The Annals of Probability
1989, Vol. 17, No. 3, 1255–1263

CENTRAL LIMIT THEOREMS FOR INFINITE URN MODELS

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An urn model is defined as follows: n balls are independently placed in an infinite set of urns and each ball has probability $p_k > 0$ of being assigned to the kth urn. We assume that $p_k \geq p_{k+1}$ for all k and that $\sum_{k=1}^{\infty} p_k = 1$. A random variable Z_n is defined to be the number of occupied urns after n balls have been thrown. The main result is that Z_n, when normalized, converges in distribution to the standard normal distribution. Convergence to $N(0,1)$ holds for all sequences $\{p_k\}$ such that $\lim_{n\to\infty} \operatorname{Var} Z_{N(n)} = \infty$, where $N(n)$ is a Poisson random variable with mean n. This generalizes a result of Karlin.

1. Introduction.

An urn model is defined in the following way: n balls are placed independently in an infinite set of urns and each ball has probability $p_k > 0$ of being assigned to the kth urn, for $k = 1, 2, 3, \ldots$. We assume that the urns are arranged in decreasing order, so that $p_k \geq p_{k+1}$ for all k and that $\sum_{k=1}^{\infty} p_k = 1$. We define the random variable

$$X_{nk} = \text{number of balls in the } k\text{th urn after } n \text{ throws.}$$

We will need to consider the case where the number of throws is not fixed in advance but depends on the outcome of a random experiment. Specifically, suppose that the number of balls thrown is a Poisson random variable with mean n, denoted by $N(n)$. We have $P[N(n) = r] = e^{-n}n^r/r!$.

We define

$$X_{N(n),k} = \text{number of balls in the } k\text{th urn after } N(n) \text{ throws.}$$

By calculating a joint probability distribution for any M-tuple (as in [2], page 216), it is easy to show that the random variables $\{X_{N(n),k}\}$, $k = 1, 2, 3, \ldots$, are mutually independent Poisson variables with respective means $\{np_k\}$, so that $P[X_{N(n),k} = r] = \exp(-np_k)(np_k)^r/r!$. The random variables $\{X_{nk}\}$, where the sample size n is fixed and k varies, are not independent.

We next define the random variable

$$Z_n = \sum_{k=1}^{\infty} \varphi(X_{nk}), \quad \text{where} \quad \varphi(u) = \begin{cases} 1, & u > 0, \\ 0, & u = 0. \end{cases}$$

Similarly,

$$Z_{N(n)} = \sum_{k=1}^{\infty} \varphi(X_{N(n),k}).$$

The random variable Z_n is the number of occupied urns after n balls have been thrown, and $Z_{N(n)}$ is the number of occupied urns after $N(n)$ balls have been thrown.

Received September 1987; revised October 1988.
AMS 1980 *subject classifications*. Primary 60F05; secondary 60C05.
Key words and phrases. Central limit theorem, urn model.

We will use the notation μ_n and σ_n^2 for the mean and variance of Z_n; we will use $\mu(n)$ and $\sigma^2(n)$ for the mean and variance of $Z_{N(n)}$. The standard normal distribution with mean 0 and variance 1 is denoted by $N(0,1)$.

We will prove in this paper that Z_n, when appropriately normalized, obeys a central limit law under quite general conditions—valid for the same sets $\{p_k\}$ for which the corresponding "Poissonized" random variable $Z_{N(n)}$ obeys a central limit law based on the Lindeberg conditions.

Specifically, we will prove the following result: For all $\{p_k\} \in \mathbb{A} = \{\{p_k\} | \lim_{n\to\infty} \sigma^2(n) = \infty\}$, $[Z_n - \mu_n]/\sigma(n)$ converges in distribution to $N(0,1)$ as $n \to \infty$.

This compares with Karlin's result ([4], Theorem 4, or [3], page 370), which can be stated as follows: $[Z_n - \mu_n]/b_n$ converges in distribution to $N(0,1)$ for all $\{p_k\} \in \mathbb{B} = \{\{p_k\} | \alpha(x) = x^\gamma L(x),\ 0 < \gamma \le 1\}$, where $\alpha(x) = \max\{k | p_k \ge 1/x\}$ and $L(x)$ is slowly varying, that is, $L(cx)/L(x) \to 1$ as $x \to \infty$ for any fixed $c > 0$. Thus, $\alpha(x)$ is of regular variation in the sense of Karamata. The normalizing function b_n is such that $b_n \to \infty$ and $b_n \sim \sigma_n$, $n \to \infty$. An explicit formula for b_n^2 is given in [4] (page 386). As will be shown by examples in the next section, the class $\mathbb{A}$ is wider than the class $\mathbb{B}$. It is convenient in our proofs to normalize Z_n by $\sigma(n)$ rather than by its own standard deviation σ_n. By Khintchine's convergence of types theorem ([5], page 216) any nontrivial limit law that holds for Z_n is independent of the normalizing constants used.

2. Preliminary results. We will calculate the means of $Z_{N(n)}$ and Z_n. It follows by additivity that $\mu(n) = \sum_{k=1}^\infty (1 - e^{-np_k})$ and

$$\mu_n = \sum_{k=1}^{\infty} \left[1 - (1 - p_k)^n\right].$$

The series for $\mu(n)$ converges absolutely for fixed n, since $\sum_{k=1}^\infty (1 - e^{-np_k}) \le \sum_{k=1}^\infty np_k = n$. It is clear that $\mu(n) \to \infty$ as $n \to \infty$. When n is replaced by the continuous variable t, $\mu(t)$ is differentiable and is a C^∞ function.

In order to calculate the variance of the random variable $Z_{N(n)}$, we use the representation $Z_{N(n)} = \sum_{k=1}^\infty \varphi(X_{N(n),k})$, which is a sum of independent binomial random variables each assuming the values 1 or 0. Thus,

$$\begin{aligned}\sigma^2(n) &= \sum_{k=1}^{\infty} \sigma^2\varphi\left(X_{N(n),k}\right) = \sum_{k=1}^{\infty} \left(e^{-p_k n} - e^{-2p_k n}\right)\\ &= \mu(2n) - \mu(n).\end{aligned}$$

The following result, needed later, shows that the limiting behavior of $\mu(n)$ and μ_n is the same (cf. [4], page 381).

LEMMA 1. *For any sequence $\{p_k\}$ defined as before,*

$$\lim_{n\to\infty} \left[\mu_n - \mu(n)\right] = \lim_{n\to\infty} \sum_{k=1}^{\infty} \left[e^{-p_k n} - (1 - p_k)^n\right] = 0.$$

PROOF. We use the inequality ([6], page 530)

$$0 \le e^{-x} - \left(1 - \frac{x}{n}\right)^n \le \frac{x^2}{n} e^{-x}, \qquad 0 \le x \le n,$$

which gives

$$\sum_{k=1}^{\infty} \left[e^{-np_k} - (1 - p_k)^n\right] \le \sum_{k=1}^{\infty} np_k^2 e^{-np_k} \le \sum_{k=1}^{\infty} p_k e^{-1}.$$

The sum is dominated by the convergent positive series $\Sigma p_k e^{-1}$ and therefore we can interchange the limit and summation operations, which proves the lemma. □

We conclude this section by giving examples that distinguish between the classes $\mathbb{A}$ and $\mathbb{B}$. Roughly speaking, $\mathbb{A}$ contains sequences $\{p_k\}$ possessing irregularities, where the variances $\sigma^2(n) \to \infty$ for $n \to \infty$ and the smoothness conditions $\alpha(x) = x^\gamma L(x)$, $0 < \gamma \le 1$, need not hold. Karlin has shown [4] that if $\{p_k\} \in \mathbb{B}$, then $\sigma^2(n) \to \infty$, $n \to \infty$, and therefore $\mathbb{B}$ is a subset of $\mathbb{A}$.

We need to establish a sufficient condition in order that the variances have an infinite limit, stated as follows: If $\lim_{k \to \infty} p_{k+1}/p_k = 1$, then $\lim_{n \to \infty} \sigma^2(n) = \infty$. To show this, we express $\sigma^2(n)$ as a Stieltjes integral using the definition of $\alpha(x)$ and then integrate by parts (cf. [4], page 384),

$$\begin{aligned}\sigma^2(n) &= \int_0^\infty [e^{-n/x} - e^{-2n/x}]\, d\alpha(x) \\ &= \int_0^\infty \left[\frac{2n}{x^2} e^{-2n/x} - \frac{n}{x^2} e^{-n/x}\right] \alpha(x)\, dx \\ &= \int_0^\infty \frac{n}{x^2} e^{-n/x} [\alpha(2x) - \alpha(x)]\, dx.\end{aligned}$$

The condition $\lim_{k \to \infty} p_{k+1}/p_k = 1$ implies that $\lim_{x \to \infty} [\alpha(2x) - \alpha(x)] = \infty$ ([4], page 378) and the desired conclusion follows from the following lemma.

LEMMA 2. *If* $\lim_{x \to \infty} [\alpha(2x) - \alpha(x)] = \infty$, *then* $\lim_{n \to \infty} \sigma^2(n) = \infty$, *where*

$$\sigma^2(n) = \int_0^\infty \frac{n}{x^2} e^{-n/x} [\alpha(2x) - \alpha(x)]\, dx.$$

The proof of the lemma is routine and is not shown. We contrast this sufficient condition on the variances with [4] (page 383), where it is shown that if $\limsup p_{k+1}/p_k < 1$, then $\sigma^2(n)$ is bounded for all n.

In each of the following examples, the set $\{p_k\}$ belongs to $\mathbb{A}$ but not to $\mathbb{B}$. Recall that $\alpha(x) = \max\{k | p_k \ge 1/x\}$.

EXAMPLE 1. Let $p_k = C/k^{\log k}$, where C is a normalizing constant. In this case, $\alpha(x) \sim \exp(\log Cx)^{1/2}$, $x \to \infty$. It is routine to show that $p_{k+1}/p_k \to 1$, $k \to \infty$, and this implies that $\sigma^2(n) \to \infty$, $n \to \infty$.

EXAMPLE 2. If $p_k = C/k^r$, $r > 1$, then $\alpha(x) \sim C^{1/r}x^{1/r}$, $x \to \infty$, and $\alpha(x)$ is of regular variation. However, we can combine in arbitrary ways terms of the form $1/k^2$ and $1/k^3$. For example (the normalizing constant has been omitted), let $\{p_k\}$ be defined by the following sequence: $1/2^2, 1/3^2, \dots, 1/8^2, 1/5^3, \dots, 1/9^3, 1/28^2, \dots$. Switches between the squared and cubed subsequences can be made when a perfect square integer is also a perfect cube. Thus $1/64 = 1/8^2 = 1/4^3$ and $1/729 = 1/9^3 = 1/27^2$. In order to assure that $\alpha(x)$ is not a function of regular variation, we can specify that the length of each subsequence increases rapidly with each switch, so that $\alpha(x)$ is alternately approximated by $Cx^{1/2}$ and $Cx^{1/3}$. In this case, $\lim_{k\to\infty} p_{k+1}/p_k = 1$ and $\sigma^2(n) \to \infty$, $n \to \infty$.

3. Related remarks. As was mentioned earlier, Z_n will be normalized by $\sigma(n)$ and not by its own standard deviation σ_n. The variance σ_n^2 is not representable (because of the nonindependence of $\{X_{nk}\}$) as a simple sum and is very difficult to work with in the absence of a regularity condition. Karlin ([4], page 385) first represents σ_n^2 formally, with its "mixed" terms, and then assumes that $\alpha(x) = x^\gamma L(x)$, $0 < \gamma \le 1$, which makes possible a normalizing function b_n such that $\sigma_n \sim \sigma(n) \sim b_n$, $n \to \infty$, for $\{p_k\} \in \mathbb{B}$.

The variances can, in general, exhibit erratic behavior and an example has been given ([4], page 384) where $\sigma^2(n)$ oscillates unboundedly. If $p_k = (1-\theta)\theta^{k-1}$, $0 < \theta < 1$, then $\sigma^2(n)$ is bounded as $n \to \infty$.

Karlin has studied ([4], page 399) the random variable $Z_{N(t)} - \mu(t)$ for the case $p_k = (1-\theta)\theta^{k-1}$, $0 < \theta < 1$, $k = 1, 2, \dots$, and asserts that it converges to a nondegenerate limit as the continuous variable $t \to \infty$. The assertion is based on analysis of convergence of moments and the applicability of Carleman's criterion that a distribution is uniquely determined by its moments.

However, the following example shows that the method of moments does not work. We have that $\sigma^2(t) = \sum_{k=1}^{\infty}(e^{-p_k t} - e^{-2p_k t})$. By sucessively setting $t = 1/p_k$, $k = 1, 2, \dots$, it follows that $\sigma^2(1/p_k) \ge e^{-1} - e^{-2}$. Therefore, the sequence $\sigma^2(t_k)$ is bounded below for $t_k = 1/p_k \to \infty$ and so the assertion ([4], page 385) that $\sigma^2(t)$ converges to $\log_{1/\theta} 2$ for all $\theta \in (0,1)$ cannot be true, because $\log_{1/\theta} 2$ can be made as small as desired by choosing θ small enough. This observation does not establish, of course, that the random variables in question do not converge, because a sequence of random variables can converge even if the moments do not. In fact, for the case $p_k = 1/2^k$,

$$\sigma^2(t) = \sum_{k=1}^{\infty} \left[\exp(-t2^{-k}) - \exp(-t2^{-k+1})\right] = 1 - e^{-t},$$

so that $\lim_{t\to\infty} \sigma^2(t) = 1$ exists. However, other moments do not converge for this case, and for other values of θ the variance itself does not converge. A more detailed discussion of the variance and specifically its connection with the integral representation used by Karlin ([4], pages 384–385) is given in [1]. There it is also shown that $Z_{N(t)} - \mu(t)$ does not converge in distribution, and limits are identified along convergent subsequences.

We can write $Z_{N(t)} - \mu(t) = \Sigma_{k=1}^{\infty} Y_k(t)$, where the $Y_k(t)$ are independent, $Y_k(t) = \exp[-p_k t]$ with probability $1 - \exp[-p_k t]$ and $Y_k(t) = \exp[-p_k t] - 1$ with probability $\exp[-p_k t]$. Let $p_k = (1-\theta)\theta^{k-1}$, $k = 1, 2, 3, \ldots$. Inserting the values $t_m = \theta^{-m+\gamma}$ ($m = 1, 2, 3, \ldots$ and γ is any real number) in $Z_{N(t)} - \mu(t)$ produces a two-tailed sum and shows that $Z_{N(t)} - \mu(t)$ converges in distribution along the sequence $\{t_m\}$ to a random variable which is distributed as $W(\gamma) = \Sigma_{k=-\infty}^{\infty} W_k$, where W_k are independent and

$$W_k = \begin{cases} \exp[-(1-\theta)\theta^{k+\gamma}] - 1 & \text{with probability } \exp[-(1-\theta)\theta^{k+\gamma}], \\ \exp[-(1-\theta)\theta^{k+\gamma}] & \text{with probability } 1 - \exp[-(1-\theta)\theta^{k+\gamma}]. \end{cases}$$

The distribution of $W(\gamma)$ is periodic of period 1, but is not independent of γ and the limit along all t does not exist.

4. Central limit property of $Z_{N(n)}$. The number of occupied urns after $N(n)$ balls have been thrown is $Z_{N(n)} = \Sigma_{k=1}^{\infty} \varphi(X_{N(n),k})$, where $\varphi(u) = 1$ for $u > 0$ and $\varphi(u) = 0$ for $u = 0$. $\{X_{N(n),k}\}$ is a sequence of independent Poisson variables with respective means $\{np_k\}$ and $Y_{N(n),k} = \varphi(X_{N(n),k})$ are independent binomial variables,

$$Y_{N(n),k} = \begin{cases} 0 & \text{with probability } e^{-np_k}, \\ 1 & \text{with probability } 1 - e^{-np_k}. \end{cases}$$

We define now the centered and normalized variables

$$x_{nk} = \frac{Y_{N(n),k} - (1 - e^{-np_k})}{\sqrt{\Sigma_{k=1}^{\infty} \sigma^2 Y_{N(n),k}}}.$$

The conditions for convergence to the standard normal distribution are satisfied and we state the following theorem, valid when $\lim_{n\to\infty} \sigma^2(n) = \infty$ (cf. [4], page 387).

THEOREM 1. *$[Z_{N(n)} - \mu(n)]/\sigma(n)$ converges in distribution to the normal distribution $N(0,1)$ as $n \to \infty$, for all $\{p_k\} \in \mathbb{A}$.*

PROOF. We use Lindeberg's criterion for convergence and define an infinite rectangular array of random variables $[x_{ij}]$, $i = 1, 2, \ldots$, $j = 1, 2, \ldots$. We have $[Z_{N(n)} - \mu(n)]/\sigma(n) = \Sigma_{k=1}^{\infty} x_{nk}$ and the row sums are normalized, that is, $\Sigma_{k=1}^{\infty} \sigma^2 x_{nk} = 1$.

Let F_{nk} denote the distribution function of x_{nk}. Because of the condition $\sigma^2(n) \to \infty$, $n \to \infty$, the set $\{x_{nk}\}$ is uniformly bounded. This implies that for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_k \int_{|x| \ge \varepsilon} x^2 \, dF_{nk} = 0,$$

which means that the criteria for convergence of $\Sigma_{k=1}^{\infty} x_{nk}$ to $N(0,1)$ are met ([5], page 307). This completes the proof. □

5. Central limit property of Z_n. In this section we develop the method that establishes the asymptotic normality of Z_n, valid for the same set $\mathbb{A}$ as Theorem 1.

LEMMA 3. *Let $\mu(n) = \sum_{k=1}^{\infty}(1 - e^{-p_k n})$ and $\sigma(n) = \sqrt{\mu(2n) - \mu(n)}$. Then $\lim_{n\to\infty}[\mu(n + M\sqrt{n}) - \mu(n)]/\sigma(n) = 0$ for every $M > 0$.*

PROOF. Note that the only restriction on $\{p_k\}$ in this lemma and Lemma 4 is that $\sum_{k=1}^{\infty} p_k = 1$, $p_k \geq p_{k+1}$ and $p_k > 0$ for all k. For convenience we replace n by the continuous variable t. It was previously noted that $\mu(t)$ is a C^{∞} function. We have

$$\mu'(t) = \sum_{k=1}^{\infty} p_k e^{-p_k t} \quad \text{and} \quad \mu''(t) = -\sum_{k=1}^{\infty} p_k^2 e^{-p_k t} < 0.$$

The sign of the second derivative implies that $\mu(t)$ is concave and $\mu'(t)$ is positive and decreasing. Let $f(t) = \mu'(t)$. We write

$$\frac{\mu(n + M\sqrt{n}) - \mu(n)}{\sigma(n)} = \frac{\int_n^{n+M\sqrt{n}} f(t)\,dt}{\left[\int_n^{2n} f(t)\,dt\right]^{1/2}} \leq \frac{M\sqrt{n}\,f(n)}{\sqrt{nf(2n)}}.$$

Thus Lemma 3 follows if we prove that $[\mu'(t)]^2/\mu'(2t) \to 0$, $t \to \infty$.

LEMMA 4. *Let $f(t) = \sum_{k=1}^{\infty} p_k e^{-p_k t}$, where $\sum_{k=1}^{\infty} p_k = 1$, $p_k \geq p_{k+1}$ and $p_k > 0$ for all k. Then, $\lim_{t\to\infty} f^2(t)/f(2t) = 0$.*

PROOF. Let $p_k > p_l$ be given. Then

$$\frac{e^{-p_k t}}{f(t)} \leq \frac{e^{(p_l - p_k)t}}{p_l},$$

which implies that

$$\lim_{t\to\infty} \frac{\sum_{k=N}^{\infty} p_k e^{-p_k t}}{f(t)} = 1 \quad \text{for every } N.$$

The Cauchy–Schwarz inequality implies that

$$\left[\sum_{k=N}^{\infty} p_k e^{-p_k t}\right]^2 \leq \sum_{k=N}^{\infty} p_k \sum_{k=N}^{\infty} p_k e^{-2p_k t}.$$

It follows that

$$\limsup_{t\to\infty} \frac{f^2(t)}{f(2t)} = \limsup_{t\to\infty} \frac{\left[\sum_{k=N}^{\infty} p_k e^{-p_k t}\right]^2}{\sum_{k=N}^{\infty} p_k e^{-2p_k t}} \leq \sum_{k=N}^{\infty} p_k.$$

The proof is completed by letting $N \to \infty$. □

REMARK. Under a slightly more stringent hypothesis, Lemma 3 implies that the variances $\sigma^2(t + M\sqrt{t})$ and $\sigma^2(t)$ are asymptotically equal. Let the conditions of Lemma 3 hold (i.e., there are given $p_1 \geq p_2 \geq \cdots$, where $p_k > 0$ and $\sum_{k=1}^{\infty} p_k = 1$) and suppose, in addition, that $\lim_{t\to\infty} \sigma^2(t) = \infty$.

Then, for every $M > 0$,

$$\lim_{t\to\infty} \frac{\sigma(t + M\sqrt{t})}{\sigma(t)} = 1.$$

PROOF. We note that the hypotheses are those of Theorem 1. We have

$$\begin{aligned}\frac{\sigma^2(t + M\sqrt{t})}{\sigma^2(t)} &= \frac{\mu(2t + 2M\sqrt{t}) - \mu(t + M\sqrt{t})}{\mu(2t) - \mu(t)} \\ &= \frac{\mu(2t + 2M\sqrt{t}) - \mu(2t)}{\mu(4t) - \mu(2t)} \cdot \frac{\mu(4t) - \mu(2t)}{\mu(2t) - \mu(t)} \\ &\quad - \frac{\mu(t + M\sqrt{t}) - \mu(t)}{\mu(2t) - \mu(t)} + 1 \\ &= A \cdot B - C + 1.\end{aligned}$$

The concavity of $\mu(t)$ implies that $B \leq 2$. As $t \to \infty$, the terms A and C converge to zero by Lemma 3 and the condition $\sigma^2(t) \to \infty$. □

We continue with the series of lemmas that leads to our main result, asymptotic normality for the random variable Z_n.

LEMMA 5. *For any* $M > 0$, $\lim_{n\to\infty}[\mu_{n+M\sqrt{n}} - \mu_n]/\sigma(n) = 0$.

PROOF. The result follows by applying Lemma 1 to Lemma 3. □

LEMMA 6. *For any* $\varepsilon > 0$, $\lim_{n\to\infty} P[|Z_{n+M\sqrt{n}} - Z_n|/\sigma(n) > \varepsilon] = 0$.

PROOF. By using Markov's inequality, we have

$$P\big[|Z_{n+M\sqrt{n}} - Z_n|/\sigma(n) > \varepsilon\big] \leq [\mu_{n+M\sqrt{n}} - \mu_n]/\varepsilon\sigma(n)$$

and the right-hand side goes to zero by Lemma 5. □

We can now prove the main theorem.

THEOREM 2. *The random variable* $[Z_n - \mu_n]/\sigma(n)$ *converges in distribution to the standard normal distribution* $N(0,1)$, *as* $n \to \infty$, *for all* $\{p_k\} \in \mathbb{A}$.

PROOF. We use the identity

$$(1)\qquad P\left\{\frac{Z_{N(n)}-\mu_n}{\sigma(n)}\le x\right\}=\sum_{k=0}^{\infty}P\left\{\frac{Z_k-\mu_n}{\sigma(n)}\le x\right\}\frac{n^k e^{-n}}{k!},$$

conditioning on the values of $N(n)$. The rest of the details are similar to those in Karlin ([4], page 390). With $\varepsilon > 0$ given and n specified sufficiently large, there exists a constant $C_1(\varepsilon)$ independent of n such that

$$\sum_{k=n-C_1 n^{1/2}}^{n+C_1 n^{1/2}}\frac{n^k e^{-n}}{k!}\ge 1-\varepsilon.$$

Lemma 6 implies that for $n > n_0(\varepsilon,\delta)$ and all k satisfying $|k-n|\le C_1\sqrt{n}$, the inequality

$$P\left[|Z_n - Z_k| > \delta\sigma(n)\right] < \varepsilon$$

holds, where $\delta > 0$ is arbitrary and fixed.

It follows that for $n > n_0(\varepsilon,\delta)$,

$$P\left\{\frac{Z_k-\mu_n}{\sigma(n)}\le x\right\}\le P\left\{\frac{Z_n-\mu_n}{\sigma(n)}\le x+\frac{Z_n-Z_k}{\sigma(n)}\right\}\le F_n(x+\delta)+\varepsilon,$$

where $F_n(x) = P\{Z_n - \mu_n \le x\sigma(n)\}$. We apply this to (1) and get, for $n > n_0(\varepsilon,\delta)$,

$$(2)\qquad P\left[Z_{N(n)}-\mu_n\le x\sigma(n)\right]\le F_n(x+\delta)+\varepsilon.$$

The asymptotic normality of $[Z_{N(n)} - \mu(n)]/\sigma(n)$ (Theorem 1), together with Lemma 1, gives

$$\lim_{n\to\infty}P\left\{\frac{Z_{N(n)}-\mu_n}{\sigma(n)}\le x\right\}=\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(\frac{-s^2}{2}\right)ds.$$

Taking a limit as $n \to \infty$ and applying (2) gives, for all x,

$$\Phi(x)\le\liminf_{n\to\infty}F_n(x+\delta).$$

Similarly,

$$\Phi(x)\ge\limsup_{n\to\infty}F_n(x-\delta),\quad\text{for all }x.$$

Together these two inequalities imply that

$$\lim_{n\to\infty}F_n(x)=\Phi(x).$$

This completes the proof. □

Acknowledgments. The material in this paper is part of my Ph.D. thesis. I am very grateful to L. N. Vaserstein, my adviser, for his support and encouragement of my work at Pennsylvania State University. I would like to thank D. A. Darling and J. G. Wendel for their suggestions. I am very grateful to T. M. Liggett for his careful editorial criticisms and suggestions, which resulted in shortening of several proofs and in improving the discussion in Section 3.

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