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WEAK CONVERGENCE OF SMOOTHED AND NONSMOOTHED BOOTSTRAP QUANTILE ESTIMATES

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Under fairly general assumptions on the underlying distribution function, the bootstrap process, pertaining to the sample q-quantile, converges weakly in $D_{\mathbb{R}}$ to the standard Brownian motion.

Furthermore, weak convergence of a smoothed bootstrap quantile estimate is proved which entails that in this particular case the smoothed bootstrap estimate outperforms the nonsmoothed one.

0. Introduction. Let $X_1, \dots, X_n$ be a sample of independent random variables (rv's) with common distribution function (df) F and denote by F_n the corresponding empirical df.

Consider a functional T from the space of all dfs into the real line $\mathbb{R}$. The natural nonparametric estimator of $T(F)$ is the statistical functional $T(F_n)$. As an estimator of the df $P\{T(F_n) - T(F) \le t\}$, $t \in \mathbb{R}$, the bootstrap idea, introduced by Efron (1979), suggests $P_n\{T(F_n^*) - T(F_n) \le t\}$, $t \in \mathbb{R}$, where F_n^* denotes the empirical df of a sample $X_1^*, \dots, X_n^*$ of independent rv's with common df F_n. We add the index n to the preceding probability to indicate its dependence on the outcome of F_n.

Now, $P_n\{T(F_n^*) - T(F_n) \le t\}$ is a rv before the sample $X_1, \dots, X_n$ is drawn and thus, the bootstrap error

$$Z_n(t) := P_n\{T(F_n^*) - T(F_n) \le t\} - P\{T(F_n) - T(F) \le t\}, \qquad t \in \mathbb{R},$$

defines a stochastic process with values in $D_{\mathbb{R}}$, the space of all functions on $\mathbb{R}$ that are right-hand continuous and have left-hand limits. We call Z_n the bootstrap process based on T.

Up to now mainly strong laws were obtained for the bootstrap error in the statistical literature, i.e., it was proved for various choices of T that $\sup_{t \in \mathbb{R}} |Z_n(t)|$ converges to zero with probability 1 [see, for example, the paper by Bickel and Freedman (1981)] or satisfies laws of the iterated logarithm [Singh (1981)]. A survey of bootstrap results is given by Beran (1984a).

Convoluting Z_n with a normal density, Beran [(1984b), Theorem 3] obtained weak convergence in sup norm of the resulting process, suitably standardized, to a degenerate Gaussian process. This is achieved under the assumption that the functional T is locally quadratic, i.e., that it is a rather smooth functional.

However, as Beran (1982) points out in connection with certain optimality properties of the bootstrap estimate, it is an open question whether such results can be obtained without smoothing.

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Now, for the particular functional $T(F) := F^{-1}(q) = \inf\{s \in \mathbb{R}: F(s) \geq q\}$, i.e., the q-quantile of F, $q \in (0,1)$, we will prove in the first section of the present paper that the nonsmoothed process Z_n, suitably standardized, converges weakly in $D_{\mathbb{R}}$ to the standard Brownian motion on $\mathbb{R}$, i.e., to $(Z(t))_{t\in\mathbb{R}} = ((B_1(-t))_{t\leq 0}, (B_2(t))_{t>0})$, where B_1, B_2 are two independent standard Brownian motions on $[0,\infty)$.

This result indicates that the asymptotic behavior of the nonsmoothed bootstrap process Z_n is completely different from that of a smoothed version. This is underlined by the results of the second section of the present paper in which we estimate $P\{F_n^{-1}(q) - F^{-1}(q) \leq t\}$ by a smoothed bootstrap $P_n\{F_n^{*-1}(q) - \hat{F}_n^{-1}(q) \leq t\}$. This time the observations $X_1^*, \dots, X_n^*$ are generated according to a smoothed version $\hat{F}_n$ of F_n.

Although this approach is different from smoothing the process Z_n, it turns out that the pertaining limiting process is again a degenerate one, unlike in the nonsmooth case.

The striking advantage of smoothing F_n before bootstrapping in this particular case is the rate of convergence of the resulting estimator of $P\{F_n^{-1}(q) - F^{-1}(q) \leq t\}$, $t \in \mathbb{R}$, which is considerably better than in the nonsmooth case.

Hence, smoothing F_n before bootstrapping may be of great practical importance and thus, we obtain a partial answer to the problem: What are the advantages of a smoothed bootstrap? Moreover, one might expect corresponding results in general for those statistical functionals which depend on the local behavior of the underlying df.

1. Weak convergence of the bootstrap quantile process. At first we define weak convergence in $D_{\mathbb{R}}$. Consider therefore for $K > 0$ the space D_K of those functions on $[-K, K]$ which are right-hand continuous and have left-hand limits. In complete analogy to the well-known space $(D_{[0,1]}, d)$, where d denotes the Skorohod metric, we can equip D_K with the Skorohod metric d_K and all results on the metric space $(D_{[0,1]}, d)$ carry over to (D_K, d_K) [see Chapter 3 of Billingsley (1968) for details].

In particular, we know that d_K is smaller or equal to the supremum metric on D_K and their restrictions to the subspace of continuous functions in D_K generate the same topology.

Now, a sequence W_n, $n \in \mathbb{N}$, of stochastic processes with values in $D_{\mathbb{R}}$ is said to be weakly convergent to W, denoted by $W_n \to_{\mathscr{D}} W$, if for any $K > 0$ the restriction $W_n^K := (W_n(t))_{-K\leq t\leq K}$ converges weakly in (D_K, d_K) to $W^K := (W(t))_{-K\leq t\leq K}$.

Now we are ready to formulate our first main result. Suppose that F is differentiable near $F^{-1}(q)$ and denote its derivative by f. Moreover, suppose that $f(F^{-1}(q)) > 0$ and define

$$\tilde{Z}_n(t) := c_t n^{1/4}\left[P_n\left\{n^{1/2}\left(F_n^{*-1}(q) - F_n^{-1}(q)\right) \leq t\right\} - P\left\{n^{1/2}\left(F_n^{-1}(q) - F^{-1}(q)\right) \leq t\right\}\right], \tag{1.1}$$

where

$$(1.2)\quad c_t := \left(q(1-q)\right)^{1/2} f\left(F^{-1}(q)\right)^{-1/2} \Big/ \varphi\left\{\left(q(1-q)\right)^{-1/2} f\left(F^{-1}(q)\right)t\right\},$$

and φ denotes the density of the standard normal df Φ. Then, $\tilde{Z}_n \in D_{\mathbb{R}}$ and we have Theorem 1.3.

1.3 THEOREM. *If F is continuous and differentiable near $F^{-1}(q)$ such that $f = F'$ is Hölder-continuous of order $\delta > 1/2$ and $f(F^{-1}(q)) > 0$, then*

$$\tilde{Z}_n \to_{\mathscr{D}} Z := \left(\left(B_1(-t)\right)_{t \le 0}, \left(B_2(t)\right)_{t>0}\right),$$

where B_1, B_2 are independent standard Brownian motions on $[0, \infty)$.

Lemma 1.4 will be quite useful for the derivation of the preceding result. It follows from Reiss (1986), elementary computations and the probability integral transformation theorem.

1.4 LEMMA. *Suppose that F is continuous and let F_n be the empirical df pertaining to a sample of n independent rv's identically distributed according to F. Then, for $q \in (0,1)$ and $u \in \mathbb{R}$, the conditional distribution of the process $(F_n(t + F_n^{-1}(q)))_{t \in \mathbb{R}}$ given $F_n^{-1}(q) = u$ equals the distribution of the process*

$$W_{n,u}(t) := \begin{cases} ((m-1)/n)\overline{F}_{m-1}(F(t+u)/F(u)), & t<0, \\ m/n, & t=0, \\ m/n + ((n-m)/n)\overline{G}_{n-m}\left((F(t+u) - F(u))\big/(1-F(u))\right), & t>0, \end{cases}$$

where $m = nq$ if $nq \in \mathbb{N}$, $m = [nq] + 1$ else (here $[x]$ denotes the integral part of $x \in \mathbb{R}$) and $\overline{F}_{m-1}$, $\overline{G}_{n-m}$ denote the empirical df's of two independent samples of independent and uniformly on $(0,1)$ distributed rv's.

Replacing t by $t/n^{1/2}$ and u by $u_n = F^{-1}(q) + un^{-1/2}$, we deduce from Lemma 1.4 together with the bound for the oscillation behavior of empirical processes due to Stute [(1982), Lemma 2.3] the consequence:

1.5 COROLLARY. *Suppose that F satisfies the assumptions of Theorem 1.3. Then, uniformly for t, u in finite intervals of $\mathbb{R}$,*

$$W_{n,u_n}(t/n^{1/2}) = W_n(t) + o_P(n^{-3/4}),$$

where

$$W_n(t) := \begin{cases} q\overline{F}_m\left\{1 + f\left(F^{-1}(q)\right)t/\left(qn^{1/2}\right)\right\}, & t<0, \\ q, & t=0, \\ q + (1-q)\overline{G}_{n-m}\left\{f\left(F^{-1}(q)\right)t\big/\left((1-q)n^{1/2}\right)\right\}, & t>0. \end{cases}$$

Now we are ready to prove Theorem 1.3.

PROOF OF THEOREM 1.3. Choose $g\colon (D_K, d_K) \to \mathbb{R}$ uniformly continuous and bounded. We have to prove

$$\int g(\tilde{Z}_n^K)\,dP \underset{n\in\mathbb{N}}{\longrightarrow} \int g(Z^K)\,dP. \tag{1.6}$$

The proof will be based on the normal approximation to the distribution of sample quantiles. By $\bar{F}_n$ we denote again the empirical df pertaining to a sample of n independent and uniformly on $(0,1)$ distributed rv's. Put $\Phi_q(x) := \Phi(x/(q(1-q))^{1/2})$ for $x \in \mathbb{R}$ and $q \in (0,1)$. Then, by the probability integral transformation theorem we obtain

$$\begin{aligned}
\tilde{Z}_n(t) &= c_t n^{1/4}\big[P_n\{n^{1/2}(F_n^{-1}(\bar{F}_n^{-1}(q)) - F_n^{-1}(q)) \le t\} \\
&\qquad - P\{n^{1/2}(F^{-1}(\bar{F}_n^{-1}(q)) - F^{-1}(q)) \le t\}\big] \\
&= c_t n^{1/4}\big[P_n\{\bar{F}_n^{-1}(q) \le F_n(tn^{-1/2} + F_n^{-1}(q))\} \\
&\qquad - P\{\bar{F}_n^{-1}(q) \le F(tn^{-1/2} + F^{-1}(q))\}\big] \\
&= c_t n^{1/4}\big[P_n\{n^{1/2}(\bar{F}_n^{-1}(q) - q) \le n^{1/2}(F_n(tn^{-1/2} + F_n^{-1}(q)) - q)\} \qquad (1.7)\\
&\qquad - P\{n^{1/2}(\bar{F}_n^{-1}(q) - q) \le n^{1/2}(F(tn^{-1/2} + F^{-1}(q)) - q)\}\big] \\
&= c_t n^{1/4}\big[\Phi_q\{n^{1/2}(F_n(tn^{-1/2} + F_n^{-1}(q)) - q)\} \\
&\qquad\qquad - \Phi_q\{f(F^{-1}(q))t\}\big] + o(1) \\
&=: \hat{Z}_n(t) + o(1)
\end{aligned}$$

uniformly for $t \in \mathbb{R}$ by the normal approximation to the distribution of sample quantiles as given in Proposition 1.5 of Reiss (1974) and a Taylor expansion of F at $F^{-1}(q)$. By the uniform continuity of g with respect to supremum metric on D_K, it suffices therefore to prove (1.6) with $\hat{Z}_n^K$ in place of $\tilde{Z}_n^K$.

Conditioning on $F_n^{-1}(q) = un^{-1/2} + F^{-1}(q) =: u_n$ we can write

$$\begin{aligned}
\int g(\hat{Z}_n^K)\,dP &= \int E(g(\hat{Z}_n^K) \mid F_n^{-1}(q) = u_n) \qquad (1.8)\\
&\qquad \times \big(P^*(n^{1/2}(F_n^{-1}(q) - F^{-1}(q)))\big)(du),
\end{aligned}$$

where P^*Y denotes the measure induced by P and a rv Y.

From Corollary 1.5 we obtain that uniformly for u in any finite interval of $\mathbb{R}$

$$\begin{aligned} E\big(g(\hat{Z}_n^K)|F_n^{-1}(q) = u_n\big) &= E\Big(g\Big\{\Big(c_t n^{1/4}\big(\Phi_q\big(n^{1/2}(W_n(t) - q)\big) \\ &\qquad - \Phi_q\big(f(F^{-1}(q))t\big)\big)\Big)_{-K\le t\le K}\Big\}\Big) + o(1) \\ &= E\Big(g\Big\{\Big(f(F^{-1}(q))^{-1/2} n^{3/4} \\ &\qquad \times\big(W_n(t) - q - f(F^{-1}(q))tn^{-1/2}\big)\Big)_{-K\le t\le K}\Big\}\Big) + o(1) \end{aligned}$$

by a Taylor expansion, the Kolmogorov–Smirnov theorem and elementary computations. Note that the preceding expectation is independent of u. Moreover, it is routine to show that the process

$$\Big(f(F^{-1}(q))^{-1/2} n^{3/4}\big(W_n(t) - q - f(F^{-1}(q))tn^{-1/2}\big)\Big)_{|t|\le K}$$

converges weakly in (D_K, d_K) to Z^K, i.e., the standard Brownian motion on $[-K, K]$. Hence, we have uniformly for u in any finite interval of $\mathbb{R}$

$$E\big(g(\hat{Z}_n^K)|F_n^{-1}(q) = u_n\big) = E\big(g(Z^K)\big) + o(1). \tag{1.9}$$

Since $n^{1/2}(F_n^{-1}(q) - F^{-1}(q))$ has a limiting normal distribution, the assertion now follows from (1.8) and (1.9). This completes the proof. □

The continuous mapping theorem [see Theorem 5.1 of Billingsley (1968)] implies the following result for the unweighted bootstrap quantile process

$$\begin{aligned} Z_n'(t) := n^{1/4}\big[&P_n\{n^{1/2}(F_n^{*-1}(q) - F_n^{-1}(q)) \le t\} \\ &- P\{n^{1/2}(F_n^{-1}(q) - F^{-1}(q)) \le t\}\big], \qquad t \in \mathbb{R}. \end{aligned}$$

1.10 COROLLARY. *Under the conditions of Theorem* 1.3

$$Z_n' \to_{\mathscr{D}} \big((c_t^{-1}B_1(-t))_{t\le 0}, (c_t^{-1}B_2(t))_{t>0}\big),$$

where B_1, B_2 *are two independent standard Brownian motions on* $[0, \infty)$.

2. Weak convergence of a smoothed bootstrap quantile estimate. In this section we investigate the approximation of $P\{F_n^{-1}(q) - F^{-1}(q) \le t\}$, $t \in \mathbb{R}$, by a smoothed bootstrap $P_n\{F_n^{*-1}(q) - \hat{F}_n^{-1}(q) \le t\}$, $t \in \mathbb{R}$. Again F_n^* denotes the empirical df of a sample $X_1^*, \dots, X_n^*$ of independent rv's, but this time these are generated according to a smoothed version $\hat{F}_n$ of F_n.

Define $\hat{F}_n(t)$ as the kernel estimator of $F(t)$, i.e., put

$$\hat{F}_n(t) := n^{-1}\sum_{i=1}^{n} K((t - X_i)/\alpha_n) = \int K((t - x)/\alpha_n)F_n(dx), \tag{2.1}$$

where the bandwidth $\alpha_n > 0$ tends to zero as n increases and the kernel function

$K: \mathbb{R} \to \mathbb{R}$ is itself a df. These estimators were extensively studied as competitors of the empirical df by Reiss (1981) and Falk (1983).

This particular choice of a smoothed empirical df allows us to generate rv's $X_1^*, \dots, X_n^*$ according to $\hat{F}_n$ in a simple way: If $Y_1^*, \dots, Y_n^*$ are independent rv's with common df F_n and $V_1, \dots, V_n$ are independent rv's with common df K and if these two samples are independent, then we may take $X_i^* := Y_i^* + \alpha_n V_i$, $i = 1, \dots, n$.

Suppose that the kernel function K has a density, say k. Define

$$\hat{f}_n(t) := (n\alpha_n)^{-1} \sum_{i=1}^{n} k((t - X_i)/\alpha_n),$$

which is the kernel density estimator. Theorem 2.2, the main theorem of this section, reveals that the stochastic behavior of the bootstrap error is now asymptotically related to that of the kernel density estimator $\hat{f}_n(F^{-1}(q))$.

2.2 **Theorem.** *Suppose that F is three times continuously differentiable near $F^{-1}(q)$ with $f(F^{-1}(q)) > 0$, $f = F'$. Let $K: \mathbb{R} \to [0,1]$ have bounded support $[-1,1]$, be three times differentiable with bounded second derivative and satisfy $\int k(x)\,dx = 1$, $\int xk(x)\,dx = 0$, where $k = K'$. Then, if $n\alpha_n^3 \to \infty$ and $n\alpha_n^5 \log^2(n) \to 0$ we have*

$$\begin{aligned}(n\alpha_n)^{1/2} \sup_{t \in \mathbb{R}} \Big| &P_n\{n^{1/2}(F_n^{*-1}(q) - \hat{F}_n^{-1}(q)) \le t\} \\ &- P\{n^{1/2}(F_n^{-1}(q) - F^{-1}(q)) \le t\} \\ &- \psi(t)(\hat{f}_n(F^{-1}(q)) - f(F^{-1}(q))\Big| \\ = o_P(1),&\end{aligned}$$

where $\psi(t) := (q(1-q))^{-1/2} t\varphi\{f(F^{-1}(q))(q(1-q))^{-1/2}t\}$, $t \in \mathbb{R}$.

Proof. At first we prove the following auxiliary results, where $\varepsilon > 0$ is chosen sufficiently small:

$$\sup_{|t - F^{-1}(q)| \le \varepsilon} |\hat{F}_n(t) - F(t)| = O_P(n^{-1/2} + \alpha_n^2). \tag{2.3}$$

Note that (2.3) immediately implies $\hat{F}_n^{-1}(q) - F^{-1}(q) = o_P(1)$.

$$\sup_{|t - F^{-1}(q)| \le \varepsilon} |\hat{f}_n(t) - f(t)| = O_P((n\alpha_n)^{-1/2} + \alpha_n^2), \tag{2.4}$$

$$\sup_{|t - F^{-1}(q)| \le \varepsilon} \left| \hat{f}_n'(t) - \int k(x) f'(t - \alpha_n x)\,dx \right| = O_P(n^{-1/2}\alpha_n^{-3/2}), \tag{2.5}$$

$$|\hat{F}_n^{-1}(q) - F^{-1}(q)| = O_P(n^{-1/2} + \alpha_n^2). \tag{2.6}$$

PROOF OF (2.3). By using integration by parts we can write, where $0 < \vartheta < 1$,

$$\begin{aligned}\hat{F}_n(t) - F(t) &= \int K((t-x)/\alpha_n)(F_n - F)(dx) \\ &\quad + \int K((t-x)/\alpha_n)F(dx) - F(t) \\ &= \alpha_n^{-1}\int k((t-x)/\alpha_n)(F_n(x) - F(x))\,dx \\ &\quad + \int k(x)(F(t-\alpha_n x) - F(t))\,dx \\ &= \int k(x)(F_n(t-\alpha_n x) - F(t-\alpha_n x))\,dx \\ &\quad + \int k(x) f'(t - \vartheta\alpha_n x)\alpha_n^2 x^2/2\,dx \\ &= O_P(n^{-1/2}) + O(\alpha_n^2)\end{aligned}$$

uniformly for t near $F^{-1}(q)$. □

PROOF OF (2.4). In analogy to the preceding steps we obtain

$$\begin{aligned}\hat{f}_n(t) - f(t) &= \alpha_n^{-1}\int k'(x)(F_n(t-\alpha_n x) - F(t-\alpha_n x))\,dx \\ &\quad + \int k(x)(f(t-\alpha_n x) - f(t))\,dx \\ &= \alpha_n^{-1}\int k'(x)\{F_n(t-\alpha_n x) \\ &\qquad - F(t-\alpha_n x) - F_n(t) + F(t)\}\,dx + O(\alpha_n^2)\end{aligned}$$

since $\int k'(x)\,dx = k(1) - k(-1) = 0$. By Lemma 2.3 in Stute (1982), the first term in the preceding sum is of order $O_P((n\alpha_n)^{-1/2})$ uniformly for t near $F^{-1}(q)$. This implies (2.4). □

PROOF OF (2.5). Put $s_1 := \hat{F}_n^{-1}(q)$, $s_2 := F^{-1}(q)$. Then, by a Taylor expansion with $\vartheta \in (0,1)$,

$$\begin{aligned}|\hat{F}_n(s_2) - F(s_2)| = |\hat{F}_n(s_2) - \hat{F}_n(s_1)| &= \hat{f}_n(s_2 + \vartheta(s_1 - s_2))|s_2 - s_1| \\ &= \hat{f}_n(s_2 + \vartheta(s_1 - s_2))|\hat{F}_n^{-1}(q) - F^{-1}(q)|.\end{aligned}$$

From (2.3) and (2.4) we conclude that $\hat{f}_n(s_2 + \vartheta(s_1 - s_2)) - f(s_2) = o_P(1)$ and hence, the assertion is immediate from (2.3) and the preceding representation. □

PROOF OF (2.6). In analogy we can write

$$\begin{aligned}
\hat{f}_n'(t) - \alpha_n^{-2}\int k'((t-x)/\alpha_n)F(dx)&\\
&= \alpha_n^{-2}\int k''(x)\{F_n(t-\alpha_n x) - F(t-\alpha_n x)\}\,dx\\
&= \alpha_n^{-2}\int k''(x)\{F_n(t-\alpha_n x) - F(t-\alpha_n x) - F_n(t) + F(t)\}\,dx\\
&= O_P\big(n^{-1/2}\alpha_n^{-3/2}\big)
\end{aligned}$$

uniformly for t near $F^{-1}(q)$. Finally, observe that

$$\alpha_n^{-2}\int k'((t-x)/\alpha_n)F(dx) = \int k(x)f'(t-\alpha_n x)\,dx \to f'(t)$$

uniformly for t near $F^{-1}(q)$. □

Now we can prove Theorem 2.2. Repeating the arguments of the first part of the proof of Theorem 1.3 we can write uniformly for $t \in \mathbb{R}$,

$$\begin{aligned}
&P_n\{n^{1/2}(F_n^{*-1}(q) - \hat{F}_n^{-1}(q)) \le t\} - P\{n^{1/2}(F_n^{-1}(q) - F^{-1}(q)) \le t\}\\
&\quad= \Phi_q\{n^{1/2}(\hat{F}_n(tn^{-1/2} + \hat{F}_n^{-1}(q)) - q)\}\\
&\qquad - \Phi_q\{n^{1/2}(F(tn^{-1/2} + F^{-1}(q)) - q)\} + O(n^{-1/2}).
\end{aligned}$$

Next we will show that it suffices to consider $|t| \le \log(n)$. For $t \le -\log(n)$ we have, if n is large with $\vartheta \in (0,1)$,

$$\begin{aligned}
&\Phi_q\{n^{1/2}(F(tn^{-1/2} + F^{-1}(q)) - q)\}\\
&\quad\le \Phi_q\{n^{1/2}(F(-\log(n)n^{-1/2} + F^{-1}(q)) - q)\}\\
&\quad= \Phi_q\{-f(F^{-1}(q) - \vartheta\log(n)n^{-1/2})\log(n)\} = o(n^{-1})
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_q\{n^{1/2}(\hat{F}_n(tn^{-1/2} + \hat{F}_n^{-1}(q)) - q)\}\\
&\quad\le \Phi_q\{-\hat{f}_n(\hat{F}_n^{-1}(q) - \vartheta\log(n)n^{-1/2})\log(n)\}\\
&\quad= o_P(n^{-1})
\end{aligned}$$

since $\hat{f}_n(\hat{F}_n^{-1}(q) - \vartheta\log(n)n^{-1/2}) \to f(F^{-1}(q))$ in probability by (2.4) and (2.6).

Equally, one handles $t \ge \log(n)$ and hence, we can restrict ourselves in the following to $|t| \le \log(n)$. By using a Taylor expansion we can write

$$\begin{aligned}
&\Phi_q\{n^{1/2}(\hat{F}_n(tn^{-1/2} + \hat{F}_n^{-1}(q)) - q)\} - \Phi_q\{n^{1/2}(F(tn^{-1/2} + F^{-1}(q)) - q)\}\\
&\quad= \Phi_q'\{n^{1/2}(F(tn^{-1/2} + F^{-1}(q)) - q)\}\\
&\qquad\times n^{1/2}\{\hat{F}_n(tn^{-1/2} + \hat{F}_n^{-1}(q)) - F(tn^{-1/2} + F^{-1}(q))\}\\
&\qquad\quad + O\Big(n\big\{\hat{F}_n(tn^{-1/2} + \hat{F}_n^{-1}(q)) - F(tn^{-1/2} + F^{-1}(q))\big\}^2\Big).
\end{aligned}$$

Now, by Taylor's formula,

$$\begin{aligned} & n^{1/2}\{\hat{F}_n(tn^{-1/2} + \hat{F}_n^{-1}(q)) - F(tn^{-1/2} + F^{-1}(q))\} \\ &\quad = n^{1/2}\{q + \hat{f}_n(\hat{F}_n^{-1}(q))tn^{-1/2} + \hat{f}_n'(\hat{F}_n^{-1}(q) + \vartheta_1 tn^{-1/2})t^2 n^{-1} \\ &\qquad -q - f(F^{-1}(q))tn^{-1/2} - f'(F^{-1}(q) + \vartheta_2 tn^{-1/2})t^2 n^{-1}\} \\ &\quad = \{\hat{f}_n(\hat{F}_n^{-1}(q)) - f(F^{-1}(q))\}t + O_P(\log^2(n)n^{-1/2}) \end{aligned}$$

if $\hat{f}_n'$ is bounded in probability near $F^{-1}(q)$. But this is immediate from (2.5) and the assumption $n\alpha_n^3 \to \infty$.

Moreover, by (2.5) and (2.6),

$$\begin{aligned} \hat{f}_n(\hat{F}_n^{-1}(q)) &= \hat{f}_n(F^{-1}(q)) + \hat{f}_n'\{F^{-1}(q) + \vartheta(\hat{F}_n^{-1}(q) - F^{-1}(q))\} \\ &\qquad \times \{\hat{F}_n^{-1}(q) - F^{-1}(q)\} \\ &= \hat{f}_n(F^{-1}(q)) + O_P(n^{-1/2} + \alpha_n^2) \end{aligned}$$

and thus, we obtain altogether

$$\begin{aligned} & n^{1/2}\{\hat{F}_n(tn^{-1/2} + \hat{F}_n^{-1}(q)) - F(tn^{-1/2} + F^{-1}(q))\} \\ &\quad = \{\hat{f}_n(F^{-1}(q)) - f(F^{-1}(q))\}t + O_P\{\log^2(n)n^{-1/2} + \log(n)\alpha_n^2\} \\ &\quad = \{\hat{f}_n(F^{-1}(q)) - f(F^{-1}(q))\}t + o_P((n\alpha_n)^{-1/2}) \end{aligned}$$

uniformly for $|t| \le \log(n)$.

Finally, observe that

$$\begin{aligned} & \Phi_q'\{n^{1/2}(F(tn^{-1/2} + F^{-1}(q)) - q)\} \\ &\qquad \to (q(1-q))^{-1/2}\varphi\{f(F^{-1}(q))(q(1-q))^{-1/2}t\} \end{aligned}$$

uniformly for $|t| \le \log(n)$ and that $(n\alpha_n)^{1/2}(\hat{f}_n(F^{-1}(q)) - f(F^{-1}(q)))$ has a limiting normal distribution with mean zero and variance $f(F^{-1}(q))\int k^2(x)\,dx$. This completes the proof of Theorem 2.2. □

The preceding result immediately entails the following consequences for the bootstrap process $\hat{Z}_n$ pertaining to the smoothed empirical quantile, i.e., with

$$\hat{Z}_n(t) := P_n\{n^{1/2}(F_n^{*-1}(q) - \hat{F}_n^{-1}(q)) \le t\} - P\{n^{1/2}(F_n^{-1}(q) - F^{-1}(q)) \le t\},$$

our final result is Corollary 2.7.

2.7 COROLLARY. *Under the assumptions of the preceding theorem we have*

(i) $(n\alpha_n)^{1/2}\hat{Z}_n \to_{\mathscr{D}} \psi(\cdot)\{f(F^{-1}(q))\int k^2(x)\,dx\}^{1/2}X$, *where* X *is a standard normal random variable and*

(ii) $(n\alpha_n)^{1/2}\sup_{t\in\mathbf{R}}|\hat{Z}_n(t)| \to_{\mathscr{D}} (f(F^{-1}(q))2\pi e)^{-1/2}|X|$.

Notice that the accuracy of the bootstrap approximation in Theorem 1.3 is of order $O(n^{-1/4})$ whereas the rate in Corollary 2.7 is roughly $O(n^{-2/5})$ for an appropriate choice of α_n.

Final remark. In an earlier version of the paper the smoothed empirical df was defined via a smoothed empirical quantile function. By using that approach one can show that an iterated bootstrap leads to an asymptotically consistent estimator of the df of the maximum bootstrap error.

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