \sum_{i=1}^{m_n} |Z_i(f, n)| > (4q + 1)\varepsilon a_{m_n}\right\} \le \left(\frac{K_0}{q}\right)^k + \mathbb{P}\left\{\sum_{i=1}^{k} \|Z_i\|^* > \varepsilon a_{m_n}\right\}.$$

If we now choose $q = 2K_0$ and $k = k_n$ as in the proof of Theorem 4.2 using the integrability condition $\mathbb{E}(\|X\|^2/LL\|X\|) < \infty$, we get that

$$\sum_{n \in L} \mathbb{P}\left\{\sup_{f \in U} \sum_{i=1}^{m_n} |Z_i(f, n)| > (8K_0 + 1)\varepsilon a_{m_n}\right\} < \infty. \tag{5.10}$$

Combining (5.8) and (5.10), we see that in order to establish (5.7) and conclude the proof of the theorem, we have to show that for some numerical $C > 0$,

$$\sum_{n \notin L} \mathbb{P}\left\{\sup_{f \in U_n} \left|\sum_{i=1}^{m_n} f(X_i)\right| > C\varepsilon a_{m_n}\right\} < \infty. \tag{5.11}$$

We follow very much the pattern of the case $n \in L$. Let now $c_n' = m_n/4\varepsilon a_{m_n}$ and define $Y_i'(f, n)$, $Z_i'(f, n)$ as $Y_i(f, n)$, $Z_i(f, n)$ before but with c_n' instead of c_n. We observe, now because $\sigma \le 1$, that

$$\mathbb{E}|Z_i'(f, n)| \le \frac{8\varepsilon a_{m_n}}{m_n}.$$

Exactly as what we described before for $Z_i(f, n)$, we can get from the isoperimetric inequality that

$$\sum_{n} \mathbb{P}\left\{\sup_{f \in U} \sum_{i=1}^{m_n} |Z_i'(f, n)| > C\varepsilon a_{m_n}\right\} < \infty. \tag{5.12}$$

Concerning $Y_i'(f,n)$, the exponential inequality of Lemma 5.3 shows that

$$\mathbb{P}\left\{\sup_{f\in U_n}\left|\sum_{i=1}^{m_n} Y_i'(f,n)\right| > \varepsilon a_{m_n}\right\} \le 2\,\mathrm{card}\,U_n \exp\left(-\varepsilon^2(2-\sqrt{e})LLm_n\right)$$

$$\le 2\exp\left(-\left(\frac{\varepsilon^2}{4}-\alpha_n\right)LLm_n\right),$$

where we have used (5.6) in the last inequality. Now, if $n \notin L$,

$$LLm_n > \frac{\varepsilon}{2C_2(\rho,\delta)\beta_n}$$

where $\sum_n \beta_n < \infty$. Since $\alpha_n \to 0$, we clearly get that

$$\sum_{n\notin L}\mathbb{P}\left\{\sup_{f\in U_n}\left|\sum_{i=1}^{m_n} Y_i'(f,n)\right| > \varepsilon a_{m_n}\right\} < \infty,$$

from which, together with (5.12), (5.11) follows. This completes the proof of Theorem 5.1. □

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