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ON THE EXISTENCE OF SELF-INTERSECTIONS FOR QUASI-EVERY BROWNIAN PATH IN SPACE[1]

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The set of self-intersections of a Brownian path $b(t)$ taking values in $\mathbb{R}^3$ has Hausdorff dimension 1, for almost every such path, with respect to Wiener measure, a result due to Fristedt. Here we prove that this result (together with the corresponding result for paths in $\mathbb{R}^2$) in fact holds for quasi-every path with respect to the infinite-dimensional Ornstein–Uhlenbeck process, a diffusion process on Wiener space whose stationary measure is Wiener measure. We do this using Rosen's self-intersection local time, first proving that this exists for quasi-every path.

0. Introduction. Recently, there has been much interest in proving that certain properties of Brownian motion in d dimensions hold quasi-everywhere (q.e.) with respect to the Ornstein–Uhlenbeck process on Wiener space, as described in Fukushima (1984). What we show here is that if $d = 2$ or $d = 3$, much of the theory due to Rosen (1983), on the existence and properties of the self-intersection local time of almost every Brownian path, carries over to results for quasi-every Brownian path.

In particular, we generalise the classical result of Dvoretzky, Erdös and Kakutani (1950), that almost every Brownian path in $\mathbb{R}^3$ intersects itself, to a quasi-everywhere result. This contrasts with the converse problem in $\mathbb{R}^4$ or $\mathbb{R}^5$, where it is not true that quasi-every Brownian path is self-avoiding, even though almost every such path is [Lyons (1986)]. A proof of this result using methods related to those here appears in Mountford (1989).

Motivation for considering the material discussed here is provided by the problem of finding a measure on the set of self-avoiding Brownian paths in $\mathbb{R}^3$ which might serve as a suitable model for the statistical properties of long polymer chains [see, for example, Freed (1981)]. If the set of self-avoiding paths has positive capacity with respect to some Dirichlet form on Wiener space [e.g., one of the (r, p)-capacities described by Takeda (1984)], then the equilibrium measure of the set of self-avoiding paths with respect to that Dirichlet form might be a suitable polymer measure.

Our results fall into four main parts. First, we prove (in Section 3) that (relative to a suitable base set) the self-intersection local time exists quasi-everywhere and that it is continuous in the time-parameter of the Ornstein–Uhlenbeck process on Wiener space. This is the analogue for self-intersection local time to a result of Shigekawa (1984) on the quasi-everywhere existence of

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the ordinary local time for one-dimensional Brownian motion (the methods we use here provide an alternative proof for this result). We prove our results by applying Kolmogorov's lemma to the expression of the local time, viewed as a (random) function on $\mathbb{R}^d$, as an improper integral of its Fourier transform.

Second, we prove (in Section 4) by Borel–Cantelli arguments that the self-intersection local time has the same order of Hölder continuity quasi-everywhere as it does almost everywhere.

Third, we examine (in Section 5) the probability distribution of the intersection local time of two independent Brownian motions with the same starting point, as studied by Geman, Horowitz and Rosen (1984), and Le Gall (1986). In particular, we estimate the probability that this local time is very small. The results in this section have some intrinsic interest and can be read independently of the rest of this paper, as they are not concerned with quasi-everywhere results.

Fourth, we exhibit (in Section 6) a base set B_0 in the bounded upper triangle $\{(s, t): 1 \geq t > s \geq 0\}$, such that the self-intersection local time of quasi-every Brownian motion relative to B_0 is strictly positive. The proof uses the results of Section 5 (but if we allow B_0 to be unbounded, a simpler proof is available, as was pointed out by a referee). Using our Hölder conditions on the local time, we then deduce (Section 7) that the Hausdorff dimension of the set of self-intersections (up to time 1) of quasi-every path in $\mathbb{R}^d$ ($d = 2$ or $d = 3$) is $4 - d$. In particular, quasi-every Brownian path in $\mathbb{R}^3$ intersects itself before time 1 (or in any time-interval).

In the case $d = 2$, Komatsu and Takashima (1984) have proved the quasi-everywhere existence of an "intersectional local time" defined in a somewhat different way from ours.

1. Preliminaries.

The Ornstein–Uhlenbeck process. Let W_0^d be d-dimensional Wiener space, the space of all continuous paths $B: [0, \infty) \to \mathbb{R}^d$ vanishing at 0, with the topology of uniform convergence on compact intervals. Let $(w(\tau, t))_{\tau \geq 0, t \geq 0}$ be a two-parameter Wiener process (or "Brownian sheet") in $\mathbb{R}^d$. An *Ornstein–Uhlenbeck process* $(B_\tau)_{\tau \geq 0}$ in W_0^d, with initial distribution given by Wiener measure, can be constructed [see Meyer (1982) or Fukushima (1984)] by setting

$$B_\tau(\cdot) = e^{-\tau/2} w(e^\tau, \cdot), \qquad \tau \geq 0.$$

Thus the path $B_\tau = B_\tau(\cdot)$ is an element of W_0^d, and B_τ travels in W_0^d as τ varies. We say that Brownian motion in $\mathbb{R}^d$ has some property *quasi-everywhere* if

$$P[B_\tau \in A, \tau \geq 0] = 1,$$

where A is the set of paths in W_0^d with that property.

Here, instead of B_τ we shall often find it more convenient to consider the path $b_\tau = b_\tau(\cdot)$, given by

$$b_\tau(\cdot) = w(\tau, \cdot), \qquad \tau \geq 1.$$

Local times. We recall some definitions [see Geman and Horowitz (1980)]. If B is a Borel set in $\mathbb{R}_+^N$, we say that a Borel function $X: \mathbb{R}_+^N \to \mathbb{R}^d$ *has a local time relative to* B (or more concisely, X *has a local time in* B) if the *occupation measure* μ_B of X relative to B, given by

$$\mu_B(A) = \lambda^N(B \cap X^{-1}(A)), \qquad A \subset \mathbb{R}^d$$

(where λ^N is Lebesgue measure), is absolutely continuous with respect to λ^d. Then the Radon–Nikodym derivative $(\alpha(x, B), x \in \mathbb{R}^d)$ of μ_B with respect to λ^d is called the *local time* of X relative to B. So $\alpha(x, B)$ is defined only a.e. (dx). A *version* of the local time is a particular choice of $\alpha(\cdot, B)$.

If $b = b(\cdot)$ is an element of W_0^d and B is a Borel set in $\mathbb{R}_+^2$, then if the function $X: \mathbb{R}_+^2 \to \mathbb{R}^d$ given by

$$X(s,t) = b(t) - b(s), \qquad (s,t) \in \mathbb{R}_+^2, \tag{1.1}$$

has a local time in B, we shall refer to the local time in B of X as the *self-intersection local time* in B of b. The self-intersection local time of a Brownian motion was first studied by Rosen (1983).

Fourier analysis of local time and the integral $J_X(k, \gamma, B)$. Following Berman (1969) and others, we here study occupation measure μ_B via its Fourier transform μ_B^*, given by

$$\mu_B^*(u) = \int_{\mathbb{R}^d} e^{iu\cdot x}\mu_B(dx) = \int_{\underline{t}\in B} \exp(iu \cdot X(\underline{t})\, d\underline{t}), \qquad u \in \mathbb{R}^d.$$

In this work [as in Geman, Horowitz and Rosen (1984) and Rosen (1983)] an important role is played by the integral $J_X(k, \gamma, B)$ defined as follows: Given an $\mathbb{R}^d$-valued, N-parameter Gaussian stochastic process $X: \mathbb{R}_+^N \to \mathbb{R}^d$ and given $k \in Z_+$, $\gamma \ge 0$ and Borel B in $\mathbb{R}_+^N$, we define $J_X(k, \gamma, B)$ by

$$\begin{aligned} J_X(k,\gamma,B) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} E \prod_{j=1}^k \{|u_j|^\gamma \mu_B^*(u_j)\}\, du_1 \cdots du_k \\ &= \int_{\bar{u}\in\mathbb{R}^{dk}} \int_{(\bar{t})\in B^k} \left(\prod_{l=1}^k |u_l|^\gamma \right) \exp\left\{ -\tfrac{1}{2}\mathrm{Var}\left(\sum_{j=1}^k u_j \cdot X(\underline{t}_j) \right) \right\} d\bar{t}\, d\bar{u}, \end{aligned}$$

where $\bar{t} = (\underline{t}_1, \underline{t}_2, \dots, \underline{t}_k)$ and $\bar{u} = (u_1, \dots, u_k)$ (note that since X is Gaussian, the integrand is real and positive). Formally, $J_X(k, \gamma, B)$ is the kth moment of an improper integral of the Fourier transform of the occupation measure μ_B of X relative to B against the weight $|u|^\gamma$. Indeed, for every $k \in Z_+$, $(J_X(2k, \gamma, \cdot))^{1/2k}$ is a countably subadditive set function. For

$$J_X(2k,\gamma,B) = \lim_{n\to\infty} E\left(\int_{|u|\le n} |u|^\gamma \mu_B^*(u)\, du \right)^{2k},$$

the integral of $|u|^\gamma \mu_B^*(u)$ over $\{|u| \le n\}$ is real for all n, so

$$(J_X(2k, \gamma, B))^{1/2k} = \lim_{n\to\infty} \left\| \int_{|u|\le n} |u|^\gamma \mu_B^*(u)\, du \right\|_{L^{2k}(dP)}.$$

Hence $(J_X(2k, \gamma, B))^{1/2k}$ is finitely subadditive in B and hence countably subadditive in B.

We shall use the following estimate of $J_X(k, \gamma, B)$.

Lemma 1. *Suppose that $b(\cdot)$ is a Brownian motion in $\mathbb{R}^d$ ($d = 2$ or $d = 3$), and the random function $X \in C(\mathbb{R}_+^2 \to \mathbb{R}^d)$ is given by*

$$X(s, t) = b(t) - b(s), \qquad (s, t) \in \mathbb{R}_+^2.$$

Let $k \in Z_+$ and $\gamma \in [0, 2 - d/2)$. Then there exists a constant c depending only on γ such that for every rectangle B, with sides parallel to the axes, in the upper triangular set $\Delta = \{(s, t): 0 \le s \le t \le 1\}$,

$$J_X(2k, \gamma, B) \le c^k((2k)!)^2(\lambda^2(B))^{(2-(d/2)-\gamma)k},$$

where $\lambda^2(B)$ is the area of B. In particular, the rectangle may touch the diagonal.

Proof. For B as described, $(X(s, t), (s, t) \in B)$ can be thought of as the difference of two independent Brownian motions. The $d = 3$ result follows from the calculation in Section 4 of Geman, Horowitz and Rosen (1984). The $d = 2$ result follows from a similar computation. □

Remark. Let $\mathbb{R}_+^2(\varepsilon) = \{(s, t): 0 \le s < t, t - s \ge \varepsilon\}$. If B is a bounded subset of $\mathbb{R}_+^2(\varepsilon)$ then B is contained in a union of finitely many rectangles in the upper triangle. So by Lemma 1, $J_X(2k, \gamma, B)$ is finite for all $k \in Z_+$ if $\gamma \in [0, 2 - d/2)$. This is Lemma 2 of Rosen (1983).

2. Statement of results.

2. Statement of results. Our first result concerns the quasi-everywhere existence of a self-intersection local time for a Brownian motion in $\mathbb{R}^d$ ($d = 2$ or $d = 3$). This is equivalent to showing the almost sure existence, simultaneously for all $\tau \ge 1$, of a self-intersection local time for the function $b^\tau(\cdot)$, where $(b^\tau(t), \tau \ge 0, t \ge 0)$ is a two-parameter Wiener process in $\mathbb{R}^d$. In the statement of this result we denote the family of Borel subsets of a set B_0 by $\mathscr{B}(B_0)$, and for (s_1, t_1) and (s_2, t_2) in $\mathbb{R}_+^2$, we denote by $R(s_1, t_1, s_2, t_2)$ the open rectangle which has opposite corners at (s_1, t_1) and (s_2, t_2) (in this paper the term "rectangle" is always taken to mean "rectangle with sides parallel to the axes" and similarly for "square" or "cube").

Hypothesis for Theorems 1–3. Let $d = 2$ or $d = 3$. Let $(b^\tau(t), \tau \ge 0, t \ge 0)$ be a two-parameter Wiener process in $\mathbb{R}^d$. For $(s, t) \in \mathbb{R}_+^2$, set

$$X^\tau(s, t) = b^\tau(t) - b^\tau(s)$$

and

$$X(s, t) = X^1(s, t).$$

Let B_0 be an open bounded subset of the triangular set $\Delta = \{(s,t): 0 \le s < t \le 1\}$ (the restriction $t \le 1$ is of no real significance), such that $J_X(2k, \gamma, B_0) < \infty$ for any $k \in Z_+$ and any $\gamma \in [0, 2 - d/2)$.

REMARK. By Lemma 1 we know that B_0 satisfies the above hypothesis if it is a rectangle (possibly touching the diagonal) or if it is strictly separated from the diagonal.

THEOREM 1. *There exist random functions*

$$\varphi(\tau, x, B) = \varphi(\omega, \tau, x, B), \qquad \tau \ge 1, x \in \mathbb{R}^d, B \in \mathscr{B}(\Delta),$$

and

$$\phi(\tau, x) = \phi_{B_0}(\omega, \tau, x), \qquad \tau \ge 1, x \in \mathbb{R}^d,$$

such that the following hold almost surely:

(i) *For each $B \in \mathscr{B}(\Delta)$ and $\tau \ge 1$, $\varphi(\tau, \cdot, B)$ is a (not necessarily continuous) version of the self-intersection local time in B of the path $b^\tau(\cdot)$ (i.e., the local time in B of X^τ).*

For each $\tau \ge 1$, $\phi(\tau, \cdot)$ is a continuous version of the self-intersection local time in B_0 of the path $b^\tau(\cdot)$.

(ii) *For each $\tau \ge 1$ and $x \in \mathbb{R}^d$, $\varphi(\tau, x, \cdot)$ is a sigma-finite measure on the Borel subsets of Δ.*

(iii) *The function h defined by*

$$h(\tau, x, s_1, t_1, s_2, t_2) = \varphi(\tau, x, R(s_1, t_1, s_2, t_2))$$

on $\{(\tau, x, s_1, t_1, s_2, t_2): \tau \ge 1, x \in \mathbb{R}^d, R(s_1, s_2, t_1, t_2) \subset \Delta\}$ is jointly Hölder continuous of any order less than $1 - d/4$ (as we shall see below, the Hölder continuity of h in the x, s_i and t_i directions is stronger than this).

Also, the function $(\tau, x) \mapsto \phi(\tau, x)$, defined on $\{\tau \ge 1, x \in \mathbb{R}^d\}$, is Hölder continuous of any order less than $1 - d/4$.

REMARK. Property (i) implies that for all τ, $\varphi(\tau, x, B_0) = \phi(\tau, x)$ for almost all x.

Theorem 1 shows that the statements of Theorems 1 and 3 of Rosen (1983) hold quasi-everywhere. Since we are here concerned with quasi-everywhere results, we now give results demonstrating that the statements of Theorems 2 and 4 of Rosen (1983), which are concerned with Hölder continuity of the self-intersection local time in x and B, respectively, hold quasi-everywhere. Our results concern the Hölder continuity of $\varphi(\tau, x, B)$ in x or B for all τ and are stronger than those which would be obtained by direct application of Theorem 1. In the next two theorems B_0 is as in the hypothesis for Theorem 1, and φ and ϕ are as described in Theorem 1.

THEOREM 2. *The function $\phi(\tau, \cdot)$ is Hölder continuous of any order less than $2 - d/2$ for all $\tau \ge 1$, almost surely.*

THEOREM 3. *Given $\tau_1 \geq 1$, there are a finite constant C and a finite random variable δ such that with probability* 1, *for every square $B \subset \Delta$ of the form*

$$B = (p2^{-n}, (p+1)2^{-n}) \times (q2^{-n}, (q+1)2^{-n}),$$

where p and q are integers and $2^{-n} < \delta$, we have

$$\varphi(\tau, x, B) < C(\lambda^2(B))^{1-d/4}|\log \lambda^2(B)|^2, \quad \text{all } x \in \mathbb{R}^d,\ \tau \in [1, \tau_1].$$

COROLLARY. *Let B_1 be a compact set in Δ and let $\tau_1 \geq 1$. Then there exists a finite constant C and a positive random variable δ such that with probability* 1, *for every square B of the form $B = (a, a+h) \times (b, b+h) \subset B_1$ with $h < \delta$ we have*

$$\varphi(\tau, x, B) < C(\lambda^2(B))^{1-d/4}|\log \lambda^2(B)|^2, \qquad x \in \mathbb{R}^d,\ \tau \in [1, \tau_1].$$

REMARK. This last result is a global Hölder condition in the set variable B, corresponding to Theorem 3 of Geman, Horowitz and Rosen (1984); our bound, holding simultaneously for all τ, differs by only a constant from theirs. However, our methods do not give us any local Hölder conditions on B such as in Theorem 2 of Geman, Horowitz and Rosen (1984) or Theorem 5 of Rosen (1983).

Our next result states that we can find an open bounded set B_0 such that self-intersection local time of a Brownian path in B_0 exists and is strictly positive for quasi-every path.

THEOREM 4. *There exists an open set B_0 in the bounded triangular set $\Delta = \{(s, t): 0 \leq s < t \leq 1\}$, such that B_0 satisfies the hypothesis of Theorems* 1–3, *and*

$$\phi(\tau, x) = \varphi(\tau, x, B_0) \quad \text{all } x \in \mathbb{R}^d,\ \tau \geq 1, \text{ almost surely},$$

and

$$\phi(\tau, 0) > 0 \qquad \tau \geq 1, \text{ almost surely}.$$

From Theorem 4, we can deduce that quasi-every Brownian path in $\mathbb{R}^d$ intersects itself before time 1 (or in any time-interval). In fact, the statements of Theorems 6 and 7 of Rosen (1983) on the Hausdorff dimension of the set of self-intersections, originally due to Taylor (1966) ($d = 2$) and Fristedt (1967) ($d = 3$), hold quasi-everywhere: Again setting $\Delta = \{(s, t): 0 \leq s < t \leq 1\}$, we have

THEOREM 5. *The following holds for quasi-every path $b(\cdot)$ in W_0^d ($d = 2$ or $d = 3$):*

$$\dim\{(s, t) \in \Delta: b(s) = b(t)\} = 2 - d/2,$$

$$\dim\{x: x = b(s) = b(t), \text{ some } (s, t) \in \Delta\} = 4 - d.$$

3. Proof of Theorem 1. We write $(\overline{s,t})$ for $(s_1, t_1, \dots, s_{2k}, t_{2k})$ and $\bar{u}$ for $(u_1, \dots, u_{2k})$. For $K \in Z_+$, $\gamma > 0$ and $B \subset \Delta$, we have by definition

$$J_{X^\tau}(2k, \gamma, B) = \int_{\mathbb{R}^{2kd}} \int_{B^{2k}} \left(\prod_{l=1}^{2k} |u_l|^\gamma \right) \times \exp\left\{ -\left(\tfrac{1}{2}\right) \mathrm{Var}\left(\sum_{j=1}^{2k} u_j \cdot X^\tau(s_j, t_j) \right) \right\} d(\overline{s,t})\, d\bar{u}. \tag{3.1}$$

$X^\tau(\cdot)$ has the same law as $\tau^{1/2} X^1(\cdot) = \tau^{1/2} X(\cdot)$ by a scaling property for the Brownian sheet. Hence by the change of variable $u_j' = \tau^{1/2} u_j$,

$$J_{X^\tau}(k, \gamma, B) \le J_X(k, \gamma, B), \qquad \tau \ge 1. \tag{3.2}$$

Let $\mathscr{R}(\Delta)$ denote the set of rectangles in Δ. Given B in $\mathscr{R}(\Delta)$ and $\tau \ge 1$, X^τ has a local time relative to B, since its occupation measure has square-integrable Fourier transform almost surely [see the proof of Rosen (1983), Theorem 1].

By Lemma 1 $J_X(2k, 0, B)$ is finite $(k \in Z_+)$, so by (3.2) $J_{X^\tau}(2k, 0, B) < \infty$ for all $\tau \ge 1$. Hence, by the definition (3.1) (with $\gamma = 0$) and Theorem 4.1 of Berman (1969) (generalised to two parameters, $d > 1$ and $k > 1$), we may for each $x \in \mathbb{R}^d$ and $\tau \ge 1$ obtain a random variable $\varphi(\tau, \mathrm{x}, \mathrm{B})$ defined by

$$\varphi(\tau, x, B) = \int_{u \in \mathbb{R}^d} \int_{(s,t) \in B} \exp\{iu \cdot (X^\tau(s,t) - x)\}\, ds\, dt\, du \tag{3.3}$$

in the sense that for $k \in Z_+$, $\varphi(\tau, x, B)$ is an $L^{2k}(dP)$ limit of real-valued random variables given by restricting the integral in (3.3) to $\{|u| \le m\}$. It follows that in taking the $2k$th moment of φ we may take the expectation inside the multiple improper integral; we shall do this below without comment.

Define $\phi(\tau, x)$ in the same way [i.e., by (3.3), with B replaced by B_0].

Define the distance between two rectangles to be the maximum distance between corresponding corners. We shall use Kolmogorov's lemma to show that $\varphi(\tau, x, B)$ can be modified to be jointly continuous in x, τ and B. Fix $k \in Z_+$. We make the following estimates of $(2k)$th moments.

First, as in Rosen [(1983), page 332], for x and y in $\mathbb{R}^d$, $\tau \ge 1$, $B \in \mathscr{R}(\Delta)$ and $0 \le \gamma \le 1$,

$$\begin{aligned} E|\varphi(\tau, x, B) - \varphi(\tau, y, B)|^{2k} &\le 2^{2k} |y - x|^{2k\gamma} J_{X^\tau}(2k, \gamma, B) \\ &\le 2^{2k} |y - x|^{2k\gamma} J_X(2k, \gamma, B). \end{aligned} \tag{3.4}$$

Second, for $y \in \mathbb{R}^d$, $\tau > \sigma \ge 1$ and $B \in \mathscr{R}(\Delta)$,

$$\begin{aligned} &E|\varphi(\tau, y, B) - \varphi(\sigma, y, B)|^{2k} \\ &\quad = \int_{\mathbb{R}^{2kd}} \int_{B^{2k}} \left(\prod_{l=1}^{2k} \exp(-iu_l \cdot y) \right) \\ &\qquad \times E \prod_{j=1}^{2k} \left[\exp\{iu_j \cdot X^\tau(s_j, t_j)\} - \exp\{iu_j \cdot X^\sigma(s_j, t_j)\} \right] d(\overline{s,t})\, d\bar{u}. \end{aligned} \tag{3.5}$$

The first term in the integrand in (3.5) has unit modulus, while the second term is equal to the expectation of

$$\left[\sum_{j=1}^{2k} \exp\{iu_j \cdot X^{\sigma}(s_j, t_j)\}\right] \prod_{l=1}^{2k} \left(\exp\{iu_l \cdot (X^{\tau}(s_l, t_l) - X^{\sigma}(s_l, t_l))\} - 1\right). \tag{3.6}$$

The two factors in (3.6) are independent of one another and the expectation of the first factor is $\exp\{-\frac{1}{2}\operatorname{Var}(\Sigma_{j=1}^{2k} u_j \cdot X^{\sigma}(s_j, t_j))\}$.

By the estimate $|e^{i\theta} - 1| \le 2|\theta|^{\gamma}$, $0 \le \gamma \le 1$, the modulus of the second factor in (3.6) is no greater than 2^{2k} times

$$\prod_{l=1}^{2k} |u_l|^{\gamma} |X^{\tau}(s_l, t_l) - X^{\sigma}(s_l, t_l)|^{\gamma}.$$

Now $(\tau - \sigma)^{1/2}(b^{\tau}(t) - b^{\sigma}(t))_{t \ge 0}$ is a Brownian motion, so

$$\begin{aligned} &\sup\left\{E \prod_{l=1}^{2k} (X^{\tau}(s_l, t_l) - X^{\sigma}(s_l, t_l))\right\} \\ &\qquad = (\tau - \sigma)^{k\gamma} \sup\left\{E \prod_{l=1}^{2k} |X(s_l, t_l)|^{\gamma}\right\}, \end{aligned} \tag{3.7}$$

where the supremum on each side of (3.7) is over $\{\overline{(s, t)}: (s_l, t_l) \in \Delta, 1 \le l \le 2k\}$. Hence the modulus of the expectation of the second factor in (3.7) is bounded by a constant multiple of $|\tau - \sigma|^{k\gamma}$. Putting together our estimates for the integrand in (3.5) and applying (3.2), we have

$$\begin{aligned} E|\varphi(\tau, y, B) - \varphi(\sigma, y, B)|^{2k} &\le c J_X(2k, \gamma, B)|\tau - \sigma|^{k\gamma}, \\ & y \in \mathbb{R}^d,\ \tau > \sigma \ge 1, \end{aligned} \tag{3.8}$$

where c depends only on k and γ.

Third, suppose $y \in \mathbb{R}^d$, $\sigma \ge 1$ and $B \in \mathscr{R}(B_0)$. Then by definition

$$E|\varphi(\sigma, y, B)|^{2k} \le J_X(2k, 0, B).$$

By Lemma 1,

$$E|\varphi(\sigma, y, B)|^{2k} \le c_0^{2k}((2k)!)^2 (\lambda^2(B))^{(2-(d/2))k}, \tag{3.9}$$

where c_0 depends only on d and τ_1.

By the above three estimates (3.4), (3.8) and (3.9), combined with Minkowski's inequality in L^{2k}(prob.), we find that if $0 \le \gamma < 2 - d/2$, then for $\tau \ge \sigma \ge 1$, x and y in $\mathbb{R}^d$ and rectangles $P = R(s_1, t_1, s_2, t_2)$ and $P' = R(s_1', t_1', s_2', t_2')$ in $\mathscr{R}(\Delta)$,

$$\begin{aligned} E|\varphi(\tau, y, P') - \varphi(\sigma, x, P)|^{2k} &< \text{const.}\big(|y - x|^{\gamma} + |\tau - \sigma|^{\gamma/2} \\ &\qquad + (\operatorname{dist.}(P, P'))^{1-d/4}\big)^{2k} \\ &\le \text{const.}|(\tau, y, s_1', t_1', s_2', t_2') \\ &\qquad - (\sigma, x, s_1, s_2, t_1, t_2)|^{\gamma k}. \end{aligned}$$

Here the constant depends only on k and γ.

Take a modification of φ (also denoted φ) so that h given by $h(\tau, x, s_1, s_2, t_1, t_2) \doteq \varphi(\tau, x, R(s_1, s_2, t_1, t_2))$ is a separable process [see Doob (1953), Theorem 2.4]. By Kolmogorov's lemma [see Meyer (1981) and Garsia (1971)], h is almost surely Hölder continuous (on the domain of interest in $\mathbb{R}^{5+d}$) of any order less than $(k\gamma - 5 - d)/2k$, and hence (by allowing $k \to \infty$ and $\gamma \to 2 - d/2$), of any order less than $1 - d/4$.

In a similar way we can take the random function $\phi(\tau, x)$ to be (Hölder) continuous on $\{(\tau, x): \tau \geq 1, x \in \mathbb{R}^d\}$, so property (iii) in the statement of the theorem holds.

Let $\mathscr{Q}(\Delta)$ denote those rectangles in Δ with rational corners. The following argument holds for almost all ω. For all $B \in \mathscr{Q}(\Delta)$ and rational $\tau \geq 1$, $\varphi(\tau, x, B)$ is the local time of X^τ relative to B [Berman (1969), Theorem 4.3], so

$$(3.10) \qquad \int_B f(X^\tau(s,t))\, ds\, dt = \int_{\mathbb{R}^d} f(x)\varphi(\tau, x, B)\, dx, \qquad f \in C_0^\infty(\mathbb{R}^d).$$

Following Shigekawa (1984), we deduce from continuity in τ that (3.10) holds for all $\tau \geq 1$ and $B \in \mathscr{Q}(\Delta)$. Hence $\varphi(\tau, \cdot, B)$ is the local time of X^τ relative to B for all $\tau \geq 1$ and $B \in \mathscr{Q}(\Delta)$. Similarly, $\phi(\tau, \cdot)$ is the local time of X^τ relative to B_0 for all $\tau \geq 1$.

By the continuity of $\varphi(\tau, x, B)$ in the corners of B and routine measure theory, for each τ and x $(\varphi(\tau, x, B), B \in \mathscr{Q}(\Delta))$ extends to a sigma-finite measure on $\mathscr{B}(\Delta)$ [also denoted $\varphi(\tau, x, B)$]. By a monotone class argument on the class of sets for which (3.10) holds, $\varphi(\tau, \cdot, B)$ is the desired local time for all $B \in \mathscr{B}(\Delta)$ and $\tau \geq 1$.

4. Proof of Theorems 2 and 3.

Proof of Theorem 2. Let K be a cube in $\mathbb{R}^d$, $\tau_1 > 1$ and $\gamma < 2 - d/2$. We shall show that $\sup\{|\phi(\tau, x) - \phi(\tau, y)|/|x - y|^\gamma: x \in K, y \in K, 1 \leq \tau \leq \tau_1\}$ is finite.

Choose γ', $\gamma < \gamma' < 2 - d/2$. For $n \in Z_+$, define

$$D_n = \{x \in K: x \text{ has coordinates of the form } m2^{-n}, m \in Z\}$$

and

$$E_n = \{\tau \in [1, \tau_1]: \tau = m2^{-3n}, \text{some } m \in Z\}.$$

By Chebyshev's inequality, for all x and y in K and $k \in Z_+$,

$$(4.1) \qquad \begin{aligned} P[|\phi(\tau, x) - \phi(\tau, y)| \geq |x - y|^\gamma] &\leq E|\phi(\tau, x) - \phi(\tau, y)|^{2k}|x - y|^{-2\gamma k} \\ &\leq \text{const.}|x - y|^{2k(\gamma' - \gamma)} \end{aligned}$$

by the estimate (3.4) from the proof of Theorem 1.

Define A_n to be the event that $|\phi(\tau, x) - \phi(\tau, y)| > |x - y|^\gamma$, for some $\tau \in E_n$ and some neighboring x and y on the lattice D_n. The number of such triples

(τ, x, y) is $O(2^{3n} \times 2^{dn})$ as $n \to \infty$, so by (4.1),

$$P(A_n) \le \text{const.}2^{(3+d)n} \times (2^{-n})^{2k(\gamma'-\gamma)}.$$

Hence if k is chosen to be large enough, $\Sigma_n P(A_n)$ converges and so by Borel–Cantelli, there exists almost surely n_0 such that for all $n \le n_0$, $\tau \in E_n$ and neighboring x and y in the lattice D_n,

$$|\phi(\tau, x) - \phi(\tau, y)| \le |x - y|^{\gamma} = 2^{-\gamma n}. \tag{4.2}$$

Fix an arbitrary τ in $[1, \tau_1]$. There exists a sequence $\tau(n)$ such that for all n, $\tau(n) \in E_n$ and $|\tau(n) - \tau| < 2^{-3n}$. ϕ is jointly Hölder continuous of order $\gamma/2$ in x and τ [Theorem 1, property (iii)], so for a suitable constant, for all $x \in K$ and all n, we have

$$|\phi(\tau, x) - \phi(\tau(n), x)| < \text{const.}(2^{-3n})^{\gamma/2}. \tag{4.3}$$

By applying (4.2) and (4.3), we find that for all $n \ge n_0$ and $\tau \in [1, \tau_1]$, and all neighboring x and y in the lattice D_n,

$$\begin{aligned} |\phi(\tau, x) - \phi(\tau, y)| &< \text{const.}2^{-\gamma n} \\ &= \text{const.}|x - y|^{\gamma}. \end{aligned} \tag{4.4}$$

We can now deduce that (4.4) still holds (with a new constant) for arbitrary x and y in K such that $|x - y| < 2^{-n}$. This is done by a standard "binary expansion" argument [see, for example, the end of Section 1.6 of McKean (1969)]. □

PROOF OF THEOREM 3. Our proof is broadly similar to that of Theorem 3 of Geman, Horowitz and Rosen (1984). In the proof of Theorem 1 we obtained a uniform estimate (3.9) for the $2k$th moment of $\varphi(\tau, x, B)$ for rectangular B. This leads to a uniform estimate for $E\exp(\zeta\varphi(\tau, x, B)/(\lambda^2(B))^{1-d/4})$ for rectangular B and suitable choice of ζ; by Chebyshev's inequality we may find positive constants c and c' such that

$$P\Big[\varphi(\tau, x, B) \ge z^2\big(\lambda^2(B)\big)^{1-d/4}\Big] \le ce^{-c'z} \tag{4.5}$$

for all rectangles $B \subset \Delta$, all $\tau \in [1, \tau_1]$, $x \in \mathbb{R}^d$ and $z \ge 0$. [See Lemma 3.14 of Geman, Horowitz and Rosen (1984) for details.]

Let Z^d be the integer lattice in $\mathbb{R}^d$. Let

$$D_n = \{x \in 2^{-n}Z^d : |x| \le n\}, \ E_n = \{\tau \in 2^{-3n}Z : 1 \le \tau \le \tau_1\}.$$

Let $\mathscr{S}_n$ be the collection of all squares B in Δ of the form $B = (i2^{-n}, (i+1)2^{-n}) \times (j2^{-n}, (j+1)2^{-n})$. Then the cardinalities of $\mathscr{S}_n$ satisfy $\#(\mathscr{S}_n) = O(2^{2n})$ as $n \to \infty$. For $k \in Z_+$, $\tau \in [1, \tau_1]$ and $B \in \mathscr{S}_n$, we have by (4.5)

$$\begin{aligned} &P\Big[\varphi(\tau, x, B) \ge C^2\big(\lambda^2(B)\big)^{1-d/4}|\log\lambda^2(B)|^2\Big] \\ &\qquad \le c\exp\{-c'C|\log\lambda^2(B)|\} \le c2^{-2c'Cn}. \end{aligned}$$

Hence

$$P\left[\varphi(\tau, x, B) \geq C^2(\lambda_2(B))^{1-d/4}|\log \lambda^2(B)|^2, \text{some } B \in \mathscr{S}_n, \tau \in E_n, x \in D_{3n}\right]$$
$$\leq \text{const. } n^d 2^{(5+3d-2c'C)n},$$

so provided C is chosen large enough, we may apply Borel–Cantelli. Then there exists almost surely $n_0 \in Z_+$ such that for $n \geq n_0$, $B \in \mathscr{S}_n$, $\tau \in E_n$ and $x \in D_n$,

$$\begin{aligned}\varphi(\tau, x, B) &\leq C(\lambda^2(B))^{1-d/4}|\log(\lambda^2(B))|^2 \\ &= C2^{-(2-d/2)n}(2n\log 2)^2.\end{aligned} \tag{4.6}$$

We can also take n_0 so that $|X^\tau(s,t)| \leq n_0$ for all $\tau \in [1, \tau_1]$ and $(s,t) \in \Delta$ [using the joint continuity of $X^\tau(s,t)$ in τ, s and t]. For almost all ω we may argue as follows: For $|x| \geq n_0$ and $\tau \in [1, \tau_1]$, $\varphi(\tau, x, B) = 0$ for all squares $B \subset \Delta$. Now take arbitrary fixed $\tau \in [1, \tau_1]$ and x in $\{|x| \leq n_0\}$. For each n greater than n_0, take elements $\tau(n)$ and $x(n)$ of E_n and D_{3n}, respectively, such that $|\tau(n) - \tau| < 2^{-3n}$ and $|x(n) - x| < 2^{-3n}$. By the joint Hölder continuity (of any order γ less than $1 - d/4$) of $\varphi(\tau, x, B)$ [property (iii) in the statement of Theorem 1], we have

$$|\varphi(\tau, x, B) - \varphi(\tau(n), x(n), B)| < \text{const.} 2^{-3n\gamma}, \qquad n \geq n_0, B \in \mathscr{S}_n, \tag{4.7}$$

where the constant in (4.7) depends only on γ, τ_1 and ω. By taking γ so $3\gamma > 2 - d/2$, the right-hand side of (4.7) is dominated by that of (4.6) as $n \to \infty$. Hence by combining the two we have, for some n_1 (independent of x and τ) and a new constant C,

$$\begin{aligned}\varphi(\tau, x, B) &\leq C2^{-(2-d/2)n}(2n\log 2)^2 \\ &= C(\lambda^2(B))^{1-d/4}|\log \lambda^2(B)|^2, \quad \text{all } n \geq n_1, B \in \mathscr{S}_n. \qquad \square\end{aligned}$$

Proof of the corollary to Theorem 3. If B_1 is compact and $B_1 \subset \Delta$, then some ε-neighbourhood of B_1 is contained in Δ. For any square B in B_1 of side $h < \varepsilon/2$, take ν so $2^{-\nu-1} \leq h < 2^{-\nu}$. Then B is contained in the union of at most four rectangles in $\mathscr{S}_\nu$, and the result then follows from Theorem 3. □

5. The probability distribution of the intersection local time of two Brownian motions. In this section we prove some results on the probability distribution of the intersection local time of two independent Brownian motions starting at the origin, as introduced by Geman, Horowitz and Rosen (1984). In particular we consider the probability that the local time (relative to the unit square in the time domain) at 0 is very small. A scaling argument allows one to relate these results to results on the local time relative to small (or large) squares in the time domain.

We are here concerned with probability distributions; for two random variables X_1 and X_2, we shall say that $X_1 \geq_L X_2$ (or $X_2 \leq_L X_1$) if for all $x \in \mathbb{R}$, $P(X_1 > x) \geq P(X_2 > x)$. We shall say that $X_1 =_L X_2$ if $X_1 \geq_L X_2$ and $X_2 \geq_L X_1$.

Let d be 2 or 3 and let b_1 and b_2 be independent Brownian motions in $\mathbb{R}^d$. Define the random function $Y: \mathbb{R}^2_+ \to \mathbb{R}^d$ by

$$Y(s,t) = b_2(t) - b_1(s), \qquad s \geq 0,\ t \geq 0.$$

In the terminology of Geman, Horowitz and Rosen (1984), Y is a confluent Brownian motion. Given a bounded Borel set B in $\mathbb{R}^2_+$, define the random function

$$\alpha(x,B), \qquad x \in \mathbb{R}^d,$$

to be the continuous version of the local time relative to B of the confluent Brownian motion Y. Such a local time exists almost surely [see Geman, Horowitz and Rosen (1984)]. This is a special case of our Theorem 1. In fact the local time of the confluent Brownian motion Y, relative to the unit square $(0,1) \times (0,1)$, plainly has the same distribution as the self-intersection local time of a single Brownian motion in $\mathbb{R}^d$, relative to any unit square in $\mathbb{R}^2_+$ with sides parallel to the axes and lower right-hand corner on the diagonal $\{s = t\}$.

For $h > 0$, define the set $Q_h \subset \mathbb{R}^2_+$ by

$$Q_h = (0,h) \times (0,h), \qquad h > 0.$$

We have the scaling property:

LEMMA 5.1. *For all $\tau \geq 1$ and $h > 0$, the random function*

$$h^2(\tau h)^{-d/2}\alpha\big((\tau h)^{-1/2}x, Q_1\big), \qquad x \in \mathbb{R}^d,$$

has the same distribution in $C(\mathbb{R}^d)$ as the continuous version of the local time, relative to Q_h, of the process

$$Y^\tau(s,t) = b_2(\tau,t) - b_1(\tau,s), \qquad s \geq 0,\ t \geq 0,$$

where b_1 and b_2 are independent two-parameter Wiener processes in $\mathbb{R}^d$.

In particular, for all $x \in \mathbb{R}^d$ and $h > 0$,

$$h^{2-d/2}\alpha\big(h^{-1/2}x, Q_1\big) =_L \alpha(x, Q_h). \tag{5.1}$$

PROOF. By routine scaling of the Brownian sheet, the random functions

$$Y^\tau(s,t) \quad \text{and} \quad (\tau h)^{1/2}Y^1(h^{-1}s, h^{-1}t)$$

have the same distribution, considered as random elements of $C(\mathbb{R}^2_+)$. The result follows by a change of variables. □

LEMMA 5.2. *For $h > 0$ and $x \in \mathbb{R}^d$,*

$$E[\alpha(x,Q_h)] = (2\pi)^{-d}\int_{\mathbb{R}^d}\cos(u\cdot x)\big(2\{1 - \exp(-h|u|^2/2)\}/|u|^2\big)^2\,du. \tag{5.2}$$

In particular $P[\alpha(x,Q_h) > 0] > 0$.

Proof. As in the proof of Theorem 1, α can be expressed by the formal Fourier inversion formula,

$$\alpha(x, Q_h) = (2\pi)^{-d} \int_{u \in \mathbb{R}^d} \int_{(s,t) \in Q_h} \exp\{iu \cdot (Y(s,t) - x)\}\, ds\, dt\, du$$

in the sense of an L^{2k} limit of integrals over $|u| \leq m$ $(m \to \infty)$. Taking expectations, we have

$$E[\alpha(x, Q_h)] = \int_{\mathbb{R}^d} e^{-iu \cdot x} \left(\int_0^h \int_0^h \exp(-|u|^2(t+s)/2)\, ds\, dt \right) du$$

and (5.2) follows. The integrand in the right-hand side of (5.2) is the product of $\cos(u \cdot x)$ and a strictly decreasing function of $|u|$. It follows that $E[\alpha(x, Q_h)] > 0$, so that $P[\alpha(x, Q_h) > 0] > 0$. □

We now show that $\alpha(0, Q_h)$ is strictly positive, almost surely.

Lemma 5.3. *For all $h > 0$, $P[\alpha(0, Q_h) = 0] = 0$.*

This is Theorem 5.3(a) of Geman, Horowitz and Rosen (1984). We outline a simpler proof as follows: By the scaling property (5.1) the probability in question is independent of h. But

$$P[\alpha(0, Q_{2h}) = 0] \leq P[\alpha(0, Q_h) = 0, \alpha(0, (h, 2h) \times (h, 2h)) = 0].$$

Now condition on $\Sigma\{b_1(t), b_2(s): 0 \leq s, t \leq h\}$. By the Markov property of Brownian motion, $\alpha(0, (h, 2h) \times (h, 2h))$ depends on this sigma-field only via $(b_2(h) - b_1(h))$ [see Lemma 5.5 of Geman, Horowitz and Rosen (1984)]. Applying Lemma 5.2 gives us $P[\alpha(0, Q_{2h}) = 0] < P[\alpha(0, Q_h) = 0]$ unless $P[\alpha(0, Q_h) = 0] = 0$.

Our next result, which we prove by a coupling argument, states that $\alpha(\cdot, Q_h)$ is stochastically monotone in h.

Proposition 5.4. *Let $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$. If $|x| \geq |y|$, then $\alpha(y, Q_h) \geq_L \alpha(x, Q_h)$ for all $h > 0$.*

Proof. By the rotational invariance of Brownian motion, if $|x| = |y|$, then $\alpha(y, Q_h) =_L \alpha(x, Q_h)$. Hence it suffices to consider the case $|x| > 0$, $y = \lambda x$ for $0 \leq \lambda < 1$.

Given two paths b and $b': \mathbb{R}_+ \to \mathbb{R}^d$ such that the function $((s,t) \mapsto b'(t) - b(s))$ has a local time, relative to the set B, with a continuous version, denote the value of this version of the local time at 0 by $\beta(b, b', B)$.

We are required to prove that if $b(\cdot)$, $b_x(\cdot)$ and $b_y(\cdot)$ are Brownian motions starting at 0, x and y, respectively, where $y = \lambda x$ and $0 \leq \lambda < 1$, and if the path b is independent of the paths b_1 and b_2, then

$$\beta(b, b_x, Q_h) \leq_L \beta(b, b_y, Q_h).$$

Let $P = \{z \in \mathbb{R}^d: |z - x| = |z - y|\}$. Let $\rho: \mathbb{R}^d \to \mathbb{R}^d$ be reflection in P. Without loss of generality, b_y is coupled to b_x by defining the stopping time

$T = \inf\{t: b_x(t) \in P\}$ and setting

$$b_y(t) = \rho(b_x(t)), \qquad t \le T,$$
$$b_y(t) = b_x(t), \qquad t \ge T.$$

Then b_x and b_y are identical after time T, so it suffices to prove that, setting $T_h = \min\{T, h\}$,

$$\beta(b, b_x, [0, h] \times [0, T_h]) \le_L \beta(b, b_y, [0, h] \times [0, T_h]).$$

Define the stopping times $S = \inf\{s: b(s) \in P\}$, $S_h = \min\{S, h\}$, and the path $\bar{b}$ by

$$\bar{b}(s) = b(s), \qquad s \le S,$$
$$\bar{b}(s) = \rho(b(s)), \qquad s \ge S.$$

Then the random path $\bar{b}(\cdot)$ has the same law in $C([0,\infty) \to \mathbb{R}^d)$ as $b(\cdot)$. By definition of S and T, the origin is almost surely not in the compact set $\{b(s) - b_x(t): 0 \le s \le S, 0 \le t \le T\}$, so that we have almost surely

$$\beta(b, b_x, [0, S_h] \times [0, T_h]) = 0.$$

Hence we have almost surely

$$\begin{aligned}\beta(b, b_x, [0, h] \times [0, T_h]) &= \beta(b, b_x, [S_h, h] \times [0, T_h]) \\ &= \beta(\bar{b}, b_y, [S_h, h] \times [0, T_h])\end{aligned}$$

by the definitions of $\bar{b}$ and b_y as reflections. Hence [since $\bar{b}$ and b have the same law in $C([0,\infty) \to \mathbb{R}^d)$],

$$\begin{aligned}\beta(b, b_x, [0, h] \times [0, T_h]) &=_L \beta(b, b_y, [S_h, h] \times [0, T_h]) \\ &\le \beta(b, b_y, [0, h] \times [0, T_h])\end{aligned}$$

as desired. □

We are now able to obtain an estimate (possibly not sharp) for the rate at which the probability that $\alpha(0, Q_1) < \varepsilon_n \to 0$ for a particular sequence $\varepsilon_n \to 0$. This will be needed for the proof of Theorem 4.

LEMMA 5.5. *There exist finite positive constants c_1, c_2 and c_3 such that if we set*

$$\varepsilon_n = c_1^n (n!)^{-2+d/2},$$

then

(5.3) $$P[\alpha(0, Q_1) < \varepsilon_n] < c_2 \exp(-c_3 n).$$

REMARK. The right-hand side of (5.3) is majorized for all n by

$$c_4(\varepsilon_n)^{c_5/\log|\log \varepsilon_n|},$$

for some positive finite constants c_4 and c_5. It is an open question whether there are constants c_4 and c_5 such that

$$P[\alpha(0, Q_1) < \varepsilon] = O\left(c_4 \varepsilon^{c_5/\log|\log \varepsilon|}\right), \qquad \varepsilon \to 0.$$

PROOF OF LEMMA 5.5. Let $a_n = \lambda^n n!$, where λ is a finite positive constant to be chosen later. Let

$$P_n = P\left[\alpha(0, Q_{a_n}) < 1\right].$$

By the scaling property (5.1),

$$P_n = P\left[\alpha(0, Q_1) < a_n^{-2+d/2}\right]. \tag{5.4}$$

Define the square subsets A_n, R_n and D_n of $\mathbb{R}^2_+$ by

$$A_n = Q_{a_n} = (0, a_n) \times (0, a_n),$$
$$R_n = (a_{n-1}, a_n) \times (a_{n-1}, a_n),$$
$$D_n = Q_{(a_n - a_{n-1})}.$$

Let $\mathscr{F}_n$ be the σ-field generated by $\{(b_1(s), b_2(t)) : (s, t) \in A_n\}$. Let

$$U_n = b_2(a_n) - b_1(a_n).$$

Then for all $n \geq 1$, the event $\{\alpha(0, A_{n+1}) < 1\}$ is contained in the union of the events

$$\left\{|U_n| > a_{n+1}^{1/2}\right\} \quad \text{and} \quad \left\{|U_n| \leq a_{n+1}^{1/2}\right\} \cap \left\{\alpha(0, A_n) < 1\right\} \cap \left\{\alpha(0, R_{n+1}) < 1\right\}.$$

Taking probabilities and conditioning on $\mathscr{F}_n$, we have

$$P_{n+1} \leq P\left[|U_n| > a_{n+1}^{1/2}\right] + \int_{\Omega_n} P\left[\alpha(0, R_{n+1}) < 1 | \mathscr{F}_n\right] dP, \tag{5.5}$$

where

$$\Omega_n = \left\{|U_n| \leq a_{n+1}^{1/2}\right\} \cap \left\{\alpha(0, A_n) < 1\right\} \in \mathscr{F}_n.$$

Since $\alpha(0, R_{n+1})$ depends on $\mathscr{F}_n$ only via U_n,

$$P\left[\{\alpha(0, R_{n+1}) < 1\} | \mathscr{F}_n\right] = P\left[\alpha(x, D_{n+1}) < 1\right] \quad \text{a.s.},$$

where $x = -U_n$, so $|x| \leq a_{n+1}^{1/2}$ for $\omega \in \Omega_n$. But for all $x \in \mathbb{R}^d$, the scaling property (5.1) implies that

$$\begin{aligned} &P\left[\alpha(x, D_{n+1}) < 1\right] \\ &\quad = P\left[\alpha\left((a_{n+1} - a_n)^{-1/2} x, Q_1\right) < (a_{n+1} - a_n)^{-2+d/2}\right]. \end{aligned} \tag{5.6}$$

Since $\alpha(x, Q_1)$ is stochastically decreasing in $|x|$ (Proposition 5.4), for large n and all x such that $|x| \leq a_{n+1}^{1/2}$ the right-hand side of (5.6) is majorized by

$$P\left[\alpha(x_0, Q_1) < (a_{n+1} - a_n)^{-2+d/2}\right],$$

where x_0 is an arbitrary fixed vector in $\mathbb{R}^d$ of length greater than 1, so that for large n, $|x_0| > [a_{n+1}/(a_{n+1} - a_n)]^{1/2}$.

Now $(a_{n+1} - a_n)^{-2+d/2} \to 0$ as $n \to \infty$, so since $\alpha(x_0, Q_1) > 0$ with positive probability (Lemma 5.2), there exists $c > 0$ such that

$$\lim_{n \to \infty} P\left[\alpha(x_0, Q_1) < (a_{n+1} - a_n)^{-2+d/2}\right] = P\left[\alpha(x_0, Q_1) = 0\right] < e^{-c}.$$

Hence for all large enough n and almost all $\omega \in \Omega_n$, the integrand in the second term of the right-hand side of (5.5) is less than e^{-c}, and so for large enough n,

$$\int_{\Omega_n} P\left[\alpha(0, R_{n+1}) < 1 | \mathscr{F}_n\right] < e^{-c} P(\Omega_n) \le e^{-c} P_n. \tag{5.7}$$

As for the first term in the right-hand side of (5.5),

$$P\left[|b_2(a_n) - b_1(a_n)| > a_{n+1}^{1/2}\right] \le \text{const.} \int_{u_n}^{\infty} x^2 \exp(-x^2/2)\, dx,$$

where $u_n = (a_{n+1}/2a_n)^{1/2} = (\lambda/2)^{1/2}(n+1)^{1/2}$. So integrating by parts,

$$P\left[|b_2(a_n) - b_1(a_n)| > a_{n+1}^{1/2}\right] \le \text{const.}(n+1)^{1/2} \exp\{-\lambda(n+1)/4\}, \quad n \to \infty. \tag{5.8}$$

Applying the estimates (5.7) and (5.8) to (5.5), we obtain

$$P_{n+1} \le c'(n+1)\exp\{-\lambda(n+1)/4\} + e^{-c} P_n, \quad n \text{ large}. \tag{5.9}$$

Let $Q_n = e^{nc} P_n$. Applying (5.9), we find that $\Sigma_1^{\infty}(Q_{n+1} - Q_n)_+ < \infty$, provided that $\lambda > 4c$.

Hence $\{Q_n : n \ge 1\}$ is bounded and there exists $c' > 0$ such that $P_n \le c' e^{-cn}$. By (5.4), the result (5.3) is proved with $\varepsilon_n = (\lambda^{-2+d/2})^n (n!)^{-2+d/2}$. □

6. Proof that the intersection local time is strictly positive.

REMARK. As was pointed out by a referee, it is easy to deduce from Theorem 1 that for quasi-every path in $\mathbb{R}^d$ ($d = 2$ or $d = 3$), the self-intersection local time relative to the unbounded upper triangle is strictly positive. Set

$$A_n = \left[n, n + \tfrac{1}{2}\right] \times \left[n + \tfrac{1}{2}, n + 1\right], \qquad n = 1, 2, 3, \ldots.$$

Let $(B_\tau(\cdot))_{\tau \ge 0}$ denote an Ornstein–Uhlenbeck process in W_0^d, with initial distribution given by Wiener measure. Denote by $\Psi(\tau, \cdot, A_n)$ the (jointly continuous) self-intersection local time of $B_\tau(\cdot)$ relative to A_n, as obtained in Theorem 1. Then $\Psi(\tau, 0, A_1)$ is continuous in τ and $\Psi(0, 0, A_1) > 0$ with positive probability (see Lemma 5.2). So for some $h > 0$,

$$P\{\Psi(\tau, 0, A_1) > 0, \tau \in [0, h]\} > 0.$$

For $n = 2, 3, 4, \ldots$, the processes $(\Psi(\tau, 0, A_n), \tau \ge 0)$ are independent copies of $(\Psi(\tau, 0, A_1), \tau \ge 0)$. Hence

$$P\left[\bigcup_{n=1}^{\infty} \{\Psi(\tau, 0, A_n) > 0, \tau \in [0, h]\}\right] = 1,$$

and similarly for the intervals $[h, 2h]$, $[2h, 3h]$, and so on. Hence for quasi-every Brownian path there exists n such that the self-intersection local time $\Psi(x, A_n)$ satisfies

$$\Psi(0, A_n) > 0.$$

However, to show that the self-intersection local time relative to a bounded time-set is strictly positive, we need a different argument (the familiar scaling arguments of the "almost everywhere" theory do not carry over to the "quasi-everywhere" theory).

PROOF OF THEOREM 4. B_0 is defined as follows. Let

$$B_n = (2 \times 2^{-n}, 3 \times 2^{-n}) \times (3 \times 2^{-n}, 4 \times 2^{-n}), \qquad n = 2, 3, 4, \dots,$$

and

$$B_0 = \bigcup_{n=2}^{\infty} B_n.$$

Suppose $(b^{\tau}(t), t \geq 0, \tau \geq 0)$ is a Brownian sheet in $\mathbb{R}^d$ and (as in earlier sections) set

$$X^{\tau}(s, t) = b^{\tau}(t) - b^{\tau}(s), \qquad X(s, t) = X^1(s, t).$$

By Lemma 1, for $k \in Z_+$ and $0 \leq \gamma < 2 - d/2$, the estimating integral $J_X(k, \gamma, B_0)$ satisfies

$$\begin{aligned}(J_X(2k, \gamma, B_0))^{1/2k} &\leq \sum_{n=2}^{\infty} (J_X(2k, \gamma, B_n))^{1/2k} \\ &\leq c \sum_{n=2}^{\infty} (\lambda^2(B_n))^{(2-(d/2)-\gamma)/2} \\ &< \infty,\end{aligned}$$

so B_0 satisfies the hypothesis of Theorem 1. Let

$$\varphi(\tau, \cdot, B), \qquad (\tau \geq 1, B \text{ Borel in } \Delta) \quad \text{and} \quad \phi(\tau, \cdot), \; \tau \geq 1,$$

be the self-intersection local times of X^{τ} obtained by applying Theorem 1 (using this particular B_0).

We now prove that given $\tau_1 > 1$, $\varphi(\tau, x, B_0)$ is continuous in x for all $\tau \in [1, \tau_1]$, almost surely; for almost all fixed ω the following holds for all $\tau \in [1, \tau_1]$.

Since $\varphi(\tau, x, \cdot)$ is a measure [property (ii) in Theorem 1], we have for all $x \in \mathbb{R}^d$,

$$\varphi(\tau, x, B_0) = \sum_{2}^{\infty} \varphi(\tau, x, B_n).$$

By our uniform Hölder estimate of $\varphi(\tau, x, B)$ for dyadic rectangles B (Theorem 3) and the definition of B_n,

$$\sum_{2}^{\infty} \sup\{\varphi(\tau, x, B_n) : x \in \mathbb{R}^d\} < \infty.$$

Moreover, for all n $\varphi(\tau, \cdot, B_n)$ is continuous by Theorem 1, property (iii), because B_n is a rectangle. Hence, $\varphi(\tau, \cdot, B_0)$ is continuous since it is a uniform limit of continuous functions.

Thus $\phi(\tau, \cdot)$ and $\varphi(\tau, \cdot, B_0)$ are both continuous versions of the local time relative to B_0 of X^τ; hence they are identical, i.e., $\varphi(\tau, x, B_0) = \phi(\tau, x)$ for all x in $\mathbb{R}^d$, $\tau \in [1, \tau_1]$. In particular, $\varphi(\cdot, 0, B_0)$ is Hölder continuous of any order less than $1 - d/4$.

We now prove that $\varphi(\tau, 0, B_0) > 0$ for all $\tau \geq 1$, almost surely. We do this by a Borel–Cantelli argument, using the Hölder continuity of $\varphi(\tau, 0, B_0)$ in τ.

It suffices to prove that for $\tau_1 > 1$,

$$\varphi(\tau, 0, B_0) > 0, \qquad 1 \leq \tau \leq \tau_1, \text{ a.s.}$$

$\varphi(\tau, 0, B_n)$ can be viewed as the intersection local time at the origin of two independent Brownian motions scaled by $\tau^{1/2}$, relative to the square $(0, 2^{-n}) \times (0, 2^{-n})$. Using the notation of Section 5 and the scaling property (Lemma 5.1),

$$\text{(6.1)} \qquad \begin{aligned} \varphi(\tau, 0, B_n) &=_L \tau^{-d/2} 2^{-n(2-d/2)} \alpha(0, Q_1) \\ &\geq_L \tau_1^{-d/2} 2^{-n(2-d/2)} \alpha(0, Q_1), \end{aligned}$$

where $\alpha(0, Q_1)$ is the intersection local time of two Brownian motions relative to the unit square $(0,1) \times (0,1)$.

Let ε_n be as defined in Lemma 5.5, i.e., $\varepsilon_n = c_1^n (n!)^{-2+d/2}$, some suitable $c_1 > 0$. Let $\delta_n = 2^{-n(2-d/2)} \tau_1^{-d/2} \varepsilon_n$. Hence

$$\text{(6.2)} \qquad \begin{aligned} P[\varphi(\tau, 0, B_n) < \delta_n] &\leq P[\alpha(0, Q_1) < \varepsilon_n] \\ &\leq c_2 \exp(-c_3 n), \quad \text{all } n \geq 2, 1 \leq \tau \leq \tau_1, \end{aligned}$$

where c_2 and c_3 are positive constants, by Lemma 5.5.

The random variables $\varphi(\tau, 0, B_n)$, $n \geq 2$, are mutually independent and for $j \leq n$, $\varphi(\tau, 0, B_j) \geq_L \varphi(\tau, 0, B_n)$. So for all $\tau \in [1, \tau_1]$ and $n \geq 2$,

$$\text{(6.3)} \qquad \begin{aligned} P[\varphi(\tau, 0, B_0) < \delta_n] &\leq P[\varphi(\tau, 0, B_j) < \delta_n, 2 \leq j \leq n] \\ &= \prod_{j=2}^{n} P[\varphi(\tau, 0, B_j) < \delta_n] \\ &\leq (P[\varphi(\tau, 0, B_n) < \delta_n])^{n-1} \\ &\leq c_2^{n-1} \exp(-c_3(n^2 - n)), \end{aligned}$$

by (6.2). Let F_n be the set of all τ in $[1, \tau_1]$ of the form $\tau = m\delta_n^5$, where $m \in Z$. Then by (6.3)

$$\text{(6.4)} \qquad \begin{aligned} &P\Big[\bigcup_{\tau \in F_N} \{\varphi(\tau, 0, B_0) < \delta_n\}\Big] \\ &\quad \leq (\tau_1 - 1)\delta_n^{-5} \sup_{1 \leq \tau \leq \tau_1} P[\varphi(\tau, 0, B_0) < \delta_n] \\ &\quad \leq \text{const.}(\text{const.})^n (n!)^{5(2-d/2)} \exp(-c_3(n^2 - n)). \end{aligned}$$

The right-hand side of (6.4) is summable in n. By the Borel–Cantelli lemma, there exists a.s. some n_0 such that

$$\varphi(\tau, 0, B_0) \geq \delta_n, \quad \text{all } n \geq n_0, \ \tau \in F_n. \tag{6.5}$$

But $\varphi(\cdot, 0, B_0)$ is almost surely Hölder continuous of any order less than $1 - d/4$. Hence we have almost surely,

$$\limsup_{\varepsilon \to 0} \{|\varphi(\tau, 0, B_0) - \varphi(\sigma, 0, B_0)|/|\tau - \sigma|^5 : 1 \leq \tau \leq \sigma \leq \tau_1, |\tau - \sigma| < \varepsilon\} = 0. \tag{6.6}$$

Together, (6.5) and (6.6) imply that for all $\tau \in [1, \tau_1]$, $\varphi(\tau, 0, B_0) > 0$, so Theorem 4 is proved. □

7. Proof of Theorem 5. It suffices to prove that for any $\tau_1 > 1$, we have for almost every Brownian sheet $(b^\tau(t), t \geq 0, \tau \geq 0)$ in $\mathbb{R}^d$,

$$\dim\left((X^\tau)^{-1}(0) \cap \Delta\right) = 2 - d/2, \qquad 1 \leq \tau \leq \tau_1, \tag{7.1}$$

and

$$\dim\{x : x = b^\tau(s) = b^\tau(t), \text{some } (s, t) \in \Delta\} = 4 - d, \qquad 1 \leq \tau \leq \tau_1, \tag{7.2}$$

where $X^\tau(s, t) = b^\tau(t) - b^\tau(s)$ and $\Delta = \{(s, t) : 0 < s < t < 1\}$. As for (7.1) we shall start by showing $\dim((X^\tau)^{-1}(0) \cap \Delta) \geq 2 - d/2$.

Let B_0 be the set described in Theorem 4. We can argue as follows for almost all fixed ω and all fixed $\tau \in [1, \tau_1]$, using the fact that $\varphi(\tau, 0, \cdot)$ is a finite measure on the Borel sets in B_0. Firstly, the measure $\varphi(\tau, 0, \cdot)$ is supported by $(X^\tau)^{-1}(0)$. For given any compact rectangle R in $B_0 \setminus (X^\tau)^{-1}(0)$, $\varphi(\tau, \cdot, R)$ is the continuous version of the local time in R of X^τ (Theorem 1); hence, since $X^\tau(R)$ is disjoint from some neighbourhood of the origin, $\varphi(\tau, 0, R) = 0$.

By Theorem 4, $\varphi(\tau, 0, B_0) > 0$, so there exists compact $K \subset B_0$ such that $\varphi(\tau, 0, K) > 0$. Let $\{S_j, j \geq 1\}$ be any countable cover for $K \cap (X^\tau)^{-1}(0)$ by open squares contained in B_0. Since $(X^\tau)^{-1}(0)$ supports $\varphi(\tau, 0, \cdot)$, we have

$$\varphi(\tau, 0, K) \leq \sum_j \varphi(S_j).$$

Thus by the corollary to Theorem 3,

$$h\text{-meas}\left(K \cap (X^\tau)^{-1}(0)\right) \geq \varphi(\tau, 0, K) > 0,$$

where

$$h(t) = Ct^{2-d/2}|\log t|^2, \qquad t > 0.$$

Hence

$$\dim\left(\Delta \cap (X^\tau)^{-1}(0)\right) \geq 2 - d/2.$$

On the other hand, suppose B is a rectangle in the upper triangle Δ (so that Theorem 1 on the existence for all τ of a continuous self-intersection local time for X^τ relative to B applies). For all τ, X^τ is Hölder continuous of any order less than $\frac{1}{2}$, by the Hölder continuity of the Brownian sheet [see Orey and Pruitt

(1973)]. Hence by Lemma 7 of Adler (1978), modified to $d \neq 1$, and the continuity of the self-intersection local time of X^τ, we have

$$\dim\left[B \cap (X^\tau)^{-1}(0)\right] \leq 2 - d/2.$$

Δ is a countable union of such B, so (7.1) follows.

It remains to deduce the formula (7.2) for the Hausdorff dimension of the set of self-intersections of b^τ. To do this we need the following quasi-everywhere version of a lemma of Kaufman (1969).

LEMMA. *Let $\tau_1 > 1$. With probability* 1, *there exists n_0 such that for all $n \geq n_0$, we have the following*: *If $\tau = m2^{-2n} \in [1, \tau_1]$, where $m \in Z$, and D^* is a ball in $\mathbb{R}^d$ of radius $n^{1/2}2^{-n}$, then $b^\tau(k4^{-n}) \in D^*$ for at most $n^{3+\varepsilon}$ values of k in $\{1, \ldots, 4^n\}$.*

PROOF. This is immediate from the estimate in Kaufman (1969) of the probability of the complement of the above event for fixed τ, using Borel–Cantelli.

From the above lemma and the Hölder continuity of the Brownian sheet, we deduce that for large enough n, for all $\tau \in [1, \tau_1]$ and all balls D in $\mathbb{R}^d$ of radius 2^{-n}, $(b^\tau)^{-1}(D)$ is contained in at most $n^{3+\varepsilon}$ intervals of the form $[k4^{-n}, (k+1)4^{-n}]$, for all discs D of radius 2^{-n}. This enables us to deduce (7.2) from (7.1) as explained by Geman, Horowitz and Rosen (1984). □

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