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# NEAREST-NEIGHBOR ANALYSIS OF A FAMILY OF FRACTAL DISTRIBUTIONS[1]

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In this paper we use a central limit theorem for entropy due to Ibragimov to obtain limit theorems for linear normalizations of the log minimum distance when observations are sampled from measures belonging to a family of fractal distributions. It is shown that in almost all cases the limit distribution is Gaussian with parameters determined in part by the Hausdorff dimension associated with the underlying measure. Exceptions to this rule include absolutely continuous measures which obey the classical extreme value limit laws.

## 1. Introduction and preliminaries.

1.1. *Introduction.* The study of dynamical systems, chaos and fractals has led to an interest in the relationship between the Hausdorff dimension and the distribution of nearest neighbors. In the analysis of chaotic systems, functions of nearest neighbors are often used to obtain numerical estimates of the fractal dimensions associated with a system [see, for example, Badii and Politi (1985)]. However, the statistical properties of such estimates have not been rigorously examined. In this paper we study the nearest-neighbor behavior of a family of distributions on the unit interval $[0,1)$ whose dimension properties are well understood. These distributions may be regarded as the occupation measures associated with the mapping $T(x) = rx \,(\mathrm{mod}\, 1)$ for various choices of distribution of initial conditions in $[0,1)$. Our primary result is that, under suitable mixing conditions, in "almost all" cases a certain linear transformation of the log minimum distance asymptotically follows a Gaussian distribution whose variance is connected to the entropy and Hausdorff dimension of the original measure. This result is then extended to the class of measures absolutely continuous with respect to some member of the original family. This provides theoretical support for the conjectured or observed log normality of probabilities in many dynamical systems [see Farmer, Ott and Yorke (1983) and Badii and Politi (1985)]. Furthermore, a study of the variance term shows that the precision of dimension estimates based on nearest neighbors can vary widely over a class of measures, the degree of precision depending on the dimension and related quantities.

Later in this paper we will examine some special cases which do not exhibit the Gaussian behavior and we will establish bounds on the asymptotic distribution of the log minimum distance for these cases.

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1.2. *Local Hausdorff dimension and the log minimum distance.* We can connect the concept of Hausdorff dimension (of sets) with probability measures $\mu$ on Euclidean $N$-space by defining a related measure $\hat{\mu}$ (the dimension distribution of $\mu$) on the Borel sets of $\mathbb{R}$ by

$$\hat{\mu}([0,\alpha]) = \sup_{\dim(D) \leq \alpha} \mu(D). \tag{1.2.1}$$

Here $D$ is a Borel set in Euclidean $N$-space and $\dim(D)$ denotes its Hausdorff dimension. The measure $\hat{\mu}$ describes the manner in which the $\mu$-mass is distributed with respect to the Hausdorff dimension. Cutler (1986) and Cutler and Dawson (1989) have shown that, in fact, $\hat{\mu}$ is the distribution of the random variable $\alpha(X)$ defined by

$$\alpha(X) = \liminf_{\varepsilon \to 0^+} (\log \varepsilon)^{-1} \log \mu(B(X,\varepsilon)), \tag{1.2.2}$$

where $X$ has distribution $\mu$ and $B(X, \varepsilon)$ is the closed ball of radius $\varepsilon$ centered at $X$. The value $\alpha(x)$ at the point $X = x$ is called the *local Hausdorff dimension of* $\mu$ *at* $x$. (This quantity is sometimes called pointwise dimension in the physics literature.) Cutler and Dawson (1989) have shown that a function of the nearest-neighbor distance provides a consistent estimator of $\alpha(x)$ under the assumption that "lim inf" in (1.2.2) can be replaced by "lim" at point $x$. [In this case we say $\alpha(x)$ is *simple* at $x$.] We summarize this result in the following theorem.

THEOREM 1.2.1. *Let $\mu$ be a probability measure on the Borel sets of Euclidean N-space and suppose $\alpha(x)$ is simple at $x$. Let $X_1, X_2, \dots$ be an i.i.d. sequence of observations from $\mu$ and let $\rho_n(x) = \min_{1 \leq i \leq n} \|X_i - x\|$ denote the nearest-neighbor (or minimum) distance to $x$ from the first $n$ sample points.*

*Then* $\lim_{n \to \infty} (\log 1/n)^{-1} \log \rho_n(x) = 1/\alpha(x)$ *w.p.*1.

Measures $\mu$ for which $\hat{\mu}$ is concentrated on a single atom $\alpha$ are called *exact-dimensional measures*. [This term originated with Rogers and Taylor (1959)]. This is equivalent to stating that $\alpha(X)$ is $\mu$-a.s. constant and equal to $\alpha$. In dimension estimation in dynamical systems it is generally assumed that the measure in question is exact-dimensional (an assumption which is supported by empirical evidence and which can be proven in the case of certain ergodic systems). In this case it is not unreasonable to select the basepoint $X$ at random according to $\mu$ since $\mu$-almost all points have the same local dimension. This and Theorem 1.2.1 leads us to consider the asymptotic behavior of linear transformations of the quantity $(\log 1/n)^{-1}(\log \rho_n(X)) - 1/\alpha$, where $X, X_1, X_2, \dots$ are i.i.d. observations from $\mu$. The main work of this paper is to study this asymptotic behavior for the family of measures described in the next section.

1.3. *The family of measures.* Let $r \geq 2$ be a fixed positive integer. In constructing a family of measures on $[0,1)$ we begin with a stationary ergodic sequence $\dots, Z_{-1}, Z_0, Z_1, \dots$ of random variables with state space $S = \{0, 1, \dots, r-1\}$ defined over a probability space $(\Omega, \mathscr{F}, P)$, where $\Omega = \prod_{-\infty}^{\infty} S$ and $\mathscr{F}$ is the $\sigma$-algebra generated by the cylinder sets. (The use of a two-sided sequence is not necessary but permits a convenient notation for entropy and related quantities in the following.) We allow the possibility $P([Z_j = i]) = 0$ for some $i \in S$ but make the restriction $P(\bigcap_{j=k}^{\infty}[Z_j = i_j]) = 0$ for all $k \geq 1$ and sequences $\{i_j\}_j$. (This restriction is used to avoid trivial cases.) We use $P(Z_1, \dots, Z_n)$ to denote the random variable on $\Omega$ defined by $P(Z_1, \dots, Z_n)(\omega_0) = P([Z_1 = Z_1(\omega_0), \dots, Z_n = Z_n(\omega_0)])$. The (random) conditional likelihood of the present given the past is denoted by $P(Z_0 | Z_{-1}, Z_{-2}, \dots)$ and defined by

$r-1$

[illegible]

$i=0$

[illegible]

PROOF. The local Hausdorff dimension at $x$ can be calculated by taking limits over $r$-adic intervals [Cutler (1986)], and by comparison with (1.3.3) we then obtain

$$\begin{aligned}\alpha(x) &= \lim_{n\to\infty} (\log r^{-n})^{-1} \log \mu(C_n(x)) \quad \mu\text{-a.s.} \\ &= h/\log r \quad \mu\text{-a.s.}\end{aligned}$$

If the limit exists over $r$-adic intervals $\mu$-a.s., then it also exists over $\varepsilon$-intervals $\mu$-a.s. and so $\alpha(x)$ is simple. □

It should be noted that some of the ideas in Theorem 1.3.1 originated with Billingsley (1960, 1961, 1978).

1.4. *Summary of main results.* Let $X, X_1, X_2, \ldots$ be an i.i.d. sequence from a distribution $\nu$ which is absolutely continuous with respect to $\mu$ (where $\mu$ is constructed as in Section 1.3). Let $\alpha = h/\log r$ and define the linear transformation $L_n(X)$ by

$$\begin{aligned}L_n(X) &= (\log n)^{1/2}(\log r)^{1/2}\{(\log \rho_n(X)/\log 1/n) - 1/\alpha\} \\ &= -(\log n)^{-1/2}(\log r)^{1/2}\{\log \rho_n(X) + (\log n)/\alpha\}.\end{aligned} \tag{1.4.1}$$

Then under suitable mixing conditions on the $Z_j$ sequence (see Theorems 2.1 and 2.3) we obtain

$$L_n(X) \to_{\mathscr{D}} N(0, \sigma^2/\alpha^3),$$

provided $\sigma^2 \neq 0$, where $\sigma^2 = \text{Var}(H_0) + 2\Sigma_{j=1}^{\infty} \text{Cov}(H_0, H_j)$. (This definition of $\sigma^2$ is maintained throughout the paper.)

In addition, for certain cases of $\sigma^2 = 0$ we obtain (pointwise) extreme value distributions which asymptotically bound the transformation $-\alpha(\log \rho_n(x) + (\log n)/\alpha)$. (See Theorem 3.1.)

**2. The Gaussian case ($\sigma^2 \neq 0$).** The key element in the proof of asymptotic normality of $L_n(X)$ [as defined in (1.4.1)] is a central limit theorem for entropy [due to Ibragimov (1962)] which corresponds to the a.s. result in (1.3.1).

Let the stationary $Z_j$ sequence satisfy the strong mixing condition

$$\sup_{A \in \mathscr{F}_{-\infty}^{0},\, B \in \mathscr{F}_n^{\infty}} |P(AB) - P(A)P(B)| \le \beta(n), \tag{2.1}$$

where $\mathscr{F}_{-\infty}^{0} = \sigma(\ldots, Z_{-1}, Z_0)$, $\mathscr{F}_n^{\infty} = \sigma(Z_n, Z_{n+1}, \ldots)$ and the mixing coefficient $\beta(n) \downarrow 0$.

Also define

$$\begin{aligned}\theta(n) = \sup_{0 \le i \le r-1} E\big(|P([Z_0 = i]|Z_{-1}, Z_{-2}, \ldots) \\ -P([Z_0 = i]|Z_{-1}, \ldots, Z_{-n})|\big).\end{aligned} \tag{2.2}$$

Then $\theta(n)$ is a measure of the degree to which conditioning on the entire past can be approximated by conditioning on the previous $n$ observations.

**Theorem** 2.1 (Ibragimov). *Let the $Z_j$ sequence additionally satisfy*

$$\sum_{n=1}^{\infty} \beta(n)^{1-\delta} < \infty \quad \textit{for some } \delta > 0, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \theta(n)^{1/2-\varepsilon} < \infty \quad \textit{for some } \varepsilon > 0. \tag{2.4}$$

*Then* $\sigma^2 < \infty$ *and* $n^{-1/2}\sigma^{-1}(\log P(Z_1, \dots, Z_n) + nh) \to_{\mathscr{D}} N(0,1)$ *provided* $\sigma^2 \neq 0$.

**Proof.** See Ibragimov (1962), Theorem 2.6. Here, however, we are using natural logarithms. □

In the following $\mu$ will always denote a member of the family constructed in Section 1.3 with corresponding Hausdorff dimension $\alpha = h/\log r$. The notation $I(x, \varepsilon)$ will refer to a closed interval of radius $\varepsilon$ centered at $x$. We define the error function $\phi$ by

$$\phi(x, \varepsilon) = (\log \varepsilon)^{-1} \log \mu(I(x, \varepsilon)) - \alpha \tag{2.5}$$

and the associated error function $\tilde{\phi}$ (which is computed over $r$-adic intervals) by

$$\tilde{\phi}(x, r^{-n}) = (\log r^{-n})^{-1} \log \mu(C_n(x)) - \alpha. \tag{2.6}$$

From Theorem 1.3.1 it follows that $\lim_{n\to\infty} \tilde{\phi}(x, r^{-n}) = 0$ $\mu$-a.s. and $\lim_{\varepsilon \to 0^+} \phi(x, \varepsilon) = 0$ $\mu$-a.s.

For the remainder of this section we will assume $\sigma^2 \neq 0$ and that conditions (2.3) and (2.4) are met. In the following sequence of lemmas we assume $X$ is a random observation from $\mu$ and establish the asymptotic normality of $(\log 1/\varepsilon)^{1/2}\phi(X, \varepsilon)$.

**Lemma** 2.1. $n^{1/2}(\log r)\tilde{\phi}(X, r^{-n}) \to_{\mathscr{D}} N(0, \sigma^2)$ *as* $n \to \infty$.

**Proof.** Noting that $n^{1/2}(\log r)\tilde{\phi}(X, r^{-n}) = -n^{-1/2}(\log \mu(C_n(X)) + nh)$ has the same distribution as $-n^{-1/2}(\log P(Z_1, \dots, Z_n) + nh)$, the result follows from Theorem 2.1. □

**Lemma** 2.2. $n^{1/2}(\log r)\phi(X, r^{-n}) \to_{\mathscr{D}} N(0, \sigma^2)$ *as* $n \to \infty$.

**Proof.** Since $C_n(X) \subseteq I(X, r^{-n})$ we have $\phi(X, r^{-n}) \le \tilde{\phi}(X, r^{-n})$ and so

$$n^{1/2}(\log r)\phi(X, r^{-n}) \le n^{1/2}(\log r)\tilde{\phi}(X, r^{-n}). \tag{2.7}$$

To obtain a reverse inequality, let $C_n^-(X)$ and $C_n^+(X)$ denote the $r$-adic intervals of length $r^{-n}$ located, respectively, to the immediate left and right of $C_n(X)$. Define the ratios $R_n^-(X) = \mu(C_n^-(X))/\mu(C_n(X))$ and $R_n^+(X) = \mu(C_n^+(X))/\mu(C_n(X))$. (Note these ratios are $\mu$-a.s. finite.) We then have

$$\begin{aligned} \mu(I(X, r^{-n})) &\le \mu(C_n^-(X)) + \mu(C_n(X)) + \mu(C_n^+(X)) \\ &= \mu(C_n(X))[R_n^-(X) + R_n^+(X) + 1]. \end{aligned} \tag{2.8}$$

Hence

$$\phi(X, r^{-n}) \geq (\log r^{-n})^{-1}\big(\log\big(\mu(C_n(X)) \times [R_n^-(X) + R_n^+(X) + 1]\big)\big) - \alpha = \tilde{\phi}(X, r^{-n}) + (\log r^{-n})^{-1}\log[R_n^-(X) + R_n^+(X) + 1], \tag{2.9}$$

giving the inequality

$$n^{1/2}(\log r)\phi(X, r^{-n}) \geq n^{1/2}(\log r)\tilde{\phi}(X, r^{-n}) - n^{-1/2}\log[R_n^-(X) + R_n^+(X) + 1]. \tag{2.10}$$

In view of Lemma 2.1, (2.7) and (2.10), the lemma will be proved if we show $\lim_{n\to\infty} E(|n^{-1/2}\log[R_n^-(X) + R_n^+(X) + 1]|) = 0$.

Now $R_n^-(X) + R_n^+(X) + 1 \geq 1$ always and so applying Jensen's inequality for concave functions we obtain

$$E\big(|\log[R_n^-(X) + R_n^+(X) + 1]|\big) = E\big(\log[R_n^-(X) + R_n^+(X) + 1]\big) \leq \log E\big(R_n^-(X) + R_n^+(X) + 1\big).$$

Therefore it is certainly sufficient to show $E(R_n^-(X) + R_n^+(X) + 1)$ is bounded over $n$. But clearly

$$E(R_n^+(X)) = \sum_{\substack{i_1,\dots,i_n \\ 0\leq i_j\leq r-1}} \mu(C_n^+(i_1,\dots,i_n)) = 1 - \mu(C_n(0,0,\dots,0)).$$

Similarly, $E(R_n^-(X)) = 1 - \mu(C_n(r-1, r-1,\dots, r-1))$. Therefore $1 \leq E(R_n^-(X) + R_n^+(X) + 1) \leq 3$ for all $n$. □

To extend Lemma 2.2 to a continuous interval width $\varepsilon$, we define the integer-valued function $n(\varepsilon)$ by the relation

$$r^{-n(\varepsilon)} \leq \varepsilon < r^{-(n(\varepsilon)-1)} \tag{2.11}$$

and approximate $\phi(X, \varepsilon)$ by $\phi(X, r^{-n(\varepsilon)})$.

**Lemma 2.3.** *$(\log 1/\varepsilon)^{1/2}|\phi(X, \varepsilon) - \phi(X, r^{-n(\varepsilon)})| \to_{\mathscr{D}} 0$ as $\varepsilon \to 0^+$.*

**Proof.**

$$\begin{aligned}(\log 1/\varepsilon)^{1/2}&|\phi(X, \varepsilon) - \phi(X, r^{-n(\varepsilon)})| \\ &\leq (\log 1/\varepsilon)^{1/2}(\log 1/\varepsilon)^{-1}\big|\log\mu(I(X,\varepsilon)) - \log\mu\big(I(X, r^{-n(\varepsilon)})\big)\big| \\ &\quad + (\log 1/\varepsilon)^{1/2}\big|\log\mu\big(I(X, r^{-n(\varepsilon)})\big)\big|\,\big|(\log\varepsilon)^{-1} - \big(\log r^{-n(\varepsilon)}\big)^{-1}\big| \\ &= (\log 1/\varepsilon)^{-1/2}\big|\log\big[\mu(I(X,\varepsilon))/\mu\big(I(X, r^{-n(\varepsilon)})\big)\big]\big| \\ &\quad + (\log 1/\varepsilon)^{-1/2}\big(\log r^{-n(\varepsilon)}\big)^{-1}\log\mu\big(I(X, r^{-n(\varepsilon)})\big)\big|\log\big[r^{-n(\varepsilon)}/\varepsilon\big]\big|.\end{aligned} \tag{2.12}$$

Now as $|\log[r^{-n(\varepsilon)}/\varepsilon]| \le \log r$ and $(\log r^{-n(\varepsilon)})^{-1} \log \mu(I(X, r^{-n(\varepsilon)})) \to \alpha$ $\mu$-a.s., it follows that the second term of (2.12) converges to 0 $\mu$-a.s. Now consider the first term of (2.12). We have

$$\begin{aligned} 1 &\le \mu(I(X,\varepsilon))/\mu(I(X, r^{-n(\varepsilon)})) \le \mu(I(X, r^{-(n(\varepsilon)-1)}))/\mu(I(X, r^{-n(\varepsilon)})) \\ &\le \left[\mu(C^-_{n(\varepsilon)-1}(X)) + \mu(C_{n(\varepsilon)-1}(X)) + \mu(C^+_{n(\varepsilon)-1}(X))\right]/\mu(C_{n(\varepsilon)}(X)) \end{aligned}$$

using the same notation as in (2.8). It follows that

$$\begin{aligned} &E\left(\mu(C^-_{n(\varepsilon)-1}(X))/\mu(C_{n(\varepsilon)}(X))\right) = r\left(1 - \mu\left(C_{n(\varepsilon)-1}(r-1,\dots,r-1)\right)\right), \\ &E\left(\mu(C_{n(\varepsilon)-1}(X))/\mu(C_{n(\varepsilon)}(X))\right) = r, \\ &E\left(\mu(C^+_{n(\varepsilon)-1}(X))/\mu(C_{n(\varepsilon)}(X))\right) = r\left(1 - \mu\left(C_{n(\varepsilon)-1}(0,\dots,0)\right)\right). \end{aligned}$$

Hence

$$1 \le E\left(\mu(I(X,\varepsilon))/\mu(I(X, r^{-n(\varepsilon)}))\right) \le 3r. \tag{2.13}$$

Now by the argument used in the proof of Lemma 2.2 we conclude

$$\lim_{\varepsilon \to 0^+} (\log 1/\varepsilon)^{-1/2} E\left(\left|\log\left[\mu(I(X,\varepsilon))/\mu(I(X, r^{-n(\varepsilon)}))\right]\right|\right) = 0$$

and so the first term of (2.12) converges to 0 in mean. □

THEOREM 2.2. $(\log 1/\varepsilon)^{1/2}(\log r)^{1/2}\phi(X,\varepsilon) \to_{\mathscr{D}} N(0,\sigma^2)$ *as* $\varepsilon \to 0^+$.

PROOF. Noting that

$$n(\varepsilon)^{1/2}(\log r)\phi(X, r^{-n(\varepsilon)}) = \left(\log r^{n(\varepsilon)}\right)^{1/2}(\log r)^{1/2}\phi(X, r^{-n(\varepsilon)}),$$

we obtain

$$\begin{aligned} &\left|(\log 1/\varepsilon)^{1/2}(\log r)^{1/2}\phi(X,\varepsilon) - n(\varepsilon)^{1/2}(\log r)\phi(X, r^{-n(\varepsilon)})\right| \\ &\quad = \left|(\log 1/\varepsilon)^{1/2}(\log r)^{1/2}\phi(X,\varepsilon) - \left(\log r^{n(\varepsilon)}\right)^{1/2}(\log r)^{1/2}\phi(X, r^{-n(\varepsilon)})\right| \\ &\quad \le (\log r)^{1/2}(\log 1/\varepsilon)^{1/2}\left|\phi(X,\varepsilon) - \phi(X, r^{-n(\varepsilon)})\right| \\ &\qquad + (\log r)^{1/2}\left|\phi(X, r^{-n(\varepsilon)})\right|\left|\log\left[r^{-n(\varepsilon)}/\varepsilon\right]\right|. \end{aligned} \tag{2.14}$$

In the second term of (2.14) we have used the inequality $|x^{1/2} - y^{1/2}| \le |x - y|$ for $x, y \ge 1$. From Lemma 2.3 we know that the first term of (2.14) converges to 0 in distribution. The second term of (2.14) tends to 0 $\mu$-a.s. because $|\log[r^{-n(\varepsilon)}/\varepsilon]| \le \log r$ and $\phi(X, r^{-n(\varepsilon)}) \to 0$ $\mu$-a.s. The required result now follows by applying Lemma 2.2. □

THEOREM 2.3. *Let* $\mu$ *defined as in Section* 1.3 *satisfy the mixing conditions* (2.3) *and* (2.4) *with* $\sigma^2 \ne 0$. *Let* $X, X_1, X_2, \dots$ *be an i.i.d. sequence from* $\mu$ *and let* $L_n(X)$ *be as defined in* (1.4.1). *Then*

$$L_n(X) \to_{\mathscr{D}} N(0, \sigma^2/\alpha^3) \quad \text{as } n \to \infty.$$

PROOF. Let $P_0 = \prod_0^\infty \mu$ denote the usual product measure corresponding to the sequence $X, X_1, X_2, \ldots$. For each $y \in \mathbb{R}$, let

$$\varepsilon_n(y) = \exp\left(-y(\log n)^{1/2}(\log r)^{-1/2} - (\log n)/\alpha\right).$$

Then for fixed $x$ we have

$$\begin{aligned} P_0([L_n(x) < y]) &= P_0([\rho_n(x) > \varepsilon_n(y)]) \\ &= \{1 - \mu(I(x, \varepsilon_n(y)))\}^n \\ &= \left\{1 - \varepsilon_n(y)^{\alpha + \phi(x, \varepsilon_n(y))}\right\}^n \quad [\text{using (2.5)}] \\ &= \left\{1 - \exp(\alpha \log \varepsilon_n(y) + (\log \varepsilon_n(y))\phi(x, \varepsilon_n(y)))\right\}^n. \end{aligned}$$

Setting $Z_{y,n}(x) = -(\log 1/\varepsilon_n(y))^{1/2}(\log r)^{1/2}\phi(x, \varepsilon_n(y))$, we obtain

$$P_0([L_n(x) < y]) = f_{y,n}(Z_{y,n}(x)),$$

where

$$f_{y,n}(z) = \left\{1 - \exp\left(\alpha \log \varepsilon_n(y) + z(\log r)^{-1/2}(\log 1/\varepsilon_n(y))^{1/2}\right)\right\}^n$$

when $z \le \alpha(\log r)^{1/2}(\log 1/\varepsilon_n(y))^{1/2}$ and vanishes when

$$z > \alpha(\log r)^{1/2}(\log 1/\varepsilon_n(y))^{1/2}.$$

Note $0 \le f_{y,n} \le 1$. Randomizing over the basepoint $x$, we obtain

$$P_0([L_n(X) < y]) = \int f_{y,n}(Z_{y,n}(x))\mu(dx) = E\left(f_{y,n}(Z_{y,n})\right).$$

Note $Z_{y,n} \to_{\mathscr{D}} N(0, \sigma^2)$ by Theorem 2.2. From the Skorohod representation theorem there exist $\tilde{Z}_y$ and $\{\tilde{Z}_{y,n}\}_n$ on some probability space such that $\tilde{Z}_{y,n} \sim_{\mathscr{D}} Z_{y,n}$, $\tilde{Z}_y \sim_{\mathscr{D}} N(0, \sigma^2)$ and $\tilde{Z}_{y,n} \to \tilde{Z}_y$ w.p.1. Hence

$$P_0([L_n(X) < y]) = E\left(f_{y,n}(\tilde{Z}_{y,n})\right). \tag{2.15}$$

Now for fixed $y$ and $z$ and sufficiently large $n$ we can, by expanding $\varepsilon_n(y)$, write

$$f_{y,n}(z) = \left\{1 - n^{-1}\exp\left(-\alpha(\log r)^{-1/2}(\log n)^{1/2} \times \left(y - (z/\alpha)\left[y(\log n)^{-1/2}(\log r)^{-1/2} + (1/\alpha)\right]^{1/2}\right)\right)\right\}^n,$$

which shows that, for all $z \ne \alpha^{3/2}y$, $\lim_{n\to\infty} f_{y,n}(z) = f_y(z)$, where $f_y = I_{(-\infty, \alpha^{3/2}y)}$. Furthermore, it is easy to see that for each $\varepsilon > 0$ the convergence of $f_{y,n}$ to $f_y$ is uniform over the set $(-\infty, \alpha^{3/2}y - \varepsilon) \cup (\alpha^{3/2}y + \varepsilon, \infty)$. Since the event $[\tilde{Z}_y = \alpha^{3/2}y]$ has probability 0, we conclude $f_{y,n}(\tilde{Z}_{y,n}) \to f_y(\tilde{Z}_y)$ w.p.1. Applying the bounded convergence theorem, we obtain

$$\lim_{n\to\infty} E\left(f_{y,n}(\tilde{Z}_{y,n})\right) = E\left(f_y(\tilde{Z}_y)\right) = \int_{-\infty}^{\alpha^{3/2}y} (2\pi\sigma^2)^{-1/2}\exp(-t^2/2\sigma^2)\,dt. \tag{2.16}$$

But from (2.15) this is equivalent to $L_n(X) \to_{\mathscr{D}} N(0, \sigma^2/\alpha^3)$. □

To extend this result to measures absolutely continuous with respect to $\mu$, we will let $P^*$ denote the restriction of $P$ to the space of one-sided sequences $\Omega^* = \prod_1^\infty S$. It then follows that $\nu \ll \mu$ if and only if $\tilde{P} \ll P^*$, where $\tilde{P}$ is the distribution on $\Omega^*$ induced by $\nu$. [That is, $\tilde{P}([Z_1 = i_1, \dots, Z_n = i_n]) = \nu(C_n(i_1, \dots, i_n))$.] It will be more convenient to consider two-sided sequences so we will let $\tilde{P}$ denote any absolutely continuous extension of the one-sided $\tilde{P}$ to events on $\Omega$, so that we have $\tilde{P} \ll P$ when $\nu \ll \mu$.

If $Y$ is a random variable we will let $\|Y\|_2$ denote the $\mathscr{L}_2$ norm of $Y$, specifying the underlying probability measure when necessary. We will need the following lemma in the proof of Lemma 2.5.

LEMMA 2.4. *Let $\{F_n\}_n$ be a sequence of events in a probability space $(\Omega, \mathscr{F}, P)$ and suppose there exists $\gamma \geq 0$ such that $\lim_{n\to\infty} P(F_n \cap C) = \gamma P(C)$ for all $C$ in a $\sigma$-field generating $\mathscr{F}$. If $\tilde{P} \ll P$, then $\lim_{n\to\infty} \tilde{P}(F_n) = \gamma$.*

PROOF. See Billingsley (1968), Theorem 16.2. □

LEMMA 2.5. *Let the $Z_j$ sequence under distribution $P$ have entropy $h$ and satisfy the conditions of Theorem 2.1. Let $\sigma^2$ be the asymptotic variance obtained under $P$ in Theorem 2.1. Suppose $\tilde{P} \ll P$. If the $Z_j$ sequence is now sampled according to $\tilde{P}$, then we still obtain*

$$n^{-1/2}(\log P(Z_1, \dots, Z_n) + nh) \to_{\mathscr{D}} N(0, \sigma^2).$$

PROOF. From Lemma 2.2 of Ibragimov (1962) and the proof of Theorem 2.6 of Ibragimov (1962) we obtain

$$E\left(\left|n^{-1/2}\left(\sum_{i=1}^{n} H_i - \log P(Z_1, \dots, Z_n)\right)\right|\right) \to 0 \quad \text{under } P, \tag{2.17}$$

$$n^{-1/2}\left(\sum_{i=1}^{n} H_i + nh\right) \to_{\mathscr{D}} N(0, \sigma^2) \quad \text{under } P, \tag{2.18}$$

$$\sum_{i=1}^{\infty} \left\| H_0 - E\left(H_0 | \mathscr{F}_{-i}^{i}\right) \right\|_2 < \infty \quad \text{under } P, \tag{2.19}$$

where $\mathscr{F}_{-i}^{i} = \sigma(Z_{-i}, \dots, Z_i)$. We will first show

$$n^{-1/2}\left(\sum_{i=1}^{n} H_i + nh\right) \to_{\mathscr{D}} N(0, \sigma^2) \quad \text{under } \tilde{P}. \tag{2.20}$$

Let $S_n = n^{-1/2}(\sum_{i=1}^n H_i + nh)$ and choose a sequence of integers $p_n$ such that $p_n \uparrow \infty$ and $n^{-1/2} p_n \to 0$. Let $S_n' = n^{-1/2}(\sum_{i=p_n}^n H_i + (n - p_n + 1)h)$. As in the proof of Theorem 16.3 of Billingsley (1968) we obtain

$$|S_n - S_n'| \to_P 0, \tag{2.21}$$

$$S_n' \to_{\mathscr{D}} N(0, \sigma^2) \quad \text{under } P. \tag{2.22}$$

We wish to show $S_n' \to_{\mathscr{D}} N(0, \sigma^2)$ under $\tilde{P}$. This will use Lemma 2.4. Let $C$ be a finite-dimensional cylinder in $\mathscr{F}$ and for each Borel set $B$, let $N(B) = \int_B (2\pi\sigma^2)^{-1/2} \exp(-t^2/2\sigma^2)\,dt$. Choosing $B = (-\infty, b]$ for $b \in \mathbb{R}$ and letting $\mathscr{F}_j^k = \sigma(Z_j, Z_{j+1}, \dots, Z_k)$ for $j \le k$, we obtain

$$
\begin{aligned}
&|P([S_n' \in B] \cap C) - N(B)P(C)| \\
&\quad \le \left|P([S_n' \in B] \cap C) - P\left(\left[E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \in B\right] \cap C\right)\right| \\
&\qquad + \left|P\left(\left[E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \in B\right] \cap C\right) - P\left(\left[E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \in B\right]\right)P(C)\right| \\
&\qquad + \left|P\left(\left[E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \in B\right]\right)P(C) - N(B)P(C)\right|.
\end{aligned} \tag{2.23}
$$

Now

$$
\begin{aligned}
&\left\| S_n' - E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \right\|_2 \quad \text{under } P \\
&\le n^{-1/2} \sum_{i=p_n}^{n} \left\| H_i - E\left(H_i | \mathscr{F}_{p_n}^{2n-p_n}\right) \right\|_2 \\
&= n^{-1/2} \sum_{i=0}^{n-p_n} \left\| H_{p_n+i} - E\left(H_{p_n+i} | \mathscr{F}_{p_n}^{2n-p_n}\right) \right\|_2 \\
&\le n^{-1/2} \sum_{i=0}^{n-p_n} \left\| H_{p_n+i} - E\left(H_{p_n+i} | \mathscr{F}_{(p_n+i)-i}^{(p_n+i)+i}\right) \right\|_2 \\
&\qquad\qquad [\text{since } p_n + 2i \le 2n - p_n] \\
&= n^{-1/2} \sum_{i=0}^{n-p_n} \left\| H_0 - E\left(H_0 | \mathscr{F}_{-i}^{i}\right) \right\|_2 \\
&\qquad\qquad [\text{using stationarity of } H_i \text{ under } P] \\
&\le n^{-1/2} \sum_{i=0}^{\infty} \left\| H_0 - E\left(H_0 | \mathscr{F}_{-i}^{i}\right) \right\|_2.
\end{aligned}
$$

Thus, applying (2.19), we can conclude

$$
\lim_{n\to\infty} \left\| S_n' - E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \right\|_2 = 0 \quad \text{under } P. \tag{2.24}
$$

From (2.22) and (2.24) we then obtain

$$
E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \to_{\mathscr{D}} N(0, \sigma^2) \quad \text{under } P, \tag{2.25}
$$

which shows that the third term on the r.h.s. of (2.23) tends to 0 as $n \to \infty$. To show that the first term on the r.h.s. of (2.23) also tends to 0, we note that

$$
\begin{aligned}
&\left|P([S_n' \in B] \cap C) - P\left(\left[E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \in B\right] \cap C\right)\right| \\
&\quad \le P\left([S_n' > b] \cap \left[E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) \le b\right]\right) \\
&\qquad + P\left([S_n' \le b] \cap \left[E\left(S_n' | \mathscr{F}_{p_n}^{2n-p_n}\right) > b\right]\right).
\end{aligned} \tag{2.26}
$$

But for any positive integer $k$ we have

$$P\Big([S_n' > b] \cap \big[E\big(S_n'|\mathscr{F}_{p_n}^{2n-p_n}\big) \le b\big]\Big)$$
$$\le P\Big(\big[\big|S_n' - E\big(S_n'|\mathscr{F}_{p_n}^{2n-p_n}\big)\big| > 1/k\big]\Big) + P([S_n' \in (b, b + 1/k]]).$$

Therefore from (2.22) and (2.24) we conclude

$$\limsup_{n\to\infty} P\Big([S_n' > b] \cap \big[E\big(S_n'|\mathscr{F}_{p_n}^{2n-p_n}\big) \le b\big]\Big) \le N((b, b + 1/k]).$$

Letting $k \to \infty$ we get

$$\lim_{n\to\infty} P\Big([S_n' > b] \cap \big[E\big(S_n'|\mathscr{F}_{p_n}^{2n-p_n}\big) \le b\big]\Big) = 0.$$

This shows that the first term on the r.h.s. of (2.26) tends to 0 as $n \to \infty$. The second term on the r.h.s. of (2.26) also tends to 0 by the same argument. Therefore the first term on the r.h.s. of (2.23) tends to 0 as $n \to \infty$.

Consider the second term on the r.h.s. of (2.23). Since $C$ is a cylinder set there exists a finite positive index $k$ such that $C \in \mathscr{F}_{-\infty}^{k}$. Then from the mixing condition (2.1) under $P$ we have

$$\Big|P\Big(\big[E\big(S_n'|\mathscr{F}_{p_n}^{2n-p_n}\big) \in B\big] \cap C\Big) - P\Big(\big[E\big(S_n'|\mathscr{F}_{p_n}^{2n-p_n}\big) \in B\big]\Big)P(C)\Big| \le \beta(p_n - k),$$

which tends to 0 as $n \to \infty$. Thus we have established

$$\lim_{n\to\infty} P([S_n' \in B] \cap C) = N(B)P(C) \tag{2.27}$$

for all cylinder sets $C$ and all Borel sets of the form $B = (-\infty, b]$. By applying Lemma 2.4 we can conclude that $S_n' \to_{\mathscr{D}} N(0, \sigma^2)$ under $\tilde{P}$. Now (2.21) plus the fact $\tilde{P} \ll P$ gives $|S_n' - S_n| \to_{\tilde{P}} 0$ and so we obtain $S_n \to_{\mathscr{D}} N(0, \sigma^2)$ under $\tilde{P}$.

Now (2.17) gives $|S_n - n^{-1/2}(\log P(Z_1, \dots, Z_n) + nh)| \to_P 0$ and hence also $|S_n - n^{-1/2}(\log P(Z_1, \dots, Z_n) + nh)| \to_{\tilde{P}} 0$. Thus we obtain

$$n^{-1/2}(\log P(Z_1, \dots, Z_n) + nh) \to_{\mathscr{D}} N(0, \sigma^2)$$

under $\tilde{P}$. □

We now prove the main result.

THEOREM 2.4. *Let $\mu$ defined as in Section* 1.3 *satisfy the mixing conditions* (2.3) *and* (2.4) *with $\sigma^2 \neq 0$. If $\nu \ll \mu$ and $X, X_1, X_2, \dots$ is an i.i.d. sequence from $\nu$, then*

$$L_n(X) \to_{\mathscr{D}} N\big(0, \sigma^2/\alpha^3\big) \quad as\ n \to \infty.$$

PROOF. If the basepoint $X$ is randomly selected according to $\nu$, then the distribution of $n^{-1/2}(\log \mu(C_n(X)) + nh)$ is the same as that of $n^{-1/2}(\log P(Z_1, \dots, Z_n) + nh)$ when the $Z_j$'s are sampled according to $\tilde{P}$. Hence from Lemma 2.5 we obtain

$$n^{1/2}(\log r)\tilde{\phi}(X, r^{-n}) = -n^{-1/2}\big(\log \mu\big(C_n(X)\big) + nh\big) \to_{\mathscr{D}} N(0, \sigma^2), \tag{2.28}$$

when $X$ is distributed according to $\nu$. Letting $\tilde{\phi}_\nu$ and $\phi_\nu$ denote the error functions of $\nu$ over $r$-adic intervals and $\varepsilon$-intervals, respectively, i.e.,

$$(2.29)\qquad \begin{aligned} \tilde{\phi}_\nu(x, r^{-n}) &= (\log r^{-n})^{-1} \log \nu(C_n(x)) - \alpha, \\ \phi_\nu(x, \varepsilon) &= (\log \varepsilon)^{-1} \log \nu(I(x, \varepsilon)) - \alpha, \end{aligned}$$

we see that

$$n^{1/2}(\log r)|\tilde{\phi}(X, r^{-n}) - \tilde{\phi}_\nu(X, r^{-n})| = n^{-1/2}|\log[\nu(C_n(X))/\mu(C_n(X))]|.$$

As $\nu \ll \mu$ we have $\lim_{n\to\infty} \nu(C_n(X))/\mu(C_n(X)) = f(X)$ $\nu$-a.s., where $f$ is the Radon–Nikodym derivative of $\nu$ with respect to $\mu$. Since $f$ is $\nu$-a.s. positive and finite we conclude

$$\lim_{n\to\infty} n^{1/2}(\log r)|\tilde{\phi}(X, r^{-n}) - \tilde{\phi}_\nu(X, r^{-n})| = 0 \quad \nu\text{-a.s.}$$

Now, applying (2.28), we obtain

$$(2.30)\qquad n^{1/2}(\log r)\tilde{\phi}_\nu(X, r^{-n}) \to_{\mathscr{D}} N(0, \sigma^2) \quad \text{as } n \to \infty,$$

when $X$ is distributed according to $\nu$.

Noting that Lemmas 2.2 and 2.3 as well as Theorem 2.2 go through unchanged for $\phi_\nu$ (with $\mu$ replaced everywhere by $\nu$), we conclude that $L_n(X) \to_{\mathscr{D}} N(0, \sigma^2/\alpha^3)$ as claimed, since (as seen in the proof of Theorem 2.3) this is a direct consequence of Theorem 2.2. □

Thus we see that in the Gaussian case introduction of a density does not change the limiting distribution of $L_n(X)$ (although it may affect the rate of convergence to the limit). Essentially, the effects of a density are insignificant compared to the basic variability already present.

It should also be noted that the quantity $\sigma^2/\alpha^3$ is not bounded over the class of measures $\mu$ (even for fixed $r$). This can be seen by considering the case of independent $Z_j$'s and taking a limit as $P([Z_0 = i])$ tends to 1 for some particular $i$.

**3. Degenerate cases ($\sigma^2 = 0$).** Here we examine the asymptotic behavior of $(\log 1/n)^{-1} \log \rho_n(x)$ in certain degenerate cases. Since $\sigma^2 = \mathrm{Var}(H_0) + 2\sum_{j=1}^\infty \mathrm{Cov}(H_0, H_j)$ there are two basic types of solution to $\sigma^2 = 0$:

1. $H_0$ is $P$-a.s. constant.
2. $H_0$ is not constant but $\mathrm{Var}(H_0) = -2\sum_{j=1}^\infty \mathrm{Cov}(H_0, H_j)$.

The number of solutions is related to the dependence structure of the $Z_j$ sequence. In the case that the $Z_j$'s are independent we have $H_j = \log P(Z_j)$ and the only solutions are therefore of the type 1 variety, occurring when all states $i$ of positive probability are equally likely. In Theorem 3.1 we classify the possible behaviors in the independent case.

By $\mathrm{EV}(a, b)$ we mean an extreme value distribution with location parameter $a$ and scale parameter $b$, having distribution function $F(x) = \exp[-e^{-(x-a)/b}]$ for $-\infty < x < \infty$.

THEOREM 3.1. *Let* $\ldots Z_{-1}, Z_0, Z_1, \ldots$ *be a sequence of i.i.d. random variables with state space* $S = \{0, 1, \ldots, r-1\}$. *For each* $i \in S$ *let* $p_i = P([Z_0 = i])$. *Let* $\nu \ll \mu$, *where* $\mu$ *is constructed as in Section* 1.3 *and suppose* $X_1, X_2, \ldots$ *is an i.i.d. sequence from* $\nu$.

(a) *If* $p_i = 1/r$ *for all* $i \in S$, *then* $\mu$ *coincides with the Lebesgue measure on* $[0,1)$, $\alpha = 1$ *and*

$$-(\log \rho_n(x) + \log n) \to_{\mathscr{D}} \mathrm{EV}(\log 2f(x), 1)$$

*for* $\nu$*-almost all* $x$ *in* $[0,1)$, *where* $f$ *is the density function of* $\nu$.

(b) *Suppose* $r \geq 3$ *and there are exactly* $k$ *nonzero* $p_i$'s $(2 \leq k \leq r-1)$. *If* $p_i = 1/k$ *for each nonzero* $p_i$ *we obtain* $\alpha = \log k/\log r$ *and finite constants* $c_1 < c_2$ *such that for* $\nu$*-almost all* $x$,

$$\liminf_{n\to\infty} P_0([G_n(x) < y]) = \exp(-e^{-(y - c_2 - \log f(x))})$$

*and*

$$\limsup_{n\to\infty} P_0([G_n(x) < y]) = \exp(-e^{-(y - c_1 - \log f(x))}),$$

*where* $G_n(x) = -\alpha(\log \rho_n(x) + (\log n)/\alpha)$ *and* $f$ *is the Radon–Nikodym derivative of* $\nu$ *with respect to* $\mu$. *Thus, asymptotically,* $G_n(x)$ *oscillates between the two extreme value distributions* $\mathrm{EV}(c_1 + \log f(x), 1)$ *and* $\mathrm{EV}(c_2 + \log f(x), 1)$.

PROOF. Case (a) follows from more general results [Theorem 4.1 and Corollary 4.2 of Cutler and Dawson (1989)] concerning distributions in $N$-space which are absolutely continuous with respect to some local Hausdorff measure. In our restricted case (a) this result is completely the consequence of a specific behavior of the error function $\phi_\nu$. Since $\mu$ agrees with the Lebesgue measure on $[0,1)$ we get $\alpha = 1$ and $(\log \varepsilon)\phi_\nu(x, \varepsilon) = \log \nu(I(x, \varepsilon))/\varepsilon$. Thus the existence of a density for $\nu$ is equivalent to the $\nu$-a.s. existence of the finite pointwise limit $\lim_{\varepsilon \to 0^+}(\log \varepsilon)\phi_\nu(X, \varepsilon)$. In turn this can be shown to be equivalent to producing an extreme value limit distribution [see the technique in Case (b)]. Alternatively, the result in Case (a) can be obtained by applying classical extreme value theory [see, for example, Leadbetter, Lindgren and Rootzén (1983)] since the distribution of $|X_i - x|$ will be in the domain of attraction of a type 3 min-stable (in this case exponential) distribution when $\nu$ has a density.

For case (b) it follows that the local dimension $\alpha = \log k/\log r$ because $\mu(C_n(X)) = k^{-n}$ $\mu$-a.s. We will prove case (b) via the following lemma.

LEMMA 3.1. *If* $\mu$ *satisfies the conditions in case* (b), *then, for all* $x$ *in the support of* $\mu$, $-\log k \leq (\log \varepsilon)\phi(x, \varepsilon) \leq \log 3k$ *for all* $\varepsilon > 0$. *If* $\nu \ll \mu$ *with Radon–Nikodym derivative* $f$, *then there exist finite constants* $c_1 < c_2$ *such that*

$\liminf_{\varepsilon\to 0^+}(\log\varepsilon)\phi_\nu(x,\varepsilon) = c_1 + \log f(x)$ *$\nu$-a.s. and* $\limsup_{\varepsilon\to 0^+}(\log\varepsilon)\phi_\nu(x,\varepsilon) = c_2 + \log f(x)$ *$\nu$-a.s.*

PROOF. We have

$$(3.1)\qquad (\log\varepsilon)\phi(x,\varepsilon) = \log\mu(I(x,\varepsilon)) - (\log k/\log r)\log\varepsilon.$$

From (3.1), (2.8), (2.11) and the conditions in (b) we have

$$\begin{aligned}(\log\varepsilon)\phi(x,\varepsilon) &\le \log\mu\left(I(x, r^{-(n(\varepsilon)-1)})\right) - (\log k/\log r)\log r^{-n(\varepsilon)}\\ &\le \log 3k^{-(n(\varepsilon)-1)} + n(\varepsilon)\log k\\ &= \log 3k.\end{aligned}$$

Similarly, for $x$ in the support of $\mu$,

$$\begin{aligned}(\log\varepsilon)\phi(x,\varepsilon) &\ge \log\mu\left(I(x, r^{-n(\varepsilon)})\right) - (\log k/\log r)\log r^{-(n(\varepsilon)-1)}\\ &\ge \log k^{-n(\varepsilon)} + (n(\varepsilon)-1)\log k\\ &= -\log k,\end{aligned}$$

which proves the first part of the lemma. To prove the second part note that for $\nu$-almost all $x$ we can write

$$(\log\varepsilon)\phi_\nu(x,\varepsilon) = \log(\nu(I(x,\varepsilon))/\mu(I(x,\varepsilon))) + (\log\varepsilon)\phi(x,\varepsilon).$$

Since $\nu \ll \mu$ we have $\lim_{\varepsilon\to 0^+}\log(\nu(I(x,\varepsilon))/\mu(I(x,\varepsilon))) = \log f(x)$ $\nu$-a.s. To complete the proof, it is therefore sufficient to show the existence of $c_1 < c_2$ such that

$$(3.2)\qquad \begin{aligned}T_1(x) &= \liminf_{\varepsilon\to 0^+}(\log\varepsilon)\phi(x,\varepsilon) = c_1 \quad \mu\text{-a.s.},\\ T_2(x) &= \limsup_{\varepsilon\to 0^+}(\log\varepsilon)\phi(x,\varepsilon) = c_2 \quad \mu\text{-a.s.}\end{aligned}$$

Identifying each point $x$ with its base $r$ expansion $x_1, x_2, \dots,$ we can express $T_1(x) = T_1(x_1, x_2, \dots)$ and $T_2(x) = T_2(x_1, x_2, \dots)$. Interpreting the digits $\{x_j\}$ as a particular realization of the $\{Z_j\}$ sequence, it is then straightforward to show that each of $T_1$ and $T_2$ is $P$-a.s. equal to some function measurable with respect to the tail $\sigma$-algebra $\mathscr{T} = \bigcap_{n=1}^{\infty}\sigma(Z_n, Z_{n+1}, \dots)$. For if $n$ is any positive integer and $x$ is not an $r$-adic rational, then for sufficiently small $\varepsilon > 0$ we get $I(x,\varepsilon) \subseteq C_n(x)$ and thus

$$I(x,\varepsilon) = \left\{ y \mid y_1 = x_1, \dots, y_n = x_n \text{ and } \left|\sum_{i>n} y_i r^{-i} - \sum_{i>n} x_i r^{-i}\right| \le \varepsilon \right\}.$$

Hence

$$\mu(I(x,\varepsilon)) = P([Z_1 = x_1,\dots,Z_n = x_n])P\left(\left[\left|\sum_{i>n} Z_i r^{-i} - \sum_{i>n} x_i r^{-i}\right| \le \varepsilon\right]\right)$$

$$= k^{-n} P\left(\left[\left|\sum_{i>n} Z_i r^{-i} - \sum_{i>n} x_i r^{-i}\right| \le \varepsilon\right]\right) \quad \mu\text{-a.s.}$$

which does not depend on $x_1,\dots,x_n$. In view of (3.1) this shows that $T_1$ and $T_2$ are $P$-a.s. equal to tail functions and from the Kolmogorov 0–1 law we conclude the existence of $c_1$ and $c_2$ satisfying (3.2). We know that $c_1$ and $c_2$ are finite because of the bounds established in the first part of the lemma and we can say $c_1 \neq c_2$ because if $c_1 = c_2$, then $\mu$ is absolutely continuous with respect to Lebesgue measure which puts us in case (a). □

PROOF OF THEOREM 3.1(b). For $x \in [0,1)$ we have

$$\begin{aligned} P_0([G_n(x) < y]) &= P_0([\rho_n(x) > \exp(-(y + \log n)/\alpha)]) \\ &= \{1 - \nu(I(x, a_n(y)))\}^n \\ &\qquad [\text{where } a_n(y) = \exp(-(y+\log n)/\alpha)] \\ &= \{1 - a_n(y)^{(\alpha + \phi_\nu(x, a_n(y)))}\}^n \quad [\text{using (2.5)}] \\ &= \{1 + (e^{-y}/n)e^{(\log a_n(y))\phi_\nu(x, a_n(y))}\}^n. \end{aligned} \tag{3.3}$$

Now $a_n(y) \downarrow 0$ slowly, i.e., $\lim_{n\to\infty} a_{n+1}(y)/a_n(y) = 1$ and so from Lemma 3.1 we get

$$\liminf_{n\to\infty} (\log a_n(y))\phi_\nu(x, a_n(y)) = c_1 + \log f(x) \ \nu\text{-a.s.},$$

$$\limsup_{n\to\infty} (\log a_n(y))\phi_\nu(x, a_n(y)) = c_2 + \log f(x) \ \nu\text{-a.s.}$$

Applying this to (3.3) completes the proof. □

When dependence is introduced among the $Z_j$'s we will also obtain type 2 solutions. Although these cases have not been examined we would not expect pointwise limit laws (as in type 1) but perhaps asymptotic stable laws for certain renormalizations of $L_n(X)$.

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