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HEAT SEMIGROUP ON A COMPLETE RIEMANNIAN MANIFOLD

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Let M be a complete Riemannian manifold and $p(t, x, y)$ the minimal heat kernel on M. Let P_t be the associated semigroup. We say that M is stochastically complete if $\int_M p(t, x, y)\, dy = 1$ for all $t > 0$, $x \in M$; we say that M has the C_0-diffusion property (or the Feller property) if $P_t f$ vanishes at infinity for all $t > 0$ whenever f is so. Let $x_0 \in M$ and let $\kappa(r)^2 \geq -\inf\{\mathrm{Ric}(x): \rho(x, x_0) \leq r\}$ (ρ is the Riemannian distance). We prove that M is stochastically complete and has the C_0-diffusion property if $\int_c^\infty \kappa(r)^{-1}\, dr = \infty$ by studying the radial part of the Riemannian Brownian motion on M.

1. Introduction. Let (M, g) be a noncompact, connected Riemannian manifold and Δ the Laplace–Beltrami operator. The Riemannian Brownian motion X on M is the minimal diffusion process on M associated with the operator $\Delta/2$. The transition density $p(t, x, y)$ of the Brownian motion is the minimal fundamental solution of the heat operator $\partial/\partial t - \Delta/2$. Let $M \cup \{\partial\}$ be the one-point compactification of M. As usual, we regard X as a continuous Markov process on $M \cup \{\partial\}$. Let e be the explosion time of X, that is,

$$e = \inf\{t > 0: X_t = \partial\}.$$

When $P_x[e = \infty] = 1$ for one $x \in M$ (hence for all $x \in M$ since $P_x[e = \infty]$ is harmonic on M), we say M is stochastically complete. Intuitively, stochastic completeness means that the Brownian motion will not drift to infinity in finite amount of time. Since clearly

$$P_x[e > t] = \int_M p(t, x, y)\, dy,$$

stochastic completeness also means that the heat kernel on M is conservative:

$$\int_M p(t, x, y)\, dy = 1 \quad \text{for all } t > 0,\ x \in M.$$

Stochastic completeness is equivalent to the uniqueness of solution of heat equation with L^∞ initial data.

Let $C_0(M)$ be the space of continuous functions on M vanishing at infinity. Consider the diffusion semigroup

$$P_t f(X) = \int_M f(y) p(t, x, y)\, dy$$

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for $f \in C_0(M)$. We say that P_t is a C_0-diffusion or M has the C_0-diffusion property (or the Feller property) if $P_t C_0(M) \subset C_0(M)$, namely if the function space $C_0(M)$ is invariant under P_t for any $t > 0$. Intuitively, P_t is a C_0-diffusion if the Brownian motion starting from very far has small probability of visiting a fixed compact set before fixed time (see the precise statement in Lemma 3.1). Thus again C_0-diffusion property means the Brownian motion will not drift away from where it starts too fast. We would like to seek a geometric condition on M for stochastic completeness and the C_0-diffusion property.

We will assume that M is geometrically complete, namely, complete under its Riemannian distance ρ. Let $x_0 \in M$ be a fixed reference point on M, and $\rho(x) = \rho(x, x_0)$. Set $\rho(x, \partial) = \infty$ for all $x \in M$. Then we have $e = \inf\{t > 0: \rho(X_t) = \infty\}$. Geometrically, the speed at which the Brownian motion wanders away from its starting point is controlled by the lower bound of the Ricci curvature. A quantitative version of this statement can be found in Lemma 3.2 below. Let

$$\kappa(r)^2 \geq -\inf\{\mathrm{Ric}(x): \rho(x) \leq r\}.$$

Then it is reasonable to expect that conditions for stochastic completeness and the C_0-diffusion property can be expressed as growth conditions on the function κ.

Stochastic completeness and the C_0-diffusion property has been discussed by various authors. We mention Azencott [1] and Yau [10] (also (Doziuk [2]), who proved, among other things, that a geometrically complete manifold with Ricci curvature bounded from below by a constant [namely, $\kappa(r) \leq c < \infty$] is stochastically complete and has the C_0-diffusion property. Azencott [1] pointed out that if a Cartan–Hadamard manifold has sectional curvature bounded from above by $-\rho(x)^{2+\varepsilon}$, then it is not stochastically complete. He also noted that every Cartan–Hadarmard manifold has the C_0-diffusion property. The work of Hsu and March [5] implies that M is stochastically complete if $\kappa(r) \leq L(1 + r)$ for some $L > 0$. Karp and Li [7] in an unpublished article showed that M is stochastically complete if the volume of the geodesic ball $B_R(x_0)$ of radius R centered at x_0 satisfies the growth condition $\mathrm{vol}(B_R(x_0)) \leq \exp[cR^2]$ for some constant c. This was improved by Grigor'yan [4] to $\mathrm{vol}(B_R(x_0)) \leq \exp f(R)$ for any increasing f such that $\int_c^\infty rf(r)^{-1}\,dr = \infty$. It was also shown in Karp and Li [7] that M has the C_0-diffusion property if there exists a constant L such that $\kappa(r) \leq L(1 + r)$.

Our result in this article can be simply stated as follows.

Theorem. *If M is complete and if $\int_c^\infty \kappa(r)^{-1}\,dr = \infty$, then M is stochastically complete and has the C_0-diffusion property.*

The stochastic completeness part of our result is implied by the result of Grigor'yan [4]. Our result of the C_0-diffusion property improves the condition of Karp and Li [7]. We show our results by studying the radial part of the Riemannian Brownian motion of M.

2. Stochastic completeness. Consider the radial process $\rho(X_t)$. According to Kendall [8], there exists a standard Brownian motion β_t and a *nondecreasing* process L_t with initial value zero which increases only when X_t belongs to the cut-locus $C(x_0)$ of x_0, such that for $t < e$,

$$\rho(X_t) = \beta_t + \frac{1}{2}\int_0^t \Delta\rho(X_s)\,ds - L_t. \tag{2.1}$$

Now $\Delta\rho(x)$ is smooth on $M \setminus C(x_0)$. By the Hessian comparison theorem (Greene and Wu [3], pages 19–28), $\Delta\rho$ is locally bounded away from point x_0. We also notice that the Riemannian volume measure of $C(x_0)$ is zero. As a consequence the Brownian motion spends zero amount of time on $C(x_0)$ and the term $\int_0^t \Delta\rho(X_s)\,ds$ in (2.1) is well defined. (2.1) is essentially the result of applying Itô's formula to function $\rho(x)$. See the appendix to this section for a proof.

Without loss of generality, we assume that function κ satisfies the following three extra conditions:

(i) $\kappa(c) > 0$.
(ii) κ is nondecreasing.
(iii) $\lim_{s\to\infty} \kappa(s) = \infty$.

Let $G\colon [0, \infty] \to [0, \infty)$ be the unique solution of the equation

$$G''(r) = \kappa(r)^2 G(r), \qquad G(0) = 0, \qquad G'(0) = 1.$$

Let (R^n, g^*) be the Riemannian manifold with the metric $g^* = dr^2 + G(r)^2\,d\theta^2$ [(ρ, θ) is the usual polar coordinates on R^n]. The radial Ricci curvature of (R^n, g^*) is

$$\operatorname{Ric}(\dot\rho, \dot\rho) = -(n-1)\frac{G''(\rho)}{G(\rho)} = -(n-1)\kappa(\rho)^2.$$

Now we use the Laplacian comparison theorem (Greene and Wu [3], page 26) to manifolds (M, g) and (R^n, g^*), and conclude that on $M \setminus C(x_0)$,

$$\Delta\rho \le (n-1)\frac{G'(\rho)}{G(\rho)}. \tag{2.2}$$

Let ρ_t^* be the process on $[0, \infty)$ determined by the stochastic differential equation

$$\rho_t^* = \beta_t + \frac{n-1}{2}\int_0^t \frac{G'(\rho_s^*)}{G(\rho_s^*)}\,ds, \qquad \rho_0^* = 0. \tag{2.3}$$

Let e^* be the explosion time of ρ_t^*. By a comparison theorem for solutions of stochastic differential equations (Ikeda and Watanabe [6], pages 352–356) and the positivity of L_t, (2.1), (2.2) and (2.3) imply $\rho(X_t) \le \rho_t^*$ for all $t > 0$. It follows that $e^* \le e$ a.s. Thus it suffices to show that $e^* = \infty$ a.s. Now we are dealing with a one-dimensional Itô-type diffusion (2.3). The condition for nonexplosion in this case is known (Ikeda and Watanabe [6], pages 365–367): $e^* = \infty$ a.s. if and only if

$$I(G) \overset{\text{def}}{=} \int_c^\infty G(r)^{1-n}\,dr \int_c^r G(s)^{n-1}\,ds = \infty. \tag{2.4}$$

The above condition (2.4) is the best condition for the nonexplosion of Riemannian Brownian motion in general. To convert this condition into the more explicit condition on the function $\kappa(s)$ stated in our theorem, we need to show that $\int_c^\infty \kappa(r)^{-1}\,ds = \infty$ implies $I(G) = \infty$.

Observe first that $\kappa(r)^2 = G''(r)/G(r)$ is nondecreasing. Integrating by parts, we have

$$\int_0^r G(s)^2\,d\kappa(s)^2 = G''(r)G(r) - G'(r)^2 + 1.$$

Hence $G''(r)G(r) - G'(r)^2 \geq -1$, or

$$\left[\frac{G'(r)}{G(r)}\right]^2 \leq \frac{G''(r)}{G(r)} + \frac{1}{G(r)^2} \leq c_1\kappa(r)^2$$

for $r \geq c$ and $c_1 = 1 + \kappa(c)^{-1}G(c)^{-2}$. Integrating by parts again, we have

$$\int_c^r G(s)^{n-1}\,ds = \frac{1}{n}\int_c^r \frac{dG^n(s)}{G'(s)} \geq \frac{1}{n}\frac{G(r)^n}{G'(r)} - \frac{1}{n}\frac{G(c)^n}{G'(c)}.$$

Since G grows at least exponentially, we have $\int_c^\infty G(r)^{1-n}\,dr < \infty$. Therefore

$$I(G) \geq \frac{1}{n}\int_c^\infty \frac{G(r)}{G'(r)}\,dr - c_2 \geq \frac{c_1^{-1/2}}{n}\int_c^\infty \kappa(r)^{-1}\,ds - c_2 = \infty.$$

This shows M is stochastically complete.

Appendix to Section 2. We give a more direct proof of (2.1) (compare with Kendall [8]). The key to the proof is the following remarkable property of the distance function ρ. Let ϕ be a smooth, nonnegative function on M with compact support, then

$$\int_M \rho(x)\,\Delta\phi(x)\,dx \leq \int_{M\setminus C(x_0)} \phi(x)\,\Delta\rho(x)\,dx. \tag{A}$$

[Recall that $C(x_0)$ is the cut-locus of x_0.] Inequality (A) is proved in the appendix of Yau [11]. Now let $\Delta\rho$ denote the Laplacian of ρ *in the distributional sense*. The above inequality implies that the distribution

$$\mu = \Delta\rho I_{M\setminus C(x_0)} - \Delta\rho \tag{2.5}$$

is nonnegative on nonnegative test functions. It follows from the Riesz representation theorem that μ is a Radon measure on M concentrated on the cut-locus $C(x_0)$. Thus $\Delta\rho$, the distributional Laplacian of ρ, is itself realized as a measure on M. A generalized Itô formula (Meyer [9]) applied to the function ρ gives immediately formula (2.1) with L_t equal to the continuous positive additive functional associated with the measure μ.

It is interesting to point out that (2.1) and (A) are in fact equivalent. Thus Kendall's proof of (2.1) amounts to a probabilistic proof of (A).

3. C_0-diffusion property. We need two lemmas. The first is elementary and a proof can be found in Azencott [1]. The proof of the second is essentially contained in Hsu and March [5].

LEMMA 3.1. *P_t is a C_0-diffusion semigroup if and only if for any compact set K and any fixed $t > 0$, we have*

$$\lim_{\rho(x)\to\infty} P_x[T_K \le t] = 0, \tag{3.1}$$

where T_K is the first hitting time of K:

$$T_K = \inf\{t > 0 \colon X_t \in K\}.$$

LEMMA 3.2. *Let*

$$\tau = \inf\{t > 0 \colon \rho(X_0, X_t) = 1\}.$$

If the Ricci curvature of M in the geodesic ball $B_1(X_0)$ centered at X_0 with radius 1 is bounded from below by $-L^2 \le -1$, then

$$P_x\left[\tau \le c_1 L^{-1}\right] \le e^{-c_2 L}$$

for some universal positive constants c_1 and c_2.

PROOF. This is essentially Lemma 4 of Hsu and March [5]. In the present case, the geodesic ball $B_1(X_0)$ may not entirely lie within the cut-locus. However, we notice the following three facts:

(a) From (2.1) we have for any $s \le t$,

$$\rho(X_t) - \rho(X_s) \le \beta_t - \beta_s + \frac{1}{2}\int \Delta\rho(X_s)\,ds.$$

(b) Away from the cut-locus $C(X_0)$, the Laplacian comparison theorem (Greene and Wu [3], page 26) gives

$$\Delta\rho \le (n-1)L \coth L\rho.$$

(c) The Brownian motion spends zero amount of time in the cut-locus.

The above three facts enable us to carry out the proof of the lemma by repeating, mutatis mutandis, the proof of Hsu and March [5]. □

We now prove the second part of our theorem. By Lemma 3.1, it is enough to show (3.1). We may assume that $K = \overline{B_R(x_0)}$, the geodesic ball centered at x_0 with radius R. Let τ be defined as in Lemma 3.2. Consider the following stopping times:

$$\begin{aligned}
s_0 &= 0,\\
\tau_1 &= \tau,\\
s_1 &= \inf\{t \geq \tau_1 : \rho(x_0, X_t) = \rho(x) - 1\},\\
\tau_2 &= \tau \circ \theta_{s_1},\\
s_2 &= \inf\{t \geq \tau_2 + s_1 : \rho(x_0, X_t) = \rho(x) - 2\},\\
&\cdots\\
\tau_n &= \tau \circ \theta_{s_{n-1}},\\
s_n &= \inf\{t \geq \tau_n + s_{n-1} : \rho(X_t) = \rho(x) - n\}
\end{aligned}$$

(θ is the shift operator). It is clear that

$$T_K \geq s_{[\rho(x)-R]} \geq \tau_1 + \tau_2 + \cdots + \tau_{[\rho(x)-R]} \tag{3.2}$$

($[a]$ denotes the integral part of a). Now since $\rho(X_{s_{k-1}}) = \rho(x) - k + 1$, we have by Lemma 3.2

$$P_x\left[\tau_k \leq c_1\kappa(\rho(x) - k + 2)^{-1}\right] \leq e^{-c_2\kappa(\rho(x)-k+2)}. \tag{3.3}$$

Choose $n(x, t)$ to be the first integer n such that

$$\sum_{k=1}^{n} \kappa(\rho(x) - k + 2)^{-1} > \frac{t}{c_1}.$$

In view of the condition $\int_c^\infty \kappa(s)^{-1}\,ds = \infty$ such $n(x, t)$ exists for sufficiently large $\rho(x)$, and furthermore we must have $[\rho(x) - R] \geq n(x, t)$. Hence

$$\{T_K \leq t\} \subset \{\tau_1 + \tau_2 + \cdots + \tau_{[\rho(x)-R]} \leq t\} \subset \bigcup_{k=1}^{n(x,t)} \{\tau_k \leq c_1\kappa(\rho(x) - k + 2)^{-1}\}.$$

Using (3.3), we have

$$P_x[T_K \leq t] \leq \sum_{k=1}^{n(x,t)} e^{-c_2\kappa(\rho(x)-k+2)} \leq \int_{m(x,t)}^{\rho(x)+1} e^{-c_2\kappa(r)}\,dr. \tag{3.4}$$

Here we have set $m(x, T) = \rho(x) - n(x, t) + 1$ for simplicity. The choice of $n(x, t)$ implies that

$$\sum_{k=1}^{n(x,t)-1} \kappa(\rho(x) - k + 2)^{-1} \leq \frac{t}{c_1}$$

or

$$\int_{m(x,t)+2}^{\rho(x)+2} \kappa(r)^{-1} \leq \frac{t}{c_1}. \tag{3.5}$$

Using (3.4), (3.5) and the elementary inequality $e^{-c_2\kappa} \leq ae^{-c_2 a}/\kappa$ for $\kappa \geq a \geq c_2^{-1}$, we have

$$P_x[T_k \le t] \le \kappa(m(x,t))e^{-c_2\kappa(m(x,t))}\left[2\kappa(m(x,t))^{-1} + \frac{t}{c_1}\right],$$

provided that $m(x,t) \ge c_2^{-1}$. Now the assumption $\int_c^\infty \kappa(r)^{-1}\,dr = \infty$ implies by (3.5) that $m(x,t) \to \infty$ as $\rho(x) \to \infty$. (3.1) follows immediately from the above inequality by letting $\rho(x) \to \infty$. The theorem is proved.

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