The Annals of Probability
1989, Vol. 17, No. 2, 809–812

ON A PROBLEM OF CSÖRGŐ AND RÉVÉSZ

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Suppose $\{X_n\}$ is an i.i.d. sequence of random variables with mean 0, variance 1 and $S_n = \sum_{i=1}^n X_i$. Let $0 < r < 1$. It is well known that

$$S_n - W(n) = O\left((\log n)^{1/r}\right) \quad \text{a.s.}$$

when $Ee^{t_0|X_1|^r} < \infty$ for some $t_0 > 0$, where $\{W(t), t \geq 0\}$ is the standard Wiener process. We prove that $O((\log n)^{1/r})$ cannot be replaced by $o((\log n)^{1/r})$.

Let $\{X_n, n \geq 1\}$ be an i.i.d. sequence of random variables with mean 0, variance 1 and $S_n = \sum_{i=1}^n X_i$. Let $H(x) > 0$, $x \geq 0$, be a nondecreasing continuous function such that $x^{-2-\varepsilon}H(x)$ is nondecreasing for some $\varepsilon > 0$ and $x^{-1}\log H(x)$ is nonincreasing. Komlós, Major and Tusnády (1975, 1976) and Major (1976) proved that without changing the distribution of $\{S_n, n \geq 0\}$ one can redefine the process $\{S_n, n \geq 0\}$ on a richer probability space together with a standard Wiener process $\{W(t), t \geq 0\}$ such that

$$S_n - W(n) = O(\operatorname{inv} H(n)) \quad \text{a.s.} \tag{1}$$

if $EH(|X_1|) < \infty$.

It is well known that if $H(x) = e^{t_0 x}$ for some $t_0 > 0$, then $O(\cdot)$ of (1) cannot be replaced by $o(\cdot)$ (cf. [1], page 96), while this can be done if $H(x) = x^p$, $p > 2$ (cf. [1], page 109). Thus a natural problem is posed.

PROBLEM [Csörgő and Révész (1981), page 110]. Can $O(\cdot)$ in (1) be replaced by $o(\cdot)$ if $H(x) = \exp(t_0 x^r)$, $t_0 > 0$, $0 < r < 1$?

The answer is negative. We have the following theorem.

THEOREM 1. *Let $\{X_n, n \geq 1\}$ be an i.i.d. sequence of random variables with mean* 0 *and variance* 1. *Let $0 < r < 1$ and $S_n = \sum_{i=1}^n X_i$. Suppose that $Ee^{|X_1|^r} < \infty$ and $Ee^{2|X_1|^r} = \infty$. Then*

$$S_n - W(n) = O\left((\log n)^{1/r}\right) \quad a.s. \tag{2}$$

and $O(\cdot)$ in (2) *cannot be replaced by $o(\cdot)$.*

PROOF. The first part of the conclusion follows from (1). For the second one, assume that

$$S_n - W(n) = o\left((\log n)^{1/r}\right) \quad \text{a.s.} \tag{3}$$

Received February 1988; revised May 1988.
AMS 1980 *subject classification*. Primary 60F15.
Key words and phrases. Invariance principle, increments of partial sums.

Then, by Theorem 1.2.1 of [1] we can get that for every $a > 0$,

$$(4) \qquad \limsup \max_{0 \le n \le N - a_N} \max_{1 \le k \le a_N} |S_{n+k} - S_n| / (2 a_N \log N)^{1/2} = 1 \quad \text{a.s.},$$

where $a_N = [a(\log N)^{-1+2/r}]$.

This implies

$$\limsup_{N \to \infty} |X_N| / (2 a_N \log N)^{1/2} \le 1 \quad \text{a.s.}$$

Namely, we have

$$(5) \qquad P\big(|X_N| \ge 2(a_N \log N)^{1/2} \text{ i.o.}\big) = 0.$$

So, by the Borel–Cantelli lemma

$$\sum_{N=1}^{\infty} P\big(|X_1| \ge 2a(\log N)^{1/r}\big) \le \sum_{N=1}^{\infty} P\big(|X_N| \ge 2(a_N \log N)^{1/2}\big) < \infty,$$

provided $0 < a < 1$. It follows that $E\exp((\frac{1}{2}|X_1|/a)^r) < \infty$ for every $0 < a < 1$, which contradicts the hypothesis $E\exp(2|X_1|^r) = \infty$. Namely, $O(\cdot)$ in (2) cannot be replaced by $o(\cdot)$. □

REMARK 1. Indeed, the answer to the question discussed in Theorem 1 follows almost immediately from an argument of Breiman which is contained also in [1].

Theorem 1 motivates us to consider another question. It is well known that Csörgő and Révész (1981) have established some theorems on the increments of partial sums of i.i.d. r.v.'s via the almost sure invariance principles. Namely,

THEOREM A. *Let $0 < r \le 1$ and $\{X_n, n \ge 1\}$ be a sequence of i.i.d. r.v.'s with mean* 0 *and variance* 1, *satisfying also the conditions,*

(6) *there exists a $t_0 > 0$ such that $E\exp(t_0|X_1|^r) < \infty$,*

(7) *$\{a_N\}$ is a monotonically nondecreasing sequence of integers satisfying $1 \le a_N \le N$, $a_N = O(a_{N-1})$ and*

$$(8) \qquad a_N(\log N)^{1-2/r} \to \infty \quad \text{as } N \to \infty.$$

Then

$$(9) \qquad \limsup_{N \to \infty} \max_{0 \le n \le N} \max_{1 \le k \le a_N} \beta_N |S_{n+k} - S_n| = 1 \quad a.s.,$$

where $\beta_N = (2a_N(\log(N/a_N) + \log\log N))^{-1/2}$.

THEOREM B. *If the conditions* (6) *and* (8) *in Theorem A are replaced by*

$$(6') \qquad E|X_1|^p < \infty, \qquad p > 2,$$

(8′) *there is a constant $c > 0$ such that $a_N \ge cN^{2/p}/\log N$,*

respectively, then the conclusion of Theorem A *remains valid.*

REMARK 2. In the theorems of Csörgő and Révész [1], pages 115–118, n/a_n is assumed to be monotonically nondecreasing, which is not required in [5].

So it is natural to ask whether the restriction on a_N is sharp. To this question we have the following theorem.

THEOREM 2. *(9) holds for every sequence* $\{a_N\}$ *that satisfies* (7) *and* (8) *if and only if* (6) *holds.*

PROOF. We only need to prove the only if part. Assume $E\exp(t|X_1|^r) = \infty$ for every $t > 0$. Namely, $\sum_{n=1}^{\infty} P(|X_1| \geq (t\log n)^{1/r}) = \infty$. It follows that $\sum_{n=1}^{\infty} P(|X_n| \geq m(\log n)^{1/r}) = \infty$ for every positive integer m. Choose n_m recursively so that $n_m \geq 2^m$ and so that

$$\sum_{n=1+n_{m-1}}^{n_m} P\big(|X_n| \geq m(\log n)^{1/r}\big) \geq 1. \tag{10}$$

Let $m_n = m$ for $n_{m-1} < n \leq n_m$ (with $n_0 = 0$) and $a_n = [m_n(\log n)^{-1+2/r}]$. Then we can get that $\{a_n\}$ is a nondecreasing sequence satisfying (7) and (8) and that

$$P\big(|X_n| \geq m_n(\log n)^{1/r} \text{ i.o.}\big) = 1 \tag{11}$$

by (10) and the Borel–Cantelli lemma.

On the other hand, by (9) we have

$$P\Big(|X_n| \geq 2\big(a_n(\log(n/a_n) + \log\log n)\big)^{1/2} \text{ i.o.}\Big) = 0. \tag{12}$$

It follows that

$$P\big(|X_n| \geq 3m_n^{1/2}(\log n)^{1/r} \text{ i.o.}\big) = 0, \tag{13}$$

which contradicts (11) by the fact that $m_n \to \infty$ as $n \to \infty$. That is, the prior assumption is not true. Namely, there is a $t_0 > 0$ such that $E\exp(t_0|X_1|^r) < \infty$. □

THEOREM 3. *If* (9) *holds for* $a_n = [cn^{2/p}/\log n]$, *where* $c > 0$, $p > 2$, *then* (6′) *holds.*

PROOF. It is well known that $E|X|^p < \infty$ if and only if

$$\sum_{n=1}^{\infty} P\big(|X| \geq an^{1/p}\big) < \infty$$

for every $a > 0$. Assume $E|X_1|^p = \infty$. Then $\sum_{n=1}^{\infty} P(|X_n| \geq an^{1/p}) = \infty$. So $P(|X_n| \geq an^{1/p} \text{ i.o.}) = 1$ for every $a > 0$. Now use the argument of Theorem 2. □

Acknowledgment. The author wishes to express his gratitude to the Associate Editor for his valuable suggestions.

REFERENCES

[1] CSÖRGŐ, M. and RÉVÉSZ, P. (1981). *Strong Approximations in Probability and Statistics*. Academic, New York.

[2] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent rv's and the sample df. I. *Z. Wahrsch. verw. Gebiete* **32** 111–131.

[3] KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1976). An approximation of partial sums of independent rv's and the sample df. II. *Z. Wahrsch. verw. Gebiete* **34** 33–58.

[4] MAJOR, P. (1976). The approximation of partial sums of independent rv's. *Z. Wahrsch. verw. Gebiete* **35** 213–220.

[5] SHAO, Q. M. (1987). On the increments of partial sums of independent random variables. Preprint.

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