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MEASURING CLOSE APPROACHES ON A BROWNIAN PATH

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Integral tests are found for the uniform escape rate of a d-dimensional Brownian path ($d \geq 4$), i.e., for the lower growth rate of $\inf\{|X(t) - X(s)|: 0 \leq s,\ t \leq 1,\ |t - s| \geq h\}$ as $h \downarrow 0$. The gap between this uniform escape rate and the one-sided local escape rate of Dvoretsky and Erdös and the two-sided local escape rate of Jain and Taylor suggest the study of certain sets of times of slow one- or two-sided escape. The Hausdorff dimension of these exceptional sets is computed. The results are proved for a broad class of strictly stable processes.

1. Introduction. For a transient process, X, starting at 0, one can measure the rate at which X "escapes" from 0 by characterizing those increasing functions ψ for which $|X(h)| \geq \psi(h)$ for small h a.s. An integral test for ψ was found by Dvoretsky and Erdös (1951) for Brownian motion in $\mathbb{R}^d$ ($d \geq 3$), by Spitzer (1958) for Brownian motion in $\mathbb{R}^2$ and by Takeuchi (1964b) and Taylor (1967) for a large class of strictly stable processes. The related problem of determining the two-sided rate of escape of X was solved by Jain and Taylor (1973) for Brownian motion in $\mathbb{R}^d$ ($d \geq 4$). An integral test was found for the class of increasing functions ψ such that

$$|X(t+u) - X(t-v)| \geq (u+v)^{1/2}\psi(u+v),$$
$$\text{for } 0 \leq u, v \text{ small a.s. for any } t > 0.$$

Corresponding results for a large class of d-dimensional strictly stable processes of index α ($d \geq 2\alpha$) can be established by the same techniques (see Theorem 3.1). The difference between the problems of one- and two-sided escape is illustrated in the Brownian setting by the fact that the latter problem is only of interest if there are no double points ($d \geq 4$), whereas the former only requires the process not to hit points ($d \geq 2$).

In this paper we consider the uniform (in time) escape rate of a d-dimensional strictly stable process, X, of index α. More precisely, we study the asymptotic behavior of

$$W(h) = \inf\{|X(t) - X(s)|: 0 \leq s < t \leq 1,\ t - s \geq h\},$$

as $h \downarrow 0$. Assume $d \geq 2\alpha$. For the processes to be studied this is necessary and sufficient for the nonexistence of double points [see Taylor (1967), Theorem 3]. An integral test is established for the lower growth rate of W (Theorem 3.3). A

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condensation (i.e., Baire category) argument shows that if $W(h) \ll \psi(h)$ as $h \downarrow 0$, then there is an uncountable dense set of times t such that

$$\liminf_{h \downarrow 0} |X(t+h) - X(t)| \psi(h)^{-1} = 0$$

(Theorem 3.4).

As one would expect, there is a gap between the one- or two-sided local escape rate at a fixed time and the uniform escape rate. We determine the Hausdorff dimension of various sets of unusually slow one- or two-sided escape (Theorems 4.1 and 4.3).

To illustrate our results, let us consider the Brownian case $\alpha = 2$. The integral test given in Theorem 3.3 implies that for any $\varepsilon > 0$,

$$h^{1/2+(d-4)^{-1}+\varepsilon} \ll W(h), \quad \text{as } h \downarrow 0 \quad \text{but} \quad W(h_n) \ll h_n^{1/2+(d-4)^{-1}},$$
$$\text{for some } h_n \downarrow 0, \text{ if } d \geq 5,$$
$$e^{-h^{-1-\varepsilon}} \ll W(h), \quad \text{as } h \downarrow 0 \quad \text{but} \quad W(h_n) \ll e^{-h_n^{-1}},$$
$$\text{for some } h_n \downarrow 0, \text{ if } d = 4.$$

For a fixed $t > 0$, the test of Jain and Taylor (1973) shows that for any $\varepsilon > 0$,

$$(u+v)^{1/2+\varepsilon} \ll |X(t+u) - X(t-v)|, \quad \text{as } u, v \downarrow 0 \quad \text{but}$$
$$|X(t+u_n) - X(t-v_n)| \ll (u_n + v_n)^{1/2},$$
$$\text{for some } u_n, v_n \downarrow 0, \text{ if } d \geq 5,$$
$$e^{-(u+v)^{-\varepsilon}} \ll |X(t+u) - X(t-v)|, \quad \text{as } u, v \downarrow 0 \quad \text{but}$$
$$|X(t+u_n) - X(t-v_n)| \ll (u_n + v_n)^{\log\log(u_n+v_n)^{-1}},$$
$$\text{for some } u_n, v_n \downarrow 0, \text{ if } d = 4.$$

This suggests we look at the exceptional sets of times of unusually slow two-sided escape defined by

$$C_\gamma = \left\{ t \colon \liminf_{s, u \downarrow 0} |X(t+s) - X(t-u)| (s+u)^{-\gamma} = 0 \right\},$$
$$\tfrac{1}{2} \leq \gamma \leq \tfrac{1}{2} + (d-4)^{-1}, \ d \geq 5,$$
$$D_\gamma = \left\{ t \colon \liminf_{s, u \downarrow 0} |X(t+s) - X(t-u)| e^{(s+u)^{-\gamma}} = 0 \right\}, \qquad 0 \leq \gamma \leq 1, \ d = 4.$$

(Throughout this work $\liminf_{s, u \downarrow 0}$ will mean $s, u \geq 0$ but $s + u > 0$.) $\dim A$ is the Hausdorff dimension of the set A and a negative value of $\dim A$ means $A = \varnothing$. Theorem 4.1 shows $\dim C_\gamma = 1 - (\gamma - \frac{1}{2})(d-4)$ for $\gamma \geq \frac{1}{2}$ $(d \geq 5)$ and $\dim D_\gamma = 1 - \gamma$ for $\gamma \geq 0$ $(d = 4)$. Moreover, $C_{1/2+(d-4)^{-1}}$ and D_1 are uncountable dense sets (Theorem 3.4). The local one-sided escape rates of Dvoretsky and

Erdös (1951) and Spitzer (1958) lead one to the exceptional sets

$$E_\gamma = \left\{t\colon \liminf_{h \downarrow 0} |X(t+h) - X(t)| h^{-\gamma} = 0\right\},$$

$$\tfrac{1}{2} \le \gamma \le \tfrac{1}{2} + 1/(d-4)^+,\ d \ge 2.$$

Theorem 4.3 gives $\dim E_\gamma = 1 - (1 - (2\gamma)^{-1})(d/2 - 1)$ for $\gamma \ge \frac{1}{2}$, $d \ge 2$ and $E_{1/2+(d-4)^{-1}}$ is uncountable and dense for $d \ge 5$. Note that for $d \ge 5$, $E_\gamma \subset C_\gamma$ and in fact $\dim E_\gamma < \dim C_\gamma$ for $\frac{1}{2} < \gamma < \frac{1}{2} + (d-4)^{-1}$. Nonetheless, the two families of sets become empty at the same critical γ. In fact, Theorem 3.4 shows that for $d \ge 4$ a given set of times of unusually slow one-sided escape is empty if and only if the corresponding two-sided set is empty. The effect of the double points in dimensions 2 and 3 leads to

$$\lim_{\gamma \to \infty} \dim E_\gamma = \begin{cases} \frac{1}{2}, & d = 3, \\ 1, & d = 2. \end{cases}$$

These limiting values are precisely the dimensions of the times at which X is a double point. This is true more generally and is discussed at the end of Section 4.

The results are proved for type A stable processes if $\alpha \ne 1$ and for the symmetric Cauchy processes if $\alpha = 1$. For $\alpha \ne 1$ this excludes stable subordinators and their higher-dimensional analogs [the precise definition is recalled from Taylor (1967) in Section 2]. In the subordinator case, however, Hawkes (1971a) found an *exact* uniform escape rate of $ch^{1/\alpha}(\log 1/h)^{1-1/\alpha}$, as opposed to an integral test. The situation is apparently quite different in the "type B setting." The reader only interested in Brownian motion may safely ignore this degree of generality because after some well-known hitting estimates are stated in Section 2, the proofs are only slightly simpler in the Brownian case due to the continuity of paths.

2. Preliminaries. Throughout this work $X(t)$ denotes a d-dimensional stable process of index $\alpha \in (0,2]$ with zero drift. It is assumed that the law of $X(1)$ is not supported on a lower-dimensional space and hence X_t has a bounded continuous density [see Rvaceva (1962)]. Finally, we assume:

(i) If $\alpha = 1$, then X is a symmetric Cauchy process, i.e.,

$$E(e^{i\langle z, X(t)\rangle}) = \exp\{-t\lambda|z|\}$$

for all z in $\mathbb{R}^d$ and some $\lambda > 0$.

(ii) If $\alpha \ne 1$, then the density of $X(1)$ is nonzero at 0.

Processes satisfying (ii) are called stable processes of type A [see Taylor (1967)]. [(ii) will always hold if $\alpha > 1$.] $X(ct)$ and $c^{1/\alpha}X(t)$ have the same law (as processes in t) for every $c > 0$. We may (and shall) select a version of X with right-continuous paths and left limits. X is defined on a complete probability space $(\Omega, \mathscr{F}, P)$ and will start at 0 at $t = 0$, unless otherwise indicated. If $Y(t)$ is a process on this space, $\mathscr{F}_t^Y$ denotes the smallest right-continuous filtration such that $Y(t)$ is $\mathscr{F}_t^Y$-measurable and $\mathscr{F}_0^Y$ contains all the P-null sets. We write $\mathscr{F}_t$ for $\mathscr{F}_t^X$.

$\mathscr{H}$ denotes the class of real-valued functions, ψ, which are defined and strictly increasing on some $[0, \varepsilon)$, $\varepsilon > 0$, and satisfy $\psi(0) = \psi(0+) = 0$. $B(x, r)$ denotes the closed ball in $\mathbb{R}^d$ with radius r and center x. Unimportant positive constants appearing in the course of a proof are denoted by $c_1, c_2, \dots$ and constants that may reappear in subsequent arguments are denoted by k_0, $c^{(1)}$, $c^{(1)}$, $c^{(2)}, \dots$. The latter constants depend only on the law of X, unless otherwise indicated. $[x]$ denotes the integer part of x.

Assume $d > \alpha$ throughout this section. The first lemma is a well-known estimate on the number of balls exited by X on a finite interval.

NOTATION. If $t, \Delta > 0$, let $\tau_0^X(\Delta) = 0$,

$$\tau_{i+1}^X(\Delta) = \inf\left\{t \geq \tau_i^X(\Delta): \left|X_t - X_{\tau_i^X(\Delta)}\right| \geq \Delta\right\},$$

$$N^X(t, \Delta) = \max\left\{n: \tau_n^X(\Delta) \leq t\right\}.$$

LEMMA 2.1. *There are positive constants $c^{(1)}$ and λ, depending only on the law of X, such that $P(N^X(t, \Delta) \geq n) \leq e^{-\lambda n}$ whenever $n \geq c^{(1)} t \Delta^{-\alpha}$.*

PROOF. Let $\{T_i\}$ be an i.i.d. sequence with distribution equal to that of $\tau_1^X(1)$, and let $\mu = E(T_1)$. Take $c^{(1)} = 2/\mu$. If $Y_i = T_i - \mu$ and $n \geq c^{(1)} t \Delta^{-\alpha}$, then

$$\begin{aligned} P\left(N^X(t, \Delta) \geq n\right) &= P\left(\sum_{i=1}^n Y_i \leq t\Delta^{-\alpha} - \mu n\right) \\ &\leq P\left(\sum_{i=1}^n Y_i \leq -\mu n/2\right). \end{aligned}$$

The result follows from the well-known exponential bounds of Cramér [note that Y_i has an exponential tail by Lemma 5 of Taylor (1967)]. □

The following "delayed hitting probability" estimates are taken from Pruitt and Taylor (1969), Theorem 4. The hypotheses of the first result are slightly different but the same proof works. The second result (Lemma 2.3) is a trivial consequence of the first.

LEMMA 2.2. *There are positive $c^{(2)}$, $c^{(3)}$ such that*

$$c^{(2)}\left(rh^{-1/\alpha}\right)^{d-\alpha} \leq P\left(X(t) \in B(x, r) \text{ for some } t \geq h\right) \leq c^{(3)}\left(rh^{-1/\alpha}\right)^{d-\alpha},$$

where for the lower bound we assume $B(x, r) \subset B(0, h^{1/\alpha})$ and otherwise $r, h > 0$ and $x \in \mathbb{R}^d$ are arbitrary.

LEMMA 2.3. *There are a $k_0 > 1$ and $c^{(4)}$, $c^{(5)} > 0$ such that*

$$c^{(4)}\left(rh^{-1/\alpha}\right)^{d-\alpha} \leq P\left(|X(t)| \leq r \text{ for some } t \in [h, k_0 h]\right) \leq c^{(5)}\left(rh^{-1/\alpha}\right)^{d-\alpha},$$

whenever $0 < r \leq h^{1/\alpha}$.

We now can extend the key estimates in Jain and Taylor (1973) (Lemmas 4.1 and 4.2, respectively) to our current setting.

LEMMA 2.4. *Let $d > 2\alpha$. Assume X, Y are i.i.d. and $\Delta(h)h^{-1/\alpha} \to 0$ as $h \downarrow 0$.*

(a) *For each $k > 1$, there are a $c^{(6)}(k)$ and a $\delta = \delta(\Delta) > 0$ such that*

$$P^{x,y}\Big(\inf_{h \le u \le kh,\, v \ge 0} |X(u) - Y(v)| \le \Delta\Big) \le c^{(6)}(\Delta h^{-1/\alpha})^{d-2\alpha},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $0 < h \le \delta$.

Here $P^{x,y}((X, Y) \in A) = P((x + X, y + Y) \in A)$.

(b) *There are a $c^{(7)}$ and a $\delta = \delta(\Delta) > 0$ such that*

$$P\Big(\inf_{u, v \in [h, k_0 h]} |X(u) - Y(v)| \le \Delta\Big) \ge c^{(7)}(\Delta h^{-1/\alpha})^{d-2\alpha}, \quad \text{for } 0 < h \le \delta$$

(k_0 as in Lemma 2.3).

PROOF. (a)

$$\begin{aligned}
&P^{x,y}\Big(\inf_{h \le u \le kh,\, v \ge 0} |X(u) - Y(v)| \le \Delta\Big)\\
&\quad \le P^{x,y}\Big(\inf_{h \le u \le kh,\, v \le h} |X(u) - Y(v)| \le \Delta\Big)\\
&\qquad + P^{x,y}\Big(\inf_{h \le u \le kh,\, h \le v} |X(u) - Y(v)| \le \Delta\Big)\\
&\quad \le P^{y}\big(N^{Y}(h, \Delta) \ge c^{(1)}h\Delta^{-\alpha}\big) + P^{x}\big(N^{X}(kh, \Delta) \ge c^{(1)}kh\Delta^{-\alpha}\big)\\
&\qquad + \sum_{i=0}^{[c^{(1)}h\Delta^{-\alpha}]} E^{y}\Big(P^{x}\big(X(u) \in B\big(Y(\tau_i^{Y}(\Delta)), 2\Delta\big) \text{ for some } u \in [h, kh] \,|\, Y\big)\Big)\\
&\qquad + \sum_{i=0}^{[c^{(1)}kh\Delta^{-\alpha}]} E^{x}\Big(P^{y}\big(Y(v) \in B\big(X(\tau_i^{X}(\Delta)), 2\Delta\big) \text{ for some } v \ge h \,|\, X\big)\Big)\\
&\quad \le 2\exp\{-\lambda c^{(1)}h\Delta^{-\alpha}\} + 2\big([c^{(1)}kh\Delta^{-\alpha}] + 1\big)c^{(3)}(2\Delta h^{-1/\alpha})^{d-\alpha}\\
&\qquad\qquad \text{(Lemmas 2.1 and 2.2)}\\
&\quad \le c^{(6)}(k)(\Delta h^{-1/\alpha})^{d-2\alpha},
\end{aligned}$$

where $h < \delta(\Delta)$ is needed for the last inequality.

(b) follows by making the obvious changes (using Lemmas 2.2 and 2.3) in the derivation of Lemma 4.1 of Jain and Taylor (1973). □

LEMMA 2.5. *Let $d = 2\alpha$. Assume X, Y, $\Delta(h)$ and $P^{x,y}$ are as given previously.*

(a) *For each $k > 1$, there are a $c^{(8)}(k)$ and a $\delta = \delta(\Delta) > 0$ such that*

$$P^{x,y}\Big(\inf_{h \le u \le kh,\, v \ge 0} |X(u) - Y(v)| \le \Delta\Big) \le c^{(8)}|\log \Delta h^{-1/\alpha}|^{-1},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $0 < h \le \delta$.

(b) *There are a* $c^{(9)}$ *and a* $\delta = \delta(\Delta) > 0$ *such that*

$$P\Big(\inf_{u,\, v \in [h,\, k_0 h]} |X(u) - Y(v)| \le \Delta\Big) \ge c^{(9)} \big|\log \Delta h^{-1/\alpha}\big|^{-1}, \text{ for } 0 < h \le \delta.$$

The proof is very similar to the derivation of Lemma 4.2 in Jain and Taylor (1973). In addition to Lemmas 2.2 and 2.3, one uses the estimate

$$P^x(X(t) \in B(y, r) \text{ for some } t \ge 0) \ge c_1\Big[\big(r/|y - x|\big)^{d-\alpha} \wedge 1\Big]$$

[see Pruitt and Taylor (1969), Theorem 3]. Note that the converse inequality does not hold in general (it does if $d < 1 + 2\alpha$ or if X is symmetric stable) as is shown in Pruitt and Taylor (1969). This is why, for $\alpha = 2$, Lemma 2.4(a) is a little weaker than Lemma 4.1 of Jain and Taylor (1973).

The introduction of a general starting position leads to a simpler proof of

LEMMA 2.6. *Let* $\psi \in \mathscr{H}$ *satisfy* $\psi(h) \ll h^{1/\alpha}$ *as* $h \downarrow 0$. *If* $k_1, k_2 > 1$ *there are a* $c^{(10)}(k_1, k_2)$ *and a* $\delta(k_1, k_2) > 0$ *such that if* $0 < h_1, h_2 \le \delta$ *and* $t \ge k_1 h_1 + k_2 h_2$, *then*

$$P\Bigg(\inf_{u,\, v \in [h_1,\, k_1 h_1]} |X(t-u) - X(t+v)| \le \psi(h_1),$$

$$\inf_{\substack{k_1 h_1 \le u \le k_1 h_1 + k_2 h_2 \\ k_1 h_1 + h_2 \le v \le k_1 h_1 + k_2 h_2}} |X(t-u) - X(t+v)| \le \psi(h_2)\Bigg)$$

$$\le \begin{cases} c^{(10)} \big|\log\big(\psi(h_1) h_1^{-1/\alpha}\big)\big|^{-1} \big|\log\big(\psi(h_2) h_2^{-1/\alpha}\big)\big|^{-1}, & \text{if } d = 2\alpha, \\ c^{(10)} \big(\psi(h_1) h_1^{-1/\alpha}\big)^{d-2\alpha} \big(\psi(h_2) h_2^{-1/\alpha}\big)^{d-2\alpha}, & \text{if } d > 2\alpha. \end{cases}$$

PROOF. Consider only the case $d = 2\alpha$. Let $Y(u) = X(t-u) - X(t)$ and $Z(v) = X(t+v) - X(t)$, where $t > 0$ is fixed. If $P^{x,y}$ is as in Lemma 2.4, then the Markov property shows the probability in question equals

$$E\Bigg(I\Big(\inf_{u,\, v \in [h_1,\, k_1 h_1]} |Y(u) - Z(v)| \le \psi(h_1)\Big)$$

$$\times P^{Y(k_1 h_1),\, Z(k_1 h_1)}\Bigg(\inf_{\substack{0 \le u \le k_2 h_2 \\ h_2 \le v \le k_2 h_2}} |Y(u) - Z(v)| \le \psi(h_2)\Bigg)\Bigg)$$

$$\le c^{(8)}(k_2) \big|\log\big(\psi(h_2) h_2^{-1/\alpha}\big)\big|^{-1} P\Big(\inf_{u,\, v \in [h_1,\, k_1 h_1]} |Z(u) - Y(v)| \le \psi(h_1)\Big),$$

if $h_2 \le \delta(k_2)$ (Lemma 2.5)

$$\le c^{(8)}(k_2) c^{(8)}(k_1) \big|\log\big(\psi(h_2) h_2^{-1/\alpha}\big)\big|^{-1} \big|\log\big(\psi(h_1) h_1^{-1/\alpha}\big)\big|^{-1}$$

if $h_1 \le \delta(k_1)$ (Lemma 2.5). □

3. Integral tests for uniform escape rates. We first extend the integral test for the local two-sided escape rate of a Brownian path in Jain and Taylor (1973) to our stable setting. We include a proof for two reasons. First, the argument given there uses the Gaussian tail of $X(t)$ (if $\alpha = 2$). Second, there is a small gap in the proof, as the θ described at the top of page 542 need not exist. Fortunately, Lemma 2.6 allows one to close the gap and significantly simplify the proof.

THEOREM 3.1. *Let X be a d-dimensional stable process of index α satisfying the conditions described at the beginning of Section 2. Assume $d \geq 2\alpha$, $t > 0$ and $\psi \in \mathscr{H}$.*

For a.a. ω there is a $\delta(\omega) > 0$ such that

$$|X(t+u) - X(t-v)| > (u+v)^{1/\alpha}\psi(u+v),$$

whenever $0 < u + v \leq \delta$, $0 \leq u, v$, if and only if

$$\int_{0+} \psi(h)^{d-2\alpha} h^{-1}\, dh < \infty, \qquad d > 2\alpha, \tag{3.1}$$

$$\int_{0+} |\log \psi(h)|^{-1} h^{-1}\, dh < \infty, \qquad d = 2\alpha. \tag{3.2}$$

PROOF. We consider only the case $d = 2\alpha$ because the arguments for $d > 2\alpha$ are similar.

If (3.2) holds, the short Borel–Cantelli argument given in Jain and Taylor (1973) proves the result [use Lemma 2.5(a)]. Assume the integral in (3.2) is infinite. If k_0 is as in Lemma 2.5(b), define

$$B_n = \left\{\omega: \inf_{k_0^{-n} \leq u,\, v \leq k_0^{1-n}} |X(t+u) - X(t-v)| < k_0^{-n/\alpha}\psi(k_0^{-n})\right\}.$$

Lemma 2.5(b) shows that for large enough n, $P(B_n) \geq c^{(9)}|\log \psi(k_0^{-n})|^{-1}$ and hence our assumption implies $\sum_n P(B_n) = \infty$. If $m \leq n - 2$, apply Lemma 2.6 with $k_1 = k_0$, $h_1 = k_0^{-n}$, $h_2 = k_0^{-m} - k_0^{-n+1}$ and $k_2 = k_0(1 - k_0^{-2})(1 - k_0^{-1})^{-1}$ to see that for large enough m,

$$P(B_m \cap B_n) \leq c_1 |\log \psi(k_0^{-m})|^{-1} |\log \psi(k_0^{-n})|^{-1}.$$

The preceding lower bound on $P(B_n)$ therefore implies that

$$\limsup_{N\to\infty} \sum_{m=1}^{N} \sum_{n=1}^{N} P(B_m \cap B_n) \left(\sum_{m=1}^{N} P(B_m) \right)^{-2} \leq c_2 < \infty.$$

The Borel–Cantelli lemma of Kochen and Stone (1964) implies $P(B_n \text{ i.o.}) \geq c_2^{-1} > 0$. This probability is therefore 1 by the Blumenthal 0–1 law. The result follows. □

REMARK 3.2. Comparing this to the tests for the local one-sided escape rates obtained by Takeuchi (1964b), Theorem 1 [see also Taylor (1967) and Spitzer

(1958) for $d = \alpha = 2$ and Takeuchi and Watanabe (1964) for $d = \alpha = 1$], we see that the test for two-sided escape for X of index α is identical to the test for one-sided escape for X of index 2α. Of course, for Brownian motion this is the observation made in Jain and Taylor (1973) concerning the equivalence of the two-sided test for d-dimensional Brownian motion and the one-sided test for $(d-2)$-dimensional Brownian motion.

The main result of this section is the following integral test for the uniform two-sided escape rate of X.

THEOREM 3.3. *Let X be a d-dimensional stable process of index α, satisfying the conditions described at the beginning of Section 2. Assume $d \geq 2\alpha$ and let $\psi \in \mathscr{H}$. Then*

$$\liminf_{h \downarrow 0} \inf\{|X(t) - X(s)| : 0 \leq s, t \leq 1, |s - t| \geq h\}/\psi(h) \tag{3.3}$$

is a.s. ∞ or a.s. 0 according as

$$\int_{0+} \psi(h)^{d-2\alpha} h^{-d/\alpha}\, dh < \infty \text{ or } = \infty, \qquad d > 2\alpha, \tag{3.4}$$

$$\int_{0+} |\log \psi(h)|^{-1} h^{-2}\, dh < \infty \text{ or } = \infty, \qquad d = 2\alpha. \tag{3.5}$$

PROOF. We consider only the case $d = 2\alpha$ because a simpler argument works if $d > 2\alpha$.

Assume the integral in (3.5) is finite and define

$$\tilde{X}_h = \inf\{|X(s) - X(t)| : 0 \leq s, t \leq 1, |s - t| \geq h\}.$$

If

$$Y(s) = X(2h/3 - s) - X(2h/3), \qquad 0 \leq s \leq 2h/3,$$

and

$$Z(t) = X(2h/3 + t) - X(2h/3), \qquad t \geq 0,$$

then

$$\begin{aligned} P(\tilde{X}_h \leq \Delta) &\leq (3h^{-1}) P\Big(\inf_{s \leq h/3,\, h \leq t \leq 1} |X(s) - X(t)| \leq \Delta \Big) \\ &\leq 3h^{-1} P\Big(\inf_{h/3 \leq s \leq 2h/3,\, h/3 \leq t \leq 1} |Y(s) - Z(t)| \leq \Delta \Big) \\ &\leq c_1 h^{-1} |\log \Delta h^{-1/\alpha}|^{-1}, \end{aligned}$$

for h small enough and $\Delta \ll h^{1/\alpha}$ by Lemma 2.5. The finiteness of the integral in (3.5) implies

$$\sum_{n=N}^{\infty} 2^n |\log \psi(2^{-n})|^{-1} < \infty \quad \text{for some } N \in \mathbb{N} \tag{3.6}$$

and in particular $\psi(h) \ll h^{1/\alpha}$ as $h \downarrow 0$. Fix $K > 0$ and let $\Delta = K\psi(2^{-n+1})$, $h = 2^{-n}$ in the preceding, and use (3.6) to see that

$$\sum_{n=1}^{\infty} P\big(\tilde{X}_{2^{-n}} \le K\psi(2^{-n+1})\big) < \infty.$$

The Borel–Cantelli lemma implies

$$\liminf_{h \downarrow 0} \tilde{X}_h/\psi(h) \ge K \quad \text{a.s.}$$

Letting $K \to \infty$ we see that this lim inf is ∞ a.s.

Assume now that the integral in (3.5) is infinite. This is equivalent to

$$\sum_{j=N}^{\infty} 2^j \left|\log \psi(2^{-j})\right|^{-1} = \infty$$

$$\Leftrightarrow \sum_{j=N}^{\infty} \left(2^j \left|\log \psi(2^{-j})\right|^{-1}\right) \wedge 1 = \infty$$

$$\Leftrightarrow \sum_{j=N}^{\infty} 2^j \left|\log \tilde{\psi}(2^{-j})\right|^{-1} = \infty, \quad \text{where } \tilde{\psi}(h) = \psi(h) \wedge e^{-1/h}$$

$$\Leftrightarrow \int_{0+} \left|\log \tilde{\psi}(h)\right|^{-1} h^{-2}\, dh = \infty.$$

Hence by replacing ψ with $\tilde{\psi}$ we may assume without loss of generality that

$$\psi(h) \le e^{-1/h}. \tag{3.7}$$

Fix $k \in \mathbb{N}$, $k \ge k_0$ and let $h_j = (2k)^{-j}$,

$$I_j^{1,p} = \left[2pkh_j, (2pk + k - 1)h_j\right],$$

$$I_j^{2,p} = \left[(2pk + k + 1)h_j, 2(p + 1)kh_j\right],$$

$$I_j^p = \left[2pkh_j, 2(p + 1)kh_j\right].$$

If $\phi(h) = h^{-1}|\log \psi(h)|^{-1}$, the integral condition on ψ implies there is a sequence $\{\varepsilon_j\}$ that decreases to 0 and satisfies $\varepsilon_j > h_j$ and

$$\sum_j \varepsilon_j \phi(h_j) = \infty. \tag{3.8}$$

Define events A_j^p and A_j by

$$A_j^p = \left\{\inf\left\{|X(s) - X(t)| : s \in I_j^{1,p}, t \in I_j^{2,p}\right\} < \psi(h_j)\right\},$$

$$A_j = \bigcup_{0 \le p < \varepsilon_j (2k)^j} A_j^p.$$

An easy computation using Lemma 2.5, (3.7), $\varepsilon_j > h_j$ and the independence of $\{A_j^p : 0 \le p < \varepsilon_j(2k)^j\}$ shows there are positive constants c_2 and c_3 such that (for large j)

$$c_2 \left|\log \psi(h_j)\right|^{-1} \le P(A_j^p) \le c_3 \left|\log \psi(h_j)\right|^{-1}, \tag{3.9}$$

$$c_2 \varepsilon_j \phi(h_j) \le P(A_j) \le c_3 \varepsilon_j \phi(h_j). \tag{3.10}$$

In particular, we see that $\sum_j P(A_j) = \infty$ by (3.8). To conclude that $P(A_j \text{ i.o.}) = 1$, an upper bound on $P(A_i \cap A_j)$ is needed. Fix $i < j$ and for each nonnegative integer, q, let $p(q)$ denote the unique value of p for which $I_j^q \subset I_i^p$. If $p \ne p(q)$, then A_i^p and A_j^q are independent events and so, using (3.9) and (3.10), one easily obtains

$$P(A_i \cap A_j) \le c_4 P(A_i) P(A_j) + \sum_{0 \le q < \varepsilon_j(2k)^j} P\left(A_i^{p(q)} \cap A_j^q\right). \tag{3.11}$$

Fix $0 \le q < \varepsilon_j(2k)^j$ and $p = p(q)$. Assume first that

$$I_j^q \subset \left[(2pk + k - 1)h_i, (2pk + k + 1)h_i\right].$$

Apply Lemma 2.6 and (3.7) to see that (at least for i, j large enough)

$$P\left(A_j^q \cap A_i^p\right) \le c_5 \left|\log \psi(h_i)\right|^{-1} \left|\log \psi(h_j)\right|^{-1}. \tag{3.12}$$

Next consider the case when $I_j^q \subset I_i^{1,p}$ ($I_j^q \subset I_i^{2,p}$ may be handled by a similar argument). A_i^p is contained in the union of the following three sets:

$$B_1 = \left\{|X(s) - X(t)| < \psi(h_j) \text{ for some } s \text{ in } I_i^{1,p} \text{ to the right of } I_j^q \text{ and some } t \text{ in } I_i^{2,p}\right\},$$

$$B_2 = \left\{|X(s) - X(t)| < \psi(h_i) \text{ for some } s \text{ in } I_i^{1,p} \text{ to the left of } I_j^q \text{ and some } t \text{ in } I_i^{2,p}\right\},$$

$$B_3 = \left\{|X(s) - X(t)| < \psi(h_i) \text{ for some } s \text{ in } I_j^q \text{ and } t \text{ in } I_i^{2,p}\right\}.$$

Then B_1 and A_j^q are independent and hence (3.9) gives (for large i, j)

$$P\left(A_j^q \cap B_1\right) \le c_3^2 \left|\log \psi(h_j)\right|^{-1} \left|\log \psi(h_i)\right|^{-1}. \tag{3.13}$$

Moreover, Lemma 2.6 and (3.7) imply (for i, j large)

$$P\left(A_j^q \cap B_2\right) \le c_6 \left|\log \psi(h_j)\right|^{-1} \left|\log \psi(h_i)\right|^{-1}. \tag{3.14}$$

It remains to bound $P(A_j^q \cap B_3)$. Define i.i.d. copies of X by

$$Y(s) = X\left(2(q + 1)kh_j - s\right) - X\left(2(q + 1)kh_j\right),$$

$$Z(t) = X\left(2(q + 1)kh_j + t\right) - X\left(2(q + 1)kh_j\right).$$

If $h_i' = (2pk + k + 1)h_i - 2(q+1)kh_j \geq h_i$, then for $n \in \mathbb{N}$ we have

$$P\big(A_j^q \cap B_3\big) \leq P\big(N^Y(2kh_j, \psi(h_i)) \geq n\big) + E\Big(I\big(A_j^q, N^Y(2kh_j, \psi(h_i)) < n\big) \times P\big(|Y(s) - Z(t)| < \psi(h_i) \text{ for some } s \in [0, 2kh_j] \text{ and } t \in [h_i', h_i' + 3h_i] | Y\big)\Big).$$

On $\{N^Y(2kh_j, \psi(h_i)) < n\}$, the preceding conditional probability is bounded by

$$\sum_{m=0}^{n-1} P\big(Z(t) \in B\big(Y(\tau_m^Y(\psi(h_i))), 2\psi(h_i)\big) \text{ for some } t \in [h_i', h_i' + 3h_i] | Y\big)$$

$$\leq c^{(3)} n \big(\psi(h_i) h_i^{-1/\alpha}\big)^{d-\alpha} = c^{(3)} n \psi(h_i)^\alpha h_i^{-1} \quad \text{(Lemma 2.2)}.$$

Therefore we have shown that for any $n \in \mathbb{N}$,

$$P\big(A_j^q \cap B_3\big) \leq P\big(N^Y(2kh_j, \psi(h_i)) \geq n\big) + c^{(3)} P\big(A_j^q\big) n \psi(h_i)^\alpha h_i^{-1}. \tag{3.15}$$

Fix $r > 2\alpha$ and assume first that $h_j \geq \psi(h_i)^r$. Take $n = [c^{(1)} 2kh_j \psi(h_i)^{-\alpha}] + [\psi(h_i)^{-\alpha/2}] + 1 \geq h_j^{-\alpha/2r}$ and use (3.15) and Lemma 2.1 to obtain (for i, j large)

$$P\big(A_j^q \cap B_3\big) \leq \exp\big\{-\lambda h_j^{-\alpha/2r}\big\} + c_7 P\big(A_j^q\big)\big(h_j h_i^{-1} + \psi(h_i)^{\alpha/2} h_i^{-1}\big). \tag{3.16}$$

Assume now that $h_j < \psi(h_i)^r$. Then

$$\begin{aligned} P\big(N^Y(2kh_j, \psi(h_i)) \geq 2\big) &\leq P\big(\tau_1^Y(\psi(h_i)) \leq 2kh_j\big)^2 \\ &= P\Big(\sup_{t \leq 1} |X(t)| \geq \psi(h_i)(2kh_j)^{-1/\alpha}\Big)^2 \\ &\leq c_8 \psi(h_i)^{-2\alpha} h_j^2 \quad \text{[see Proposition 10.2 of Fristedt (1974)]} \\ &\leq c_8 h_j^{2(1-\alpha/r)}. \end{aligned}$$

Take $n = 2$ in (3.15) to see that in this case

$$P\big(A_j^q \cap B_3\big) \leq c_8 h_j^{2(1-\alpha/r)} + 2c^{(3)} P\big(A_j^q\big) \psi(h_i)^\alpha h_i^{-1}. \tag{3.17}$$

Combining (3.12), (3.13), (3.14), (3.16) and (3.17), we get that in any case (for $i < j$ large),

$$P\big(A_j^q \cap A_i^{p(q)}\big) \leq c_9\Big(\big|\log \psi(h_j)\big|^{-1} \big|\log \psi(h_i)\big|^{-1} + \exp\big\{-\lambda h_j^{-\alpha/2r}\big\} + \big|\log \psi(h_j)\big|^{-1}\big(h_j h_i^{-1} + \psi(h_i)^{\alpha/2} h_i^{-1}\big) + h_j^{2(1-\alpha/r)}\Big).$$

Substitute into (3.11) and use (3.10) to conclude that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P(A_i \cap A_j)$$

$$\le c_{10}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} P(A_i)P(A_j) + \sum_{j=1}^{n} \varepsilon_j h_j^{-1} j\left(\exp\{-\lambda h_j^{-\alpha/2r}\} + h_j^{2(1-\alpha/r)}\right)\right.$$

$$\left. + \sum_{j=1}^{n} P(A_j) h_j\left(\sum_{i<j} h_i^{-1}\right) + \left(\sum_{j=1}^{n} P(A_j)\right)\left(\sum_{i=1}^{n} \psi(h_i)^{\alpha/2} h_i^{-1}\right)\right)$$

$$\le c_{11}\left(1 + \left(\sum_{j=1}^{n} P(A_j)\right)^2 + \sum_{j=1}^{n} P(A_j)\right).$$

In the last line we have used (3.7) and the fact that $r > 2\alpha$. Recalling $\Sigma_1^\infty P(A_i) = \infty$, one concludes

$$\limsup_{n\to\infty}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} P(A_i \cap A_j)\right)\left(\sum_{j=1}^{n} P(A_j)\right)^2 \le c_{11}$$

and hence [Kochen and Stone (1964)]

$$P(A_n \text{ i.o.}) \ge c_{11}^{-1} > 0.$$

The Blumenthal 0–1 law shows the preceding probability must be 1 (this is why we introduced $\{\varepsilon_j\}$). Finally, the required result follows by simply replacing ψ with ψ/M and letting $M \to \infty$. □

It is clear that the problems of one- and two-sided uniform escape rates are one and the same. More precisely, the quantities

$$\liminf_{h\downarrow 0} \inf_{0\le s,t\le 1, |s-t|\ge h} |X(s) - X(t)|\psi(h)^{-1} \tag{3.18}$$

and

$$\liminf_{h\downarrow 0} \inf_{0\le t\le 1} |X(t+h) - X(t)|\psi(h)^{-1} \tag{3.19}$$

are equal. Less obvious is the fact that these quantities equal the seemingly larger quantities

$$\inf_{0\le t\le 1} \liminf_{s,u\downarrow 0} \left|X(t+s) - X\left((t-u)^+\right)\right|\psi(s+u)^{-1} \tag{3.20}$$

and

$$\inf_{0\le t\le 1} \liminf_{h\downarrow 0} |X(t+h) - X(t)|\psi(h)^{-1}. \tag{3.21}$$

THEOREM 3.4. *Let X be as in Theorem* 3.3. *Assume $d \ge 2\alpha$ and let $\psi \in \mathscr{H}$. The expressions* (3.18), (3.19), (3.20) *and* (3.21) *are a.s. infinite or a.s.* 0

according as the integrals in (3.4) (*if* $d > 2\alpha$) *or* (3.5) (*if* $d = 2\alpha$) *are finite or infinite. When these expressions equal* 0 *there is an uncountable dense set* (*in fact a dense* G_δ) *of times, t, in* $[0, \infty)$ *for which*

$$\liminf_{h \downarrow 0} |X(t+h) - X(t)| \psi(h)^{-1} = 0. \tag{3.22}$$

PROOF. Assume the integral in (3.4) [or (3.5) if $d = 2\alpha$] is infinite. By Theorem 3.3 we may fix ω outside a null set such that for any open interval, I, with rational endpoints,

$$\liminf_{h \downarrow 0} \inf_{s,\, t \in I,\, |s-t| \geq h} |X(s) - X(t)| \psi(h)^{-1} = 0. \tag{3.23}$$

Define a sequence of open sets by

$$G_n = \{t : \exists\, \varepsilon > 0 \text{ and } h \in (0, 1/n) \text{ such that } \forall s \in (t - \varepsilon, t + \varepsilon),$$
$$|X(s+h) - X(s)| < \psi(h) n^{-1}\}.$$

By (3.23), if n is fixed and I is an open interval with rational endpoints, there is a t in I and an $h \in (0, 1/n)$ such that $|X(t+h) - X(t)| < n^{-1}\psi(h)$. By right-continuity we see that for some $\delta > 0$, $|X(u+h) - X(u)| < n^{-1}\psi(h)$ for $u \in (t, t+\delta)$. Clearly, $(t, t+\delta) \subset G_n$ and hence $G_n \cap I \neq \varnothing$. G_n is therefore dense and hence by Baire's theorem [Dugundji (1966), page 249] $S = \bigcap_n G_n$ is a dense subset of $[0, \infty)$ of the second category, and in particular is uncountable. Clearly, (3.22) holds whenever $t \in S$. The rest of the theorem is immediate from Theorem 3.3 and the obvious ordering (3.19) = (3.18) ≤ (3.20) ≤ (3.21). □

4. Hausdorff dimensions for points of slow escape. As was pointed out in the Introduction, the gap between Theorems 3.1 and 3.3 suggests we look at the following exceptional sets of times where the two-sided escape rate is unusually slow:

$$C_\gamma = \left\{t > 0;\ \liminf_{s, u \downarrow 0} |X(t+s) - X(t-u)| (s+u)^{-\gamma} = 0\right\},$$

$$D_\gamma = \left\{t > 0:\ \liminf_{s, u \downarrow 0} |X(t+s) - X(t-u)| e^{(s+u)^{-\gamma}} = 0\right\}.$$

Theorems 3.1 and 3.4 show that if $d > 2\alpha$, C_γ is a nonempty set of Lebesgue measure 0 if $\alpha^{-1} < \gamma \leq \alpha^{-1} + (d - 2\alpha)^{-1}$ and is empty if $\gamma > \alpha^{-1} + (d - 2\alpha)^{-1}$. They also show that if $d = 2\alpha$, D_γ is a nonempty set of Lebesgue measure 0 if $0 < \gamma \leq 1$ and is empty if $\gamma > 1$. The upper bound on the Hausdorff dimensions of these sets is found by a simple covering argument, whereas the lower bound is obtained by showing the set will a.s. intersect with the range of an independent stable subordinator of appropriate index.

NOTATION. If $U(t)$ is a stable subordinator of index β' and $c > 0$, let

$$A_c = A_c^U = \{U(\tau_i^U(2^{-c})) : i = 0, 1, 2, \dots\} \cap (0, 1).$$

[The notion $\tau_i^U(\Delta)$ was introduced prior to Lemma 2.1.]

Note that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ is the closed range of U in $[0,1]$. Using the fact that $x^{\beta'} - m(A) = \infty$ a.s. [Taylor and Wendel (1966)], one can show that for a.a. ω and for any $a \in \bigcup_{n=1}^{\infty} A_n$ and $\varepsilon > 0$ there is an $N(a, \varepsilon, \omega) \in \mathbb{N}$ such that

$$\text{(4.1)} \qquad \operatorname{card}\big(A_n \cap (a, a+\varepsilon)\big) \geq \varepsilon 2^{n\beta'}, \quad \text{for } n \geq N.$$

[See the discussion following Proposition 3.4 in Barlow and Perkins (1984).]

Recall that $\dim E < 0$ implies the set E is empty.

Theorem 4.1. (a) *If* $d \geq 2\alpha$, *then*

$$\dim C_\gamma = 1 - (\gamma - \alpha^{-1})(d - 2\alpha), \quad \text{for } \gamma \geq \alpha^{-1},$$

and $C_{\alpha^{-1}+(d-2\alpha)^{-1}}$ *is an uncountable dense subset of* $[0,\infty)$.

(b) *If* $d = 2\alpha$, *then* $\dim D_\gamma = 1 - \gamma$ *for* $\gamma \geq 0$ *and* D_1 *is an uncountable dense subset of* $[0,\infty)$.

Proof. (a) In view of (b), we may assume $d > 2\alpha$, and clearly we may assume $\gamma > \alpha^{-1}$. Let

$$\mathscr{I}_n = \big\{[(j-5)2^{-n}, (j+5)2^{-n}] : 0 \leq j < 2^n, \exists s \in [(j-5)2^{-n}, (j-1)2^{-n}],$$
$$t \in [(j+1)2^{-n}, (j+5)2^{-n}] \text{ such that } |X(s) - X(t)| < 2^{-n\gamma}\big\}.$$

Let $t \in C_\gamma \cap [0,1]$, and let $s_n \uparrow t$, $u_n \downarrow t$ satisfy $|X(u_n) - X(s_n)| < |u_n - s_n|^\gamma/4$ and $u_n < 1$. Choose $m_n \in \mathbb{N}$ and $j_n < 2^{m_n}$, $j_n \in \mathbb{N}$ such that $2^{2-m_n} \leq u_n - s_n < 2^{3-m_n}$ and $|(u_n + s_n)/2 - j_n 2^{-m_n}| < 2^{-m_n}$. Then

$$\begin{aligned} 2^{-m_n} = 2^{1-m_n} - 2^{-m_n} &\leq u_n - (u_n + s_n)/2 - |(u_n + s_n)/2 - j_n 2^{-m_n}| \\ &\leq u_n - j_n 2^{-m_n} \\ &< (u_n - s_n)/2 + 2^{-m_n} < 5 \cdot 2^{-m_n}. \end{aligned}$$

Therefore $u_n \in [(j_n + 1)2^{-m_n}, (j_n + 5)2^{-m_n}]$ and similarly one shows $s_n \in [(j_n - 5)2^{-m_n}, (j_n - 1)2^{-m_n}]$. It follows that for each $n \in \mathbb{N}$,

$$t \in [(j_n - 5)2^{-m_n}, (j_n + 5)2^{-m_n}] \in \mathscr{I}_{m_n}$$

and hence

$$\text{(4.2)} \qquad C_\gamma \cap [0,1) \subset \Big\{t : t \in \bigcup\{I : I \in \mathscr{I}_n\} \text{ for infinitely many } n\Big\}.$$

If $\beta > 1 - (\gamma - \alpha^{-1})(d - 2\alpha)$, then

$$\begin{aligned} E\bigg(\sum_{I \in \mathscr{I}_n} |I|^\beta\bigg) &\leq 10^\beta 2^{-n\beta} 2^n P\big(\exists\, s \in [0, 4 \cdot 2^{-n}], t \in [6 \cdot 2^{-n}, 10 \cdot 2^{-n}] \\ &\qquad\qquad \text{such that } |X(s) - X(t)| \leq 2^{-n\gamma}\big) \\ &\leq c_1 2^{-n\beta} 2^n 2^{-n\gamma(d-2\alpha)} 2^{n(d-2\alpha)/\alpha} \quad [\text{Lemma 2.4(a)}]. \end{aligned}$$

The right side is summable by the choice of β. Therefore (4.2) shows that

$\dim C_\gamma \cap [0,1) \le \beta$. Let $\beta \downarrow 1 - (\gamma - \alpha^{-1})(d - 2\alpha)$ and replace $[0,1)$ by a sequence of intervals increasing to $[0,\infty)$ to get the required bound on $\dim C_\gamma$.

To obtain the lower bound on $\dim C_\gamma$, fix $\gamma > \alpha^{-1}$ and let U denote a stable subordinator of index $\beta' > (\gamma - \alpha^{-1})(d - 2\alpha)$, independent of X. We fix a sample path of U such that (4.1) holds and argue conditionally on U. Let $k \ge k_0 \vee 2$ and define

$$B_n = \{t \in A_n : \exists\, u \in [t - 2^{-1-n}, t - k^{-1}2^{-1-n}] \text{ and } s \in [t + k^{-1}2^{-1-n}, t + 2^{-1-n}] \text{ such that } |X(u) - X(s)| < n^{-1}2^{-n\gamma}\}.$$

For $t \in A_n$ fixed, Lemma 2.4(b) implies

$$P(t \in B_n) \ge c_2\left(n^{-1}2^{-n(\gamma-\alpha^{-1})}\right)^{d-2\alpha}.$$

If $s \ne t$ are elements of A_n, the events, $\{s \in B_n\}$ and $\{t \in B_n\}$ are independent and so (4.1) shows that for $t \in \bigcup_n A_n$, $\varepsilon > 0$ and large enough n,

$$P(B_n \cap (t, t+\varepsilon) = \varnothing) \le \left(1 - c_2 n^{2\alpha-d}2^{-n(\gamma-\alpha^{-1})(d-2\alpha)}\right)^{\varepsilon 2^{n\beta'}},$$

which is summable over n by the choice of β'. By the Borel–Cantelli lemma we may fix ω outside a null set such that for any $t \in \bigcup_n A_n$ and $\varepsilon > 0$ there is an $N(t,\varepsilon,\omega) \in \mathbb{N}$ such that $B_n \cap (t, t+\varepsilon) \ne \varnothing$ for $n \ge N$. Define a sequence of open sets by

$$G_n = \{t \in (0,1) : \exists\, h_1, h_2 \in (0, 2^{-n}) \text{ and } \delta > 0 \text{ such that for all } v \in (t-\delta, t+\delta), |X(v - h_1) - X(v + h_2)| < k^\gamma n^{-1}(h_1 + h_2)^\gamma\}.$$

We will prove $G_n \cap A$ is dense in $A \equiv$ the closed range of U in $[0,1]$. If $t \in \bigcup_n A_n$ and $\varepsilon > 0$ are fixed, then for large enough n there is a $t' \in A_n \cap (t, t+\varepsilon)$, $u \in [t' - 2^{-n-1}, t')$ and $s \in (t', t' + 2^{-n-1}]$ such that $|X(u) - X(s)| < n^{-1}k^\delta|s - u|^\gamma$. Let $h_1 = t' - u$ and $h_2 = s - t'$. Then $h_1, h_2 \in (0, 2^{-n})$ and

$$|X(t' - h_1) - X(t' + h_2)| < k^\gamma n^{-1}(h_1 + h_2)^\gamma.$$

This inequality must persist if t' is replaced by $v \in (t', t' + \eta)$ for some $\eta > 0$ by right-continuity. Since $t' \in A_n$, this open interval, which is contained in G_n, must intersect $A \cap (t, t+\varepsilon)$ and hence $G_n \cap A \cap (t, t+\varepsilon) \ne \varnothing$ for large enough n. Therefore $G_n \cap A$ is an open dense subset of the locally compact space A for all n ($G_n \downarrow$in n) and by Baire's theorem [Dugundji (1966), page 244] $(\bigcap G_n) \cap A$ is dense in A. Therefore $C_\gamma \cap A \ne \varnothing$ (for β' as previously) because $\bigcap G_n \subset C_\gamma$. A fixed Borel set, B, will not intersect A (a.s.) if $\dim B < 1 - \beta'$ [see Lemma 2 of Hawkes (1971b)]. The obvious Fubini argument implies $\dim C_\gamma \ge 1 - \beta'$. Let $\beta' \downarrow (\gamma - \alpha^{-1})(d - 2\alpha)$ to obtain the required lower bound.

The last statement of (a) is immediate from Theorem 3.4.

(b) Simply use Lemma 2.5 in place of Lemma 2.4 in the preceding argument. □

Consider now the following sets of times where there is an unusually slow rate of one-sided escape:

$$E_\gamma = \left\{t \geq 0 : \liminf_{h \downarrow 0} |X(t+h) - X(t)| h^{-\gamma} = 0\right\}.$$

If $d \geq 2\alpha$, the integral tests of Takeuchi (1964b) and Taylor (1967) for local one-sided escape rates and Theorem 3.4 show E_γ is a nonempty Lebesgue null set if $\alpha^{-1} < \gamma < \alpha^{-1} + (d - 2\alpha)^{-1}$. If $d > 2\alpha$, E_γ is nonempty for $\gamma = \alpha^{-1} + (d - 2\alpha)^{-1}$ but is empty for $\gamma > \alpha^{-1} + (d - 2\alpha)^{-1}$. E_γ may also be of interest for $\alpha \leq d < 2\alpha$ since such an X will still not hit points even though it will now have multiple points. The aforementioned local escape rates show that E_γ is still Lebesgue null for $\gamma > \alpha^{-1}$ and $d > \alpha$, but the integral tests of Spitzer (1958) for $d = \alpha = 2$ and Takeuchi and Watanabe (1964) for $d = \alpha = 1$ show E_γ is of full Lebesgue measure for any $\gamma > 0$ in these cases.

The derivation of the correct lower bound on E_γ is more involved than was the case for C_γ. We will need a much stronger result than (4.1) on the structure of the range of a stable subordinator. Recall the notation A_c introduced prior to Theorem 4.1.

LEMMA 4.2. *Let U be a stable subordinator of index β'. Fix $K \geq 1$, $0 < \beta < \beta'$, $\delta > 1$ and $\eta > \beta'$. If $a \in A_n$ and the cardinality of $A_{n\delta} \cap (a, a + 2^{-n}/K)$ exceeds $2^{\beta n(\delta - 1)}$, let*

$$S_n(a) = \mathop{\sum\nolimits' \sum\nolimits'}_{s_i \neq s_j} I\big(s_i, s_j \in A_{n\delta} \cap (a, a + 2^{-n}/K)\big)|s_i - s_j|^{-\eta},$$

where Σ' indicates the sum is only over the smallest $[2^{\beta n(\delta-1)}]$ elements of $A_{n\delta} \cap (a, a + 2^{-n}/K)$. There is an $M > 0$ such that if

$$\begin{aligned}\tilde{A}_n = \big\{a \in A_n &: \operatorname{card}\big(A_{n\delta} \cap (a, a + 2^{-n}/K)\big) > 2^{\beta n(\delta - 1)},\\ &S_n(a) \leq M 2^{n(\delta\eta + \beta(\delta - 1))}\big\},\end{aligned}$$

then for a.a. ω for all $a \in \bigcup_{n=1}^\infty A_n$ and $\varepsilon > 0$ there is an $N(a, \varepsilon, \omega) \in \mathbb{N}$ such that $\operatorname{card}(\tilde{A}_n \cap (a, a + \varepsilon)) \geq 2^{n\beta - 1}$ *for $n \geq N$.*

PROOF. It suffices to prove the conclusion for a fixed $a \in \bigcup A_n$ and $\varepsilon \in (0,1)$ and to simplify the argument slightly we will assume $a = 0$. (It will then be clear how to handle a general $a \in A_n$.) A scaling argument shows that if $\{Y_i\}$ are i.i.d. copies of $U(\tau_1^U(1))$, then $U(\tau_j(2^{-n\delta}))$ is equal in law to $2^{-n\delta}\sum_{i=1}^j Y_i$. It is well known that

$$\lim_{t \to \infty} P(Y_1 > t) t^{\beta'} = c_1 \in (0, \infty) \tag{4.3}$$

[see, for example, Theorem 4.1 of Fristedt (1974)]. It follows from the preceding that

$$
\begin{aligned}
p_n &\equiv P\big(\operatorname{card}(A_{n\delta} \cap (0, 2^{-n}/K)) \le 2^{\beta n(\delta-1)}\big) \\
&\le P\Big(U\big(\tau_{[2^{\beta n(\delta-1)}]+1}(2^{-n\delta})\big) \ge 2^{-n}/K\Big) \\
&= P\left(\sum_{i=1}^{[2^{\beta n(\delta-1)}]+1} Y_i \ge 2^{n(\delta-1)}/K \right) \\
&\le \left(\sum_{i=1}^{[2^{\beta n(\delta-1)}]+1} Y_i - 2 \ge 2^{n(\delta-1)}/2K \right) \quad (\text{if } n \ge N_0) \\
&\le P\big(U([2^{\beta n(\delta-1)}] + 1) \ge c_3 2^{n(\delta-1)}/2K\big),
\end{aligned}
$$

for some $c_3 > 0$. In the last line note that (4.3) and $P(Y_1 - 2 > 0) < 1$ show that $P(Y_1 - 2 > t) \le P(U(1) > ct)$ for all $t \in \mathbb{R}$ for some $c > 0$. Therefore if $n \ge N_0$, we have

$$
p_n \le P\big(U(1) \ge c_4 2^{n(\delta-1)(1-\beta/\beta')}/K\big) \to 0, \quad \text{as } n \to \infty. \tag{4.4}
$$

Similarly, one obtains

$$
\begin{aligned}
q_n &\equiv P\big(S_n(0) > M 2^{n(\delta\eta+\beta(\delta-1))}\big) \\
&\le M^{-1} 2^{-n(\delta\eta+\beta(\delta-1))} 2 \sum\sum_{1 \le i < j \le [2^{\beta n(\delta-1)}]} E\Big(\big(U(\tau_j(2^{-n\delta})) - U(\tau_i(2^{-n\delta}))\big)^{-\eta}\Big) \\
&\le c_5 M^{-1} 2^{-n(\delta\eta+\beta(\delta-1))} 2^{n\delta\eta} \sum\sum_{1 \le i < j \le 2^{\beta n(\delta-1)}} E\big(U(j-i)^{-\eta}\big) \quad [\text{by (4.3), as before}] \\
&\le c_5 M^{-1} E\big(U(1)^{-\eta}\big) \sum_{1 \le i \le 2^{\beta n(\delta-1)}} i^{-\eta/\beta'} \\
&\le c_6 M^{-1},
\end{aligned}
$$

where in the last line we have used $\eta > \beta'$ and $E(U(1)^{-\eta}) < \infty$ [see Lemma 1 of Hawkes (1971a)]. Choose $M > 0$ and $N_1 \in \mathbb{N}$ such that $p_n + q_n < 1/3$, if $n \ge N_1$. Therefore if

$$
r_n = P\big(\operatorname{card}(A_{n\delta} \cap (0, 2^{-n}/K)) > 2^{\beta n(\delta-1)} \text{ and } S_n(0) \le M 2^{n(\delta\eta+\beta(\delta-1))}\big),
$$

then $r_n > 2/3$ if $n \ge N_1$. It follows easily from the preceding that if

$$
\begin{aligned}
C_i^n = \big\{\omega\colon &\operatorname{card}\big(A_{n\delta} \cap \big(U(\tau_i(2^{-n})), U(\tau_i(2^{-n})) + 2^{-n}/K\big)\big) > 2^{\beta n(\delta-1)}, \\
&S_n\big(U(\tau_i(2^{-n}))\big) \le M 2^{n(\delta\eta+\beta(\delta-1))}\big\},
\end{aligned}
$$

then there is an N_2 such that $n \ge N_2$ implies

$$
P\Big(C_i^n \,|\, \mathscr{F}^U_{\tau_i(2^{-n})}\Big) > 2/3. \tag{4.5}
$$

For each $n \in \mathbb{N}$ define

$$M^n(k) = 2^{-n\beta} \sum_{i=0}^{k \wedge 2^{n\beta}} I(U(\tau_i(2^{-n})) < \varepsilon)\Big[I(C_i^n) - P\Big(C_i^n \big| \mathscr{F}^U_{\tau_i(2^{-n})}\Big)\Big].$$

Then $\{(M^n(k), \mathscr{F}^U_{\tau_{k+1}(2^{-n})}): k = 0,1,2,\dots\}$ is a martingale and

$$E\big(M^n(\infty)^2\big) = E(\langle M^n, M^n\rangle_\infty) \le 2^{-2n\beta}(2^{n\beta}+1)$$

is summable. Therefore $\lim_{n\to\infty} M^n(\infty) = 0$ a.s. and hence (4.5) shows that for a.a. ω and large enough n,

$$\begin{aligned}\sum_{i=0}^{2^{n\beta}} I(U(\tau_i(2^{-n})) < \varepsilon) I(C_i^n) &> 2/3 \sum_{i=0}^{2^{n\beta}} I(U(\tau_i(2^{-n})) < \varepsilon) - (1/6)2^{n\beta} \\ &\ge 2^{n\beta-1} \quad [\text{by } (4.1)]. \end{aligned}$$ □

THEOREM 4.3. *If $d \ge \alpha$, then*

$$\dim E_\gamma = 1 - \alpha^{-1}\big(1 - (\alpha\gamma)^{-1}\big)(d-\alpha), \quad \text{for } \gamma \ge \alpha^{-1}.$$

Moreover, if $d > 2\alpha$, $E_{\alpha^{-1}+(d-2\alpha)^{-1}}$ is an uncountable dense subset of $[0,\infty)$.

PROOF. The last statement is an immediate consequence of Theorem 3.4. Moreover, if $d = \alpha$, we have already remarked that E_γ is the complement of a Lebesgue null set for any $\gamma > 0$, so assume $d > \alpha$.

Fix $\gamma \ge \alpha^{-1}$ and for each $n \in \mathbb{N}$ inductively define a sequence of stopping times by

$$T_0^n = 0, \qquad T_{i+1}^n = \inf\{t \ge T_i^n : |X(t) - X(T_i^n)| \ge 2^{-n\gamma}\} \wedge (T_i^n + 2^{-n\gamma\alpha}).$$

A law of large numbers argument (as in the proof of Lemma 2.1) shows there is a c_1 such that if $N_n = [c_1 2^{n\gamma\alpha}] - 1$, then $T_{N_n}^n \ge 1$ for large enough n a.s. Fix such an ω and let $t \in E_\gamma \cap [0,1]$. Choose $h_n \downarrow 0$ and $m_n \in \mathbb{N}$ such that $|X(t+h_n) - X(t)| \le h_n^\gamma$, $2^{-m_n-1} \le h_n < 2^{-m_n}$. For large enough n, $T_{N_{m_n}}^{m_n} \ge 1$ and hence $T_i^{m_n} \le t < T_{i+1}^{m_n}$ for some $i \le N_{m_n}$. Note that if $h_n' = h_n + t - T_i^{m_n} \ge 2^{-m_n-1}$, then

$$\begin{aligned}|X(T_i^{m_n} + h_n') - X(T_i^{m_n})| &\le |X(h_n + t) - X(t)| + 2^{-m_n\gamma} \\ &\le 2^{1-m_n\gamma}.\end{aligned}$$

Therefore if

$$\mathscr{I}_n = \big\{[T_i^n, T_{i+1}^n] : i \le N_n, |X(T_i^n + h) - X(T_i^n)| \le 2^{1-n\gamma} \text{ for some } h \ge 2^{-n-1}\big\},$$

then $t \in [T_i^{m_n}, T_{i+1}^{m_n}] \in \mathscr{I}_{m_n}$ for large enough n, and hence

$$[0,1] \cap E_\gamma \subset \big\{t : t \in \bigcup\{I : I \in \mathscr{I}_n\} \text{ i.o.}\big\}. \tag{4.6}$$

If $\beta > 1 - \alpha^{-1}(1 - (\alpha\gamma)^{-1})(d - \alpha)$, then

$$E\left(\sum_{I \in \mathscr{I}_n} |I|^{\beta}\right) \le c_1 2^{n\gamma\alpha} 2^{-n\beta\gamma\alpha} P\left(|X(h)| \le 2^{1-n\gamma} \text{ for some } h \ge 2^{-n-1}\right)$$

$$\le c_2 2^{n\gamma\alpha(1-\beta)} 2^{-n\gamma(d-\alpha)} 2^{n(d/\alpha - 1)} \quad \text{(Lemma 2.2)}.$$

The preceding is summable by the choice of β. (4.6) now implies $\dim E_{\gamma} \cap [0,1] \le \beta$ and letting $\beta \downarrow 1 - \alpha^{-1}(1 - (\alpha\gamma)^{-1})(d - \alpha)$ we get the required upper bound on $\dim E_{\gamma} \cap [0,1]$ and hence on $\dim E_{\gamma}$.

Turning to the lower bound on $\dim E_{\gamma}$, we need only consider $\gamma > \alpha^{-1}$. Choose γ', γ'' so that $\alpha^{-1} < \gamma' < \gamma < \gamma''$ and assume $\gamma'' - \gamma'$ is small enough so that $\gamma''/\gamma' - (\alpha\gamma')^{-1} < 1$. Now select β, β' so that

$$(d/\alpha) - 1 > \beta' > \beta > \left(\gamma''/\gamma' - (\alpha\gamma')^{-1}\right)\left((d/\alpha) - 1\right). \tag{4.7}$$

Let $k > k_0 \vee 2$ (k_0 as in Lemma 2.3) and let U be a stable subordinator of index β', independent of X. Now argue conditionally on U, assuming we have fixed a sample path satisfying the conclusion of Lemma 4.2 with $\delta = \alpha\gamma'$, $\eta = d/\alpha - 1$ and $K = 2k$. If

$$B_n = \left\{t \in A_n : \exists\, u \in A_{n\alpha\gamma'} \cap \left[t, t + 2^{-n}(2k)^{-1}\right], s \in \left[t + 2^{-n}k^{-1}, t + 2^{-n}\right] \text{ such that } |X(u) - X(s)| < 2^{-n\gamma''}\right\},$$

then we claim that

$$\text{for a.a. } \omega \text{ and all } a \in \bigcup_n A_n,\ \varepsilon > 0 \text{ there is an } N(a, \varepsilon, \omega) \in \mathbb{N} \text{ such that } B_n \cap (a, a + \varepsilon) \ne \varnothing \text{ for } n \ge N(a, \varepsilon, \omega). \tag{4.8}$$

It clearly suffices to show this for a fixed $a \in A_{n_0}$ and $\varepsilon > 0$. There is an M and an N_0 such that if $\tilde{A}_n$ is as in Lemma 4.2 (with K, η, δ as before), then for $n \ge N_0$, $\tilde{A}_n \cap (a, a + \varepsilon) \supset \{t_i^n : 1 \le i \le [2^{n\beta - 1}]\}$, where $t_i^n < t_j^n$ if $i < j$. Fix $n \ge N_0$ and t_i^n as before. Let $\{r_j : j = 1, \dots, [2^{\beta n(\alpha\gamma' - 1)}]\}$, $r_j < r_{j+1}$, be the smallest $[2^{\beta n(\alpha\gamma' - 1)}]$ elements of $A_{n\alpha\gamma'} \cap (t_i^n, t_i^n + 2^{-n}/(2k))$. If

$$R_j = \left\{\omega : \exists\, s \in \left[t_i^n + 2^{-n}k^{-1}, t_i^n + 2^{-n}\right] \text{ such that } \left|X(s) - X(r_j)\right| < 2^{-n\gamma''}\right\},$$

then

$$\begin{aligned} P(t_i^n \in B_n) &> P\left(\bigcup_{j=1}^{[2^{\beta n(\alpha\gamma'-1)}]} R_j\right) \\ &\ge \sum_{j=1}^{[2^{\beta n(\alpha\gamma'-1)}]} P(R_j) \\ &\quad - \sum\sum_{1 \le j \ne l \le [2^{\beta n(\alpha\gamma'-1)}]} P(R_j \cap R_l). \end{aligned} \tag{4.9}$$

Lemma 2.3 implies that

$$P(R_j) \geq c^{(4)} 2^{-n\gamma''(d-\alpha)}\left(k^{-1}2^{-n} - \left(r_j - t_i^n\right)\right)^{-(d/\alpha - 1)} \tag{4.10}$$
$$\geq c_3 2^{-n(\gamma'' - \alpha^{-1})(d-\alpha)}.$$

To bound $P(R_j \cap R_l)$ we argue as in the derivation of (4.13) of Jain and Taylor (1973). If

$$V_j = \inf\left\{s \geq t_i^n + 2^{-n}k^{-1} : \left|X(s) - X(r_j)\right| < 2^{-n\gamma''}\right\},$$

then

$$P(R_j \cap R_l) \leq P\left(V_j < V_l \leq t_i^n + 2^{-n}\right) + P\left(V_l < V_j \leq t_i^n + 2^{-n}\right). \tag{4.11}$$

The first probability is less than or equal to

$$E\left(I(V_j < t_i^n + 2^{-n}) P\left(\left|X(s) - X(r_l)\right| < 2^{-n\gamma''} \text{ for some } s \geq V_j | \mathscr{F}_{V_j}\right)\right)$$
$$\leq E\left(I(R_j) P\left(\left|X(s) - X(V_j) - \left(X(r_l) - X(r_j)\right)\right| \leq 2^{1-n\gamma''} \text{ for some } s \geq V_j | \mathscr{F}_{V_j}\right)\right)$$
$$\leq E\left(P\left(R_j | \mathscr{F}_{t_i^n + 2^{-n}(2K)^{-1}}\right) P^{X(r_l) - X(r_j)}\left(\left|X(s)\right| \leq 2^{1-n\gamma''} \text{ for some } s \geq 0\right)\right)$$
$$\leq c_4 2^{-n(\gamma'' - \alpha^{-1})(d-\alpha)} P\left(\left|X(s)\right| \leq 2^{1-n\gamma''} \text{ for some } s \geq r_l - r_j\right) \quad \text{(Lemma 2.3)}$$
$$\leq c_5 2^{-n(2\gamma'' - \alpha^{-1})(d-\alpha)} \left|r_l - r_j\right|^{-((d/\alpha)-1)} \quad \text{(by Lemma 2.3 again)}.$$

By symmetry the second probability in (4.11) is bounded by the same quantity. Substitute this bound and (4.10) into (4.9) to see that

$$P(t_i^n \in B_n) \geq c_3\left(2^{-n((\gamma'' - \alpha^{-1})(d-\alpha) - \beta(\alpha\gamma' - 1))} - c_6 2^{-n(2\gamma'' - \alpha^{-1})(d-\alpha)} S_n\left(t_i^n\right)\right)$$
$$\geq c_3 2^{-n((\gamma'' - \alpha^{-1})(d-\alpha) - \beta(\alpha\gamma' - 1))}\left(1 - c_6 M 2^{-n(\gamma'' - \gamma')(d-\alpha)}\right)$$
$$\geq c_7 2^{-n((\gamma'' - \alpha^{-1})(d-\alpha) - \beta(\alpha\gamma' - 1))}.$$

In the last line the value of N_0 has been increased (if necessary) and in the next to last line the fact that $t_i^n \in \tilde{A}_n$ has been used to bound $S_n(t_i^n)$. The events $\{t_i^n \in B_n\}$, $1 \leq i \leq 2^{n\beta - 1}$, are independent. Therefore

$$P\left(B_n \cap (a, a + \varepsilon) = \varnothing\right) \leq \left(1 - c_7 2^{-n((\gamma'' - \alpha^{-1})(d-\alpha) - \beta(\alpha\gamma' - 1))}\right)^{[2^{n\beta - 1}]}$$
$$\leq \exp\left\{-c_8 2^{-n((\gamma'' - \alpha^{-1})(d-\alpha) - \beta\alpha\gamma')}\right\}.$$

This is summable over n by (4.7). The Borel–Cantelli lemma now gives (4.8).

Now argue as in the derivation of the lower bound of $\dim C_\gamma$, using the open sets

$$G_n = \{t \in (0,1) : \exists\, h \in (0, 2^{-n}) \text{ and } \delta > 0 \text{ such that for all } v \in (t-\delta, t+\delta), |X(v+h) - X(v)| < n^{-1}h^\gamma\}, \tag{4.12}$$

to see that $\dim E_\gamma \geq 1 - \beta'$. Let $\gamma', \gamma'' \to \gamma$ and $\beta' \downarrow (1 - (\alpha\gamma)^{-1})(d\alpha^{-1} - 1)$ to obtain the required lower bound. □

REMARK 4.4. If $\alpha \leq d < 2\alpha$, then the preceding result implies

$$\lim_{\gamma \to \infty} \dim E_\gamma = 2 - d/\alpha.$$

In the symmetric stable case this limit is precisely the dimension of the time set corresponding to the double points of X [see Section 7 of Taylor (1986) and the references cited therein, and use Theorem 1 of Pruitt (1975) to convert the spatial dimensions into temporal dimensions]. It is not hard to modify the preceding proof to show that if $\psi \in \mathscr{H}$ and

$$E_\psi = \left\{t \geq 0 : \liminf_{h \downarrow 0} |X(t+h) - X(t)| \psi(h)^{-1} = 0\right\},$$

then $\dim E_\psi \geq 2 - d/\alpha$. It is, however, more illuminating to explain this (at least in the strictly stable case) directly in terms of the double points

$$F = \{t \in (0,1) : X(t) = X(u) \text{ for some } u \neq t\}$$

and the set of "close double points,"

$$F_\varepsilon = \{t \in (0,1) : \exists\, u \text{ such that } 0 < |t - u| < \varepsilon \text{ and } X(t) = X(u)\}.$$

Then $\dim F = \dim F_\varepsilon = 2 - d/\alpha$ for any $\varepsilon > 0$. The latter equality is clear from $\dim F = 2 - d/\alpha$ because F_ε contains the double points of $X|_{(0,\varepsilon)}$. Let A be the closed range (in $[0,1]$) of an independent subordinator of index $\beta' > d/\alpha - 1$. If G_n is given by (4.12) but with $\psi(h)$ in place of h^γ, then $F_{2^{-n}} \subset G_n$ a.s. (use the fact that all points in $F_{2^{-n}}$ are a.s. continuity points of X). If $a \in \bigcup_n A_n$ (A_n as before) and $\varepsilon > 0$, then sets of dimension $> 1 - \beta'$ will intersect $A \cap (a, a+\varepsilon)$ with positive probability [Lemma 2 of Hawkes (1971b)]. Therefore $P(F_{2^{-n}} \cap A \cap (a, a+\varepsilon) \neq \varnothing) \equiv p > 0$. Let us take $a = 0$ (a similar argument goes through if $a > 0$). A scaling argument (use the scaling properties of both X and A) shows that for any $c > 0$,

$$P(F_{c2^{-n}} \cap A \cap (0, c\varepsilon) \neq \varnothing) = p.$$

By the 0–1 law we get

$$P(F_{2^{-n}k^{-1}} \cap A \cap (0, \varepsilon/k) \neq \varnothing \text{ for infinitely many } k) = 1.$$

The preceding sets are decreasing in k, so p must equal 1. Therefore $G_n \cap A$ is dense in A a.s. By Baire's theorem $(\bigcap_n G_n) \cap A \neq \varnothing$ and hence $E_\psi \cap A \neq \varnothing$ 2 a.s. As in the previous proof this shows $\dim E_\psi \geq 1 - \beta' \to 2 - d/\alpha$ as $\beta' \downarrow d/\alpha - 1$.

The corresponding problem concerning points of unusually fast escape for Brownian motion (jumps allow a very fast escape indeed) can be handled using "slow point techniques" [see Davis (1983), Greenwood and Perkins (1983) and Perkins (1983)]. Here the maximal rate is $h^{1/2}$ and, in fact,

$$\sup_t \liminf_{h \downarrow 0} |X(t+h) - X(t)| h^{-1/2} = c_d \in (0, \infty).$$

Davis (1983) shows $c_1 = 1$ and c_d can be characterized in terms of the nonnegative eigenfunction of an associated Sturm–Liouville problem using the techniques of Perkins (1983), Section 6. These methods also allow one to compute the Hausdorff dimension of times such that

$$\liminf_{h \downarrow 0} |X(t+h) - X(t)| h^{-1/2} \geq c,$$

for $0 < c \leq c_d$.

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