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AN EXAMPLE ON HIGHLY SINGULAR PARABOLIC MEASURE[1]

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For each $\delta > 0$ there is a parabolic operator in the half-plane $R_x \times R_t^+$ whose parabolic measure is supported by a boundary set of dimension $< \delta$.

0. Introduction. We consider solutions of parabolic equations $Lu = a(x,t)u_{xx} - u_t = 0$ in the region $\{t > 0\}$, where

$$0 < \Lambda_1 \le a(x,t) \le \Lambda_2 < +\infty \tag{0.1}$$

and a is of class C^2 in the region; so all solutions are classical. Because $a(x,t) \le \Lambda_2$ the Dirichlet problem (with bounded, continuous data on the line $t = 0$) has solutions and these satisfy the maximum principle. These remarks are substantiated in Appendix 1. The value of a solution at (x, t) can be treated as a linear functional of the boundary values; the value is therefore represented by a probability measure $\omega^{(x,t)}$ on the boundary—the parabolic measure.

THEOREM. *For each* $\delta > 0$ *there is some coefficient* $a(x,t)$, C^2 *in* $\{t > 0\}$, *satisfying* (0.1), *for which all parabolic measures are concentrated on a single boundary set of Hausdorff dimension* $< \delta$.

All sets on $t = 0$ of Λ_1/Λ_2 dimensional measure 0 have parabolic measure 0; see Appendix 2. Because of certain technicalities in our method, the number δ in our example satisfies $\delta < c/\log(\Lambda_2/\Lambda_1)$, but perhaps an improvement would give $\log(1/\delta) \approx \log(\Lambda_2/\Lambda_1)$.

For the region $\{t > 0\}$ and operators L with a coefficient continuous up to the boundary $t = 0$, Fabes and Kenig [3] proved that parabolic measures can be singular with respect to Lebesgue measure. Their construction is based on Riesz products in which the factors are $\phi_n(y) = (1 + 1/\sqrt{2}\cos h_n y)$. To determine the smallness of a support of the parabolic measure, it would be necessary to control the growth of the numbers h_n or the size of the gaps $h_{n+1}h_n^{-1}$. For such equations the parabolic measures vanish on all sets of dimension < 1; see Appendix 2.

The singularity shown by [3] and by the present example cannot occur for parabolic operators in divergence form [1].

As is well known, there is a close connection between certain diffusion processes and the operator L. The following elementary inequalities on parabolic

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measures can be stated in probabilistic terms and were found by experimenting with Itô's formula; but each is stated analytically.

We conclude this Introduction with a description of one difference between parabolic and elliptic equations; this will be done with the aid of a stochastic process. (This description is not part of the formal proof.) Let $W(t)$, $-\infty < t < \infty$, be a Wiener process with decreasing time, i.e., with the basic σ-algebras decreasing in time. Let $X(t)$ be a solution of the stochastic differential equation $dX = (2a(X(t), t))^{1/2}\,dW$, observing the usual measurability and the time-reversal. Let $Y = u(X(t), t)$, where u is a function of class C^2. Itô's formula yields

$$dY = u_t(X(t), t)\,dt - u_{xx}(X(t), t)a(X(t), t)\,dt$$
$$+u_x(X(t), t)(2a(X(t), t))^{1/2}\,dW$$

[2], which explains the connection with the equation $Lu = 0$. Suppose that $a(x, t)$ is small on a region $0 \le t \le 1$, $-1 \le x \le 0$. Beginning at $x = 0$, $t = 1$, a solution (visualized as a particle moving according to a certain law) has only a small chance of attaining the position $x = -\frac{1}{2}$, say, before $t = 0$. The same is true for the position $x = +\frac{1}{2}$, no matter what value $a(x, t)$ takes on $0 \le x \le 1$. Beginning at $x = +1$, $t = 1$, the particle may nevertheless attain $x = 0$ before $t = 0$ with substantial probability. These observations have no apparent analogues for elliptic equations. See the remarks after Lemma 3.

The situation for the region $\{x > 0\}$ is unclear; a possible example begins in nearly the same way as for the region $\{t > 0\}$, but the conclusion is more problematical.

The obvious fact that a line separates a plane is very useful in the example; therefore we do not know about parabolic equations in $R^2 \times R^+$, $R^3 \times R^+$, etc.

1. Elementary estimates on parabolic measures.

In any region Ω considered in the following discussion, we assume that $a(x, t)$ is C^2 in Ω, $0 < \Lambda_1 \le a(x, t) \le \Lambda_2 < +\infty$ and let $Lu = a(x, t)u_{xx} - u_t$ and ω_Ω be the parabolic measure on $\partial\Omega$ with respect to Ω.

LEMMA 1. *Let* $c_1 > c_2 > 0$, $\Omega = \{-c_2 < x < c_1,\ t > 0\}$ *and* $S = \partial\Omega \cap \{x = c_1\}$. *Then*

$$\omega_\Omega^{(0,t)}(S) \le c_2 c_1^{-1}, \quad \textit{for } t > 0.$$

PROOF. The function x is a solution of $L = 0$. Hence the value of x at $(0, t)$, which is 0, is equal to $\int_{\partial\Omega} x\,d\omega^{(0,t)}(y, s)$. Therefore

$$\int_{\partial\Omega\cap\{x\ge 0\}} x\,d\omega^{(0,t)}(y, s) = -\int_{\partial\Omega\cap\{x\le 0\}} x\,d\omega^{(0,t)}(y, s) \le c_2.$$

Hence $c_1\omega_\Omega^{(0,t)}(S) \le c_2$. □

LEMMA 2. *Let $x_0 > 0$ and Ω be the region $\{x > -x_0$ and $t > 0\}$. Suppose that $a(x, t) \leq A$ in $\Omega \cap \{-x_0 < x < 0\}$. Then*

$$\omega_\Omega^{(0,t)}(\partial\Omega \cap \{x = -x_0\}) \leq e^{-x_0^2/(144tA)},$$

for $0 < t < x_0^2/(12A)$.

PROOF. Let $\nu > 0$ and $\mu = A(6\nu x_0 + 9\nu^2 x_0^4)$. The function u on $\overline{\Omega}$, defined by $u = e^{\mu t}$ for $x \geq 0$, and $u = e^{-\nu x^3 + \mu t}$ for $-x_0 \leq x \leq 0$, satisfies $Lu \leq 0$ in Ω and $u \geq e^{\nu x_0^3}$ on $\partial\Omega \cap \{x = -x_0\}$. Hence it follows from the maximum principle that, for $0 < t < x_0^2/(12A)$,

$$\omega_\Omega^{(0,t)}(\partial\Omega \cap \{x = -x_0\}) \leq e^{\mu t - \nu x_0^3} = e^{A(6\nu x_0 + 9\nu^2 x_0^4)t - \nu x_0^3}$$

$$< e^{x_0^3(9Ax_0 t\nu^2 - 1/2\nu)}.$$

For each fixed t, the exponent takes its minimum $-x_0^2/(144tA)$ when $\nu = (36Atx_0)^{-1}$. □

LEMMA 3. *Given $0 < \varepsilon < 10^{-4}$, $k > 0$ and β real, let $\Omega = \{x < \varepsilon^\beta,\ t > 0\}$ and assume that $a(x, t) \leq \varepsilon^k$ on $\{-\varepsilon < x < 0\}$. Then for any $\gamma \geq \max\{1, \beta\}$,*

(1.1) $$\omega_\Omega^{(x,t)}(\partial\Omega \cap \{x = \varepsilon^\beta\}) < \varepsilon^{\gamma-\beta} + e^{-\varepsilon^{2\gamma-k}}/(144t),$$

whenever $x \leq 0$ and $0 < t < \varepsilon^{2\gamma-k}/12$.

PROOF. From the maximum principle, it is enough to prove (1.1) when $x = 0$. Let $D = \{-\varepsilon^\gamma < x < \varepsilon^\beta\}$. Then

$$\omega_\Omega^{(0,t)}(\partial\Omega \cap \{x = \varepsilon^\beta\}) \leq \omega_D^{(0,t)}(\partial D \cap \{x = \varepsilon^\beta\}) + \omega_D^{(0,t)}(\partial D \cap \{x = -\varepsilon^\gamma\})$$

$$\leq \varepsilon^{\gamma-\beta} + e^{-\varepsilon^{2\gamma-k}/144t},$$

and last inequality follows from Lemmas 1 and 2 and the maximum principle. □

COROLLARY. *In $\Omega = \{t > 0,\ |x| < \varepsilon/2\}$, let $a(x, t) < \varepsilon^4$ for $0 < x < \varepsilon/2$. Then*

(1.2) $$\omega_\Omega^{(0,t)}\left(|x| = \frac{\varepsilon}{2}\right) < \frac{\varepsilon}{4}, \quad \text{for } 0 < t \leq \varepsilon^{1/2}.$$

This follows from the Lemma 3 by letting $\beta = 1$, $\gamma = 2.1$ and a simple change of variables.

In view of Lemma 3 and its corollary, it is justified to call a rectangle where $a(x, t)$ is very small an obstacle for the process. The previous estimates give the probability of a particle crossing an obstacle within a given time; and they estimate, starting at a point inside an obstacle, the chance that the particle will get out in a given time period. Our construction is based on these estimations. We note that the values of Λ_1 and Λ_2 have not been used in these estimates.

LEMMA 4. *Let $a(x,t)$ be C^2 in $D = \{x > 0,\ t > 0\}$ satisfying $0 < \Lambda_1 \le a(x,t) \le \Lambda_2 < +\infty$. Given $\varepsilon, \tau > 0$, then*

$$\omega_D^{(y,s)}(\{x = 0\}) > 1 - \varepsilon,$$

whenever $\tau < s < 2\tau$ and $0 < y < \eta \equiv \varepsilon(\pi\tau\Lambda_1)^{1/2}(\int_0^1 e^{-r^2/4}\,dr)^{-1}$.

PROOF. This is based on the special function $v(\rho) = \pi^{-1/2}\int_0^\rho e^{-r^2/4}\,dr$, which satisfies $v(0) = 0$ and $v(+\infty) = 1$. On the region D, a function $u(x,t) = v(\lambda x t^{-1/2})$, $\lambda > 0$, is a subsolution ($Lu \ge 0$) if $\Lambda_2\lambda^2 \le 1$, and a supersolution ($Lu \le 0$) if $\Lambda_1\lambda^2 \ge 1$. We choose $\lambda = \Lambda_1^{-1/2}$ so that $u(x,t) = v(\lambda x t^{-1/2})$ is a supersolution. Let D_0 be the subregion of D defined by the inequalities $x > 0$, $\lambda x t^{-1/2} < 1$ or $\lambda^2x^2 < t$, and B_0 the part of the boundary defined by $t = \lambda^2x^2$. Then v is a supersolution on D_0, $u = v(1) > 0$ on B_0, so that $\omega_{D_0}^{(y,s)}(B_0) \le u(y,s)/v(1)$. But $u(y,s) < \Lambda_1^{-1/2} y s^{-1/2}$, so the parabolic measure of B_0, under the stated conditions on (y,s), is $< \Lambda_1^{-1/2}\tau^{-1/2}\eta/v(1)$. Subtraction gives a lower bound for the parabolic measure of $\{x = 0\}$. [The upper bound $a(x,t) \le \Lambda_2$ is not used in this lemma.] □

LEMMA 5. *Let $a(x,t)$ be C^2 in $H = \{t > 0\}$ satisfying $0 < a(x,t) \le \varepsilon^{-4} < +\infty$ and $Lu = au_{xx} - u_t$. Then there exists $d > 0$, depending only on ε so that*

$$\omega_H^{(0,d)}\big(\{|x| < m\varepsilon^2,\ t = 0\}\big) > 1 - \varepsilon^{5m},$$

for $m = 1, 2, 3, \ldots$.

PROOF. The parabolic measure is decreased if the region H is decreased to the region $H_m = \{|x| < m\varepsilon^2,\ t > 0\}$. We estimate the parabolic measure [at $(0,d)$] of the lateral boundaries, using Lemma 2: $x_0 = m\varepsilon^2$, $t = d$, $A = \varepsilon^{-4}$. The estimate of Lemma 2 can be applied if $d < m^2\varepsilon^8/12$, so we require $d < \varepsilon^8/12$. To obtain the stated lower bound, we need $2\exp(-m^2\varepsilon^8/144d) < \varepsilon^{-5m}$ or $d^{-1}m^2\varepsilon^8(12)^{-2} > 5m(\log\varepsilon^{-1}) + \log 2$. A solution d for $m = 1$ solves all the inequalities. □

2. Preliminary construction. Given $0 < \varepsilon < 10^{-4}$, let

$$\eta = \varepsilon^4\sqrt{\pi}\left(2\int_0^1 e^{-r^2/4}\,dr\right)^{-1}$$

(i.e., $\tau = \varepsilon^2/4$, $\Lambda_1 = \varepsilon^4$ in Lemma 4) and choose $\alpha(x)$ to be a C^2 function on $\mathbb{R}$, of period 1, which has value ε^4 on $[-2\varepsilon, 2\varepsilon]$, ε^{-4} on $[2\varepsilon + \eta, 1 - 2\varepsilon - \eta]$, and is monotonically increasing on $[2\varepsilon, 2\varepsilon + \eta]$ and monotonically decreasing on $[1 - 2\varepsilon - \eta, 1 - 2\varepsilon]$, with $\alpha(x) = \alpha(1 - x)$ for $x \in [0,1]$.

On the half-plane $H \equiv \{t > 0\}$, let $Lu \equiv \alpha(x)u_{xx} - u_t$ and ω be the parabolic measure on $\{t = 0\}$ with respect to the region H and the operator L.

PROPOSITION 1. *Let $I = [3\varepsilon^2, 1 - 3\varepsilon^2] \times \{t = 0\}$, $E = ([3\varepsilon^2, 4\varepsilon - 3\varepsilon^2] \cup [1 - 4\varepsilon + 3\varepsilon^2, 1 - 3\varepsilon^2]) \times \{t = 0\}$ and*

$$S = (3\varepsilon^2, 1 - 3\varepsilon^2) \times (0, \varepsilon^2).$$

Then for $(x, t) \in H \setminus S$,

(2.1) $$\omega^{(x,t)}(E)/\omega^{(x,t)}(I) \geq 1 - \varepsilon^{3/4}.$$

PROOF. To prove Proposition 1, it is enough to show (2.1) for $(x, t) \in \partial S \setminus \{t = 0\}$ because of the Markov property.

Let $P_1 = (3\varepsilon^2, 0)$, $P_2 = (3\varepsilon^2, \varepsilon^2)$, $P_3 = (2\varepsilon, \varepsilon^2)$, $P_4 = (2\varepsilon + \eta, \varepsilon^2)$, $P_5 = (1 - 2\varepsilon - \eta, \varepsilon^2)$, $P_6 = (1 - 2\varepsilon, \varepsilon^2)$, $P_7 = (1 - 3\varepsilon^2, \varepsilon^2)$, $P_8 = (1 - 3\varepsilon^2, 0)$, $P_9 = (2\varepsilon, \varepsilon^2/2)$, $P_{10} = (2\varepsilon + \eta, \varepsilon^2/4)$, $P_{11} = (1 - 2\varepsilon - \eta, \varepsilon^2/4)$, $P_{12} = (4\varepsilon - 3\varepsilon^2, 0)$, $P_{13} = (4\varepsilon - 3\varepsilon^2, \varepsilon^2)$, $P_{14} = (-\varepsilon, \varepsilon^2)$, $P_{15} = (\varepsilon, \varepsilon^2)$ and denote by P_jP_k the closed line segment joining P_j and P_k.

We claim that

(2.2) with respect to the rectangle $D_1 = P_4P_{10}P_{11}P_5$, the parabolic measure $\omega_{D_1}^{(x,t)}(P_4P_{10} \cup P_5P_{11}) > 1 - 4\varepsilon$, for $(x, t) \in P_4P_5$;

(2.3) with respect to the region $D_2 = \{x > 2\varepsilon, t > 0\}$, the parabolic measure $\omega_{D_2}^{(x,t)}(P_3P_9) > 1 - \varepsilon$, for $(x, t) \in P_3P_4 \cup P_4P_{10}$;

(2.4) with respect to the region $D_3 = \{x < 4\varepsilon - 3\varepsilon^2, t > 0\}$, the parabolic measure $\omega_{D_3}^{(x,t)}(P_{12}P_{13}) < \varepsilon$, for $(x, t) \in P_1P_2 \cup P_2P_3 \cup P_3P_9$; consequently, $\omega^{(x,t)}(\{x > 4\varepsilon - 3\varepsilon^2, t = 0\}) < \varepsilon$;

(2.5) with respect to H, $\omega^{(x,t)}(E) > \frac{1}{5}$, for $(x, t) \in P_1P_2 \cup P_2P_3 \cup P_3P_9$.

Let us assume these claims for the moment. Let

$$T = I \setminus E = (4\varepsilon - 3\varepsilon^2, 1 - 4\varepsilon + 3\varepsilon^2) \times \{t = 0\}.$$

If $(x, t) \in P_1P_2 \cup P_2P_3$, it follows from (2.4), (2.5) and the maximum principle that $\omega^{(x,t)}(E) > \frac{1}{5}$ and $\omega^{(x,t)}(T) < \varepsilon$. Thus (2.1) holds. By symmetry, it holds for $(x, t) \in P_6P_7 \cup P_7P_8$.

If $(x, t) \in P_3P_4 \cup P_4P_{10}$, from (2.3), (2.4), the Markov property and the maximum principle, it follows that

(2.6) $$\begin{aligned} \omega^{(x,t)}(T) &= \int_{\partial D_2} \omega^{(y,s)}(T)\, d\omega_{D_2}^{(x,t)}(y, s) \\ &\leq \varepsilon + \sup_{P_3P_9} \omega^{(y,s)}(T) \leq 2\varepsilon; \end{aligned}$$

and from (2.3) and (2.5), it follows that

(2.7) $$\begin{aligned} \omega^{(x,t)}(E) &= \int_{\partial D_2} \omega^{(y,s)}(E)\, d\omega_{D_2}^{(x,t)}(y, s) \\ &> (1 - \varepsilon) \inf_{P_3P_9} \omega^{(y,s)}(E) > \tfrac{1}{6}. \end{aligned}$$

Similarly (2.6) and (2.7) hold for $(x, t) \in P_5P_6 \cup P_5P_{11}$. Therefore (2.1) holds for $(x, t) \in P_3P_4 \cup P_5P_6$.

If $(x, t) \in P_4P_5$, from (2.2), (2.6) and the Markov property, it follows that

$$\begin{aligned}\omega^{(x,t)}(T) &= \int_{\partial D_1} \omega^{(y,s)}(T)\, d\omega_{D_1}^{(x,t)}(y,s) \\ &\leq 4\varepsilon + \sup_{P_4P_{10}\cup P_5P_{11}} \omega^{(y,s)}(T) \\ &\leq 6\varepsilon;\end{aligned}$$

and from (2.2) and (2.7), it follows that

$$\begin{aligned}\omega^{(x,t)}(E) &= \int_{\partial D_1} \omega^{(y,s)}(T)\, d\omega_{D_1}^{(x,t)}(y,s) \\ &> (1-4\varepsilon) \inf_{P_4P_{10}} \omega^{(y,s)}(E) > \tfrac{1}{10}.\end{aligned}$$

Hence (2.1) holds for $(x, t) \in P_4P_5$.

Therefore (2.1) holds on $\partial S \setminus \{t = 0\}$ and hence on $H \setminus S$.

It remains to prove the claims (2.2)–(2.5).

We recall that $\alpha \equiv \varepsilon^{-4}$ on $2\varepsilon + \eta \leq x \leq 1 - 2\varepsilon - \eta$; (2.1) will follow from the maximum principle after we prove that corresponding to the equation $\tilde{L}u = \varepsilon^{-4}u_{xx} - u_t = 0$, with respect to the region $D = \{|x| < \frac{1}{2},\ t > 0\}$, the parabolic measure

$$\omega_D^{(x,\varepsilon^2/4)}(\partial D \cap \{t = 0\}) < 4\varepsilon, \quad \text{for } |x| < \tfrac{1}{2}.$$

In fact, we let $v(x,t) = e^{1/16}\varepsilon^2(t+\varepsilon^4)^{-1/2}e^{-\varepsilon^4x^2/4(t+\varepsilon^4)}$ for $t > -\varepsilon^4$. Clearly, $\tilde{L}v = 0$ for $t > -\varepsilon^4$, and $v(x,0) \geq 1$ for $|x| \leq \frac{1}{2}$. Hence by the maximum principle, $\omega^{(x,\varepsilon^2/4)}(\partial D \cap \{t=0\}) \leq v(x, \varepsilon^2/4) < 4\varepsilon$ for $|x| < \frac{1}{2}$. This estimate is by no means the best possible; with some effort one can show that 4ε can be replaced by $e^{-c/\varepsilon}$.

Claim (2.3) follows from the choice of η, Lemma 4 and the maximum principle.

Claim (2.4) follows from (1.2) after a translation and a scale change of the region.

Fix $(x,t) \in P_1P_2 \cup P_2P_3 \cup P_3P_9$ and let $D_4 = \{x > -2\varepsilon + 6\varepsilon^2,\ t > 0\}$, $D_5 = \{-2\varepsilon + 6\varepsilon^2 < x < 2\varepsilon,\ t > 0\}$. From (2.4), the maximum principle and the Markov property, it follows that

$$\begin{aligned}\omega^{(x,t)}(\{t=0\} \setminus E) &\leq \omega_{D_3}^{(x,t)}(P_{12}P_{13}) + \omega_{D_4}^{(x,t)}(\partial D_4 \cap \{x \leq 3\varepsilon^2\}) \\ &\leq \varepsilon^{1/2} + \sup_{(3\varepsilon^2,s)\in P_1P_2} \omega_{D_4}^{(3\varepsilon^2,s)}(\partial D_4 \cap \{x \leq 3\varepsilon^2\}) \\ &\leq \varepsilon^{1/2} + \left(1 - \inf_{(3\varepsilon^2,s)\in P_1P_2} \omega_{D_5}^{(3\varepsilon^2,s)}(\{3\varepsilon^2 \leq x < 2\varepsilon,\ t=0\})\right).\end{aligned}$$

Because $a(x,t) = \varepsilon^4$ in D_5, the infimum in the previous inequality is bounded below by $\frac{1}{2}\inf_{0\leq s\leq\varepsilon^2} v(0,s)$, where v is the parabolic measure of $\{-\varepsilon < x < \varepsilon,\ t = 0\}$ with respect to the region $\{-\varepsilon < x < \varepsilon,\ t > 0\}$ and the equation $\varepsilon^4 v_{xx} -$

$v_t = 0$. After change of scales, we obtain easily that

$$\inf_{0 \le s \le \varepsilon^2} v(0, s) = \inf_{0 \le \tau \le \varepsilon^4} u(0, \tau) > \tfrac{1}{2},$$

where u is the parabolic measure of $\{-1 < x < 1,\ t > 0\}$ and the ordinary heat equation $u_{xx} - u_t = 0$. This proves that $\omega^{(x,t)}(\{t = 0\} \setminus E) \le \varepsilon^{1/2} + \frac{3}{4}$, and the estimate (2.5) follows.

This completes the proof of Proposition 1. □

3. The construction of $a(x,t)$. Given $0 < \varepsilon < 10^{-4}$, we retain $\alpha(x)$ from Section 2 and choose d as in Lemma 5.

Let $\tau_0 > s_0 > \tau_1 > s_1 > \tau_2 > s_2 > \tau_3 > \cdots > 0$ be defined as $\tau_n = \sum_{k=n}^{\infty} \varepsilon^{2k}(\varepsilon^2 + d)$ and $s_n = \tau_n - \varepsilon^{2n+2}$ and let T_n be the nth major strip $\{s_n < t \le \tau_n\}$, S_n be the nth minor strip $\{\tau_{n+1} < t \le s_n\}$. Let $a(x,t)$ be a C^2 function on $H = \{t > 0\}$, which satisfies $\varepsilon^4 \le a(x,t) \le \varepsilon^{-4}$, is defined by $a(x,t) = \alpha(x)$ on $\{t \ge s_0\}$ and $a(x,t) = \alpha(\varepsilon^{-n}x)$ on T_n; has period 1 in x, with $a(x,t) = a(1-x,t)$ on S_0, moreover, for $(x, t + \tau_{n+1}) \in S_n$, $a(x, t + \tau_{n+1}) = a(\varepsilon^{-n}x, \varepsilon^{-2n}t + \tau_1)$. Let $Lu = a(x,t)u_{xx} - u_t$ in $H \equiv \{t > 0\}$.

The introduction of minor strips allows $a(x,t)$ to be smooth, thus L has classical solutions. Minor strips can be omitted if we consider weak solutions.

In T_n, $a \equiv \varepsilon^4$ on each rectangle: $\{\varepsilon^n(k - 2\varepsilon) \le x \le \varepsilon^n(k + 2\varepsilon),\ s_n < t \le \tau_n\}$, with integer k. In view of Lemma 3 we call these rectangles obstacles.

PROPOSITION 2. *Let P be the fixed point* (0,100). *Given any integers $n \ge 1$ and k, denote by J the interval $[k\varepsilon^n, (k+1)\varepsilon^n] \times \{t = 0\}$ and by F the interval $([k\varepsilon^n, (k + 4\varepsilon)\varepsilon^n] \cup [(k + 1 - 4\varepsilon))\varepsilon^n, (k+1)\varepsilon^n]) \times \{t = 0\}$. Then*

$$\omega^P(F)/\omega^P(J) > 1 - \varepsilon^{1/2}. \tag{3.1}$$

PROOF. Fix n and k, and let

$$E = \{(x, s_n) : (k + 3\varepsilon^2)\varepsilon^n \le x \le (k + 4\varepsilon - 3\varepsilon^2)\varepsilon^n \text{ or } (k + 1 - 4\varepsilon + 3\varepsilon^2)\varepsilon^n \le x \le (k + 1 - 3\varepsilon^2)\varepsilon^n\}$$

and

$$I = \{(x, s_n) : (k + 3\varepsilon^2)\varepsilon^n \le x \le (k + 1 - 3\varepsilon^2)\varepsilon^n\}.$$

After a change of scales $x \to \varepsilon^{-n}x$ and $t \to \varepsilon^{-2n}t$, we may apply Proposition 1 to T_n and obtain

$$\omega^{(x,\tau_n)}_{\{t > s_n\}}(E)/\omega^{(x,\tau_n)}_{\{t > s_n\}}(I) > 1 - \varepsilon^{3/4},$$

for all x. The Markov property implies that

$$\omega^P_{\{t > s_n\}}(E)/\omega^P_{\{t > s_n\}}(I) > 1 - \varepsilon^{3/4}. \tag{3.2}$$

To continue the proof, we make the following claims:

$$\omega_H^{(x, s_n)}(F) > 1 - \varepsilon, \quad \text{for } (x, s_n) \in E, \tag{3.3}$$

$$\omega_H^{(x, s_n)}(F)/\omega_H^{(x, s_n)}(J/F) > (2\varepsilon)^{-1}, \tag{3.4}$$

for $(x, s_n) \in \mathbb{R} \times \{t = s_n\} \setminus I$.

From (3.3), (3.4) and the Markov property, it follows that

$$\omega^P(F) = \int_{E\cup(I\setminus E)\cup(\mathbf{R}\times\{s_n\}\setminus I)} \omega_H^{(x,s_n)}(F)\, d\omega^P_{\{t>s_n\}}(x,s_n)$$
$$> (1-\varepsilon)\omega^P_{\{t>s_n\}}(E) + (2\varepsilon)^{-1}\int_{R\times\{s_n\}\setminus I} \omega_H^{(x,s_n)}(J\setminus F)\, d\omega^P_{\{t>s_n\}}(x,s_n)$$

and that

$$\omega^P(J\setminus F) = \int_{E\cup(I\setminus E)\cup(\mathbf{R}\times\{s_n\}\setminus I)} \omega_H^{(x,s_n)}(J\setminus F)\, d\omega^P_{\{t>s_n\}}(x,s_n)$$
$$< \varepsilon\omega^P_{\{t>s_n\}}(E) + \omega^P_{\{t>s_n\}}(I\setminus E)$$
$$+ \int_{\mathbf{R}\times\{s_n\}\setminus I} \omega_H^{(x,s_n)}(J\setminus F)\, d\omega^P_{\{t>s_n\}}(x,s_n).$$

Combining these estimations with (3.2), we conclude (3.1).

Because of the periodicity of $a(x,t)$, we may assume that $n=k=0$ in the proofs of (3.3) and (3.4). Thus $I=[3\varepsilon^2, 1-2\varepsilon^2]\times\{s_0\}$, $E=([3\varepsilon^2, 4\varepsilon-3\varepsilon^2]\cup[1-4\varepsilon+3\varepsilon^2, 1-3\varepsilon^2])\times\{s_0\}$, $J=[0,1]\times\{0\}$ and $F=([0,4\varepsilon]\cup[1-4\varepsilon,1])\times\{0\}$.

We recall that the heights of strips T_n and S_n are ε^{2n+2} and $\varepsilon^{2n}d$, respectively, $a(x,t)$ has period ε^n in x on $T_n\cup S_n$ and each obstacle in T_n has width $4\varepsilon^{n+1}$. Denote by $E(z)$ the set $[3\varepsilon^2-z, 4\varepsilon-3\varepsilon^2+z]\cup[1-4\varepsilon+3\varepsilon^2-z, 1-3\varepsilon^2+z]$ formed by widening E by a width z; denote by E_n and E_n' sets on $\{t=\tau_n\}$ and $\{t=s_n\}$, respectively, defined by

$$E_0' = E, \qquad E_1 \cong E(\varepsilon^2), \qquad E_1' \cong E\left(\frac{3\varepsilon^2}{2}\right),$$
$$E_2 \cong E\left(\frac{3\varepsilon^2}{2} + 2\varepsilon^3\right), \qquad E_2' \cong E\left(\frac{5\varepsilon^2}{2} + 2\varepsilon^3\right),$$
$$E_3 \cong E\left(\frac{5\varepsilon^2}{2} + 2\varepsilon^3 + 3\varepsilon^4\right),$$
$$\cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots$$
$$E_{n-1}' \cong E\left(\frac{\varepsilon^2}{2} + 2\sum_{j=2}^{n-1}(j-1)\varepsilon^j + (n-1)\varepsilon^n\right),$$
$$E_n \cong E\left(\frac{\varepsilon^2}{2} + 2\sum_{j=2}^{n-1}(j-1)\varepsilon^j + (n-1)\varepsilon^n + n\varepsilon^{n+1}\right),$$
$$E_n' \cong E\left(\frac{\varepsilon^2}{2} + 2\sum_{j=2}^{n}(j-1)\varepsilon^j + n\varepsilon^{n+1}\right),$$

where by $\cong$ we mean that the sets on both sides have the same x-projections, and we identify these sets whenever it is more convenient. Clearly, $E_0' \subseteq E_1 \subseteq E_1' \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq E_n' \subseteq \cdots$. Since $\varepsilon^2/2 + 2\Sigma_{j=2}^{\infty}(j-1)\varepsilon^j < 3\varepsilon^2$, the projections of these sets are all contained in F.

For a fixed point $Q_0 \in E = E_0'$, we obtain from the Markov property and the maximum principle that

$$(3.5) \qquad \omega^{Q_0}(F) \geq \prod_{n=0}^{\infty} \inf_{E_n'} \omega^{Q}_{\{t>\tau_{n+1}\}}(E_{n+1}) \prod_{n=1}^{\infty} \inf_{E_n} \omega^{Q}_{\{t>s_n\}}(E_n').$$

Because for any $Q = (x_n, s_n) \in E_n'$, E_{n+1} contains the interval $\{|x - x_n| < (n+1)\varepsilon^{n+2}\} \times \{\tau_{n+1}\}$ and $s_n - \tau_{n+1} = d\varepsilon^{2n}$, it follows from Lemma 5 and a change of scale that

$$(3.6) \qquad \inf_{E_n'} \omega^{Q}_{\{t>\tau_{n+1}\}}(E_{n+1}) > 1 - \varepsilon^{5(n+1)}.$$

We note that $E_1 \cong [2\varepsilon^2, 4\varepsilon - 2\varepsilon^2] \cup [1 - 4\varepsilon + 2\varepsilon^2, 1 - 2\varepsilon^2]$ and

$$a \equiv \varepsilon^4 \text{ on } \Big(\big[-2\varepsilon^2, 2\varepsilon^2\big] \cup \big[4\varepsilon - 2\varepsilon^2, 4\varepsilon + 2\varepsilon^2\big] \cup \big[1 - 4\varepsilon - 2\varepsilon^2, 1 - 4\varepsilon + 2\varepsilon^2\big] \cup \big[1 - 2\varepsilon^2, 1 + 2\varepsilon^2\big]\Big) \times \big[s_1, \tau_1\big].$$

Starting at any point $(y, \tau_1) \in E_1$ in order for the process to reach $\mathbb{R} \times \{s_1\} \setminus E_1'$, the particle must cross at least an obstacle of width $\varepsilon^2/2$ (in x) within time ε^4. It follows from (1.2) and a change of scales that

$$(3.7) \qquad \inf_{E_1} \omega^{Q}_{\{t>s_1\}}(E_1') > 1 - \varepsilon/4.$$

Starting at any point (y, τ_n) in E_n, $n \geq 2$, in order for the process to reach $\mathbb{R} \times \{s_n\} \setminus E_n'$, the particule must cross an x-interval of width at least $(n-1)\varepsilon^n$, which is $(n-1)$ times the period of $a(\cdot, t)$ in T_n. Applying (1.2) $(n-1)$ times, we obtain

$$(3.8) \qquad \inf_{E_n} \omega^{P}_{\{t>s_n\}}(E_n') > 1 - (\varepsilon/4)^{n-1}, \quad \text{for } n \geq 2.$$

From (3.5)–(3.8), we conclude that $\omega^{Q}_{H}(F) > 1 - \varepsilon$. This proves (3.3).

To show (3.4), we consider again $n = k = 0$ and fix a point $(x_0, s_0) \in \mathbb{R} \times \{s_0\} \setminus I$; we assume, as we may, that $x_0 < 3\varepsilon^2$ because of the periodicity and the symmetry of a. From the Markov property and the maximum principle, it follows that

$$\omega_H^{(x_0, s_0)}(F)/\omega_H^{(x_0, s_0)}(J \setminus F) \geq \inf_{0<t,\, s\leq s_0} \omega_H^{(2\varepsilon, t)}(F)/\omega_H^{(2\varepsilon, d)}(J \setminus F).$$

We recall that $\omega_H^{(2\varepsilon, s_0)}(F) > 1 - \varepsilon$ from (3.3) and note that $x = 2\varepsilon$ bisects an obstacle of width $4\varepsilon^{m+1}$ in T_m for each $m \geq 1$. For $0 < t < s_0$, applying either (1.2) or Lemma 4 first, depending on whether $(2\varepsilon, t)$ is in some T_n or in S_n, then following the argument of (3.3), we may conclude that $\omega^{(2\varepsilon, t)}(F) > 1 - \varepsilon$. Therefore

$$\omega^{(x_0, s_0)}(F)/\omega^{(x_0, s_0)}(J \setminus F) > (2\varepsilon)^{-1}.$$

This shows (3.4) and the proof is completed. □

4. Conclusion. We assume that $\varepsilon = N^{-1}$ with $N \geq 10^4$ and denote by $\mathscr{A}_n$, $n = 0, 1, 2, 3, \ldots$, the σ-field of subsets of $(-\infty, \infty)$ whose atoms are the intervals $[k\varepsilon^n, (k+1)\varepsilon^n]$, $k = 0, \pm 1, \pm 2, \ldots$. Since points in $(-\infty, \infty)$ have parabolic measure 0, the end-points have no importance; because $\varepsilon = N^{-1}$, the relation $\mathscr{A}_n \subseteq \mathscr{A}_{n+1}$ is correct. Each set

$$F_{n,k} = [k\varepsilon^n, (k + 4\varepsilon)\varepsilon^n] \cup [(k + 1 - 4\varepsilon)\varepsilon^n, (k+1)\varepsilon^n]$$

belongs to $\mathscr{A}_{n+1}$. We now write ω for the parabolic measure at the point $(0, 100)$ and use this as the basis for probabilistic assertions. Because $J_{n,k} = [k\varepsilon^n, (k+1)\varepsilon^n]$ is an atom of $\mathscr{A}_n$, it follows from Proposition 2 that $F_n = \bigcup_k F_{n,k}$ has the property $\omega(F_n | \mathscr{A}_n) > 1 - \varepsilon^{1/2}$. The series $\sum_1^\infty \chi_{F_n} - E(\chi_{F_n} | \mathscr{A}_n)$ has terms bounded by 1 and orthogonal in the space $L^2(d\omega)$. The partial sums $\sum_1^\nu$ are therefore $o(\nu)$ [even $o(\nu^{2/3})$] almost everywhere for the measure ω. The measure ω is therefore concentrated on a set B described as follows.

For each $x \in B$, there is an integer $\nu(x)$ such that for $\nu \geq \nu(x)$, we have $x \in F_n$ for all integers $n = 1, 2, \ldots, \nu$ with at most $2\varepsilon^{1/2}\nu$ exceptions. [The maximum principle then shows that $\omega^{x,t}(B) = 1$ for *every* (x, t) in the upper half-plane.] It remains to obtain an upper bound for dim B.

We fix $\nu \geq 1$ and focus on the points x in $(0, 1)$, such that $\nu(x) < \nu$. By assumption $x \in F_n$ is true for at least $r = [\nu - 2\varepsilon^{1/2}\nu]$ integers, $n = 1, 2, 3, \ldots, \nu$, and these can be chosen in at most $\sum_0^{\nu - r} \binom{\nu}{s}$ ways. The latter sum has a logarithm asymptotic to

$$-\left[2\varepsilon^{1/2}\log 2\varepsilon^{1/2} + (1 - 2\varepsilon^{1/2})\log(1 - 2\varepsilon^{1/2})\right]\nu,$$

by Stirling's formula. For each x in $(0, 1)$ we use the expansion of x in the base $N = \varepsilon^{-1}$ and, in particular, the first digit up to the $\nu + 1$st. When $x \in F_n$, the $n + 1$st digit of x is restricted to 8 values. When $x \in F_n$ for a fixed choice of l integers n among $n = 1, 2, 3, \ldots, \nu + 1$, and $x \notin F_n$ for the remaining digits, the first $\nu + 1$ digits of x can be chosen in at most $8^l N^{\nu - l + 1}$ ways. Each choice determines an interval of length $N^{-\nu - 1}$. From these calculations we can see that the dimension d of B satisfies

$$\begin{aligned} d \log N \leq & -\left[2\varepsilon^{1/2}\log 2\varepsilon^{1/2} + (1 - 2\varepsilon^{1/2})\log(1 - 2\varepsilon^{1/2})\right] \\ & + \log 8 + 2\varepsilon^{1/2}\log N, \end{aligned}$$

or $d \leq c(\log N)^{-1}$.

This completes the proof of the theorem.

It may be observed that the term $\log 8$ occurs because of certain technical points in the construction and could perhaps be removed, thereby improving the estimate to $d = O(\varepsilon^{1/2})$.

APPENDIX 1

We discuss the maximum principle and barriers for operators $L = a(x, t)u_{xx} - u_t$, with the bound $0 < a(x, t) \leq \Lambda_2$ in the region $\{t > 0\}$. A basic role is played by the function $v(\rho) = \pi^{-1/2}\int_0^\rho e^{-s^2/4}\, ds$, $v(+\infty) = 1$. Then $v'' < 0$

on $(0, +\infty)$ and $v''(\rho) = -\frac{1}{2}\rho v'(\rho)$. The function $u(x, t) = v(xt^{-1/2}\Lambda_2^{-1/2})$ is a subsolution ($Lu \geq 0$) for $x > 0$, $t > 0$.

Let g be a subsolution on the region $t > 0$, let $g \leq 1$ and $\limsup g \leq 0$ everywhere on the line $t = 0$. We claim that $g \leq 0$ everywhere. To prove this, we take a large number $R > 0$ and observe that the function $h = u(R - x, t) + u(R + x, t)$ is a subsolution on the region $|x| < R$, $t > 0$. The subsolution $g + h$ has $\limsup \leq 2$ on the closed interval joining $(-R, 0)$ to $(R, 0)$, and on vertical lines $x = R$, $t > 0$, and $x = -R$, $t > 0$. Hence $g + h \leq 2$ on its domain by the maximum principle for bounded regions. Now $h \to 2$ as $R \to +\infty$ at every point (x, t) with $t > 0$. Hence $g \leq 0$.

To construct barriers, let the function v be extended to $(-\infty, +\infty)$ by $v(\rho) = 0$ for $\rho \leq 0$. This leads to the extension of u, $u(x, t) = 0$, for $x \leq 0$, and this is plainly a subsolution on $\{t > 0\}$. With this definition of u, we define

$$h(x, t) = \int_0^{1/2} u(R + x, t)\, dR + u(R - x, t)\, dR.$$

Then h is a subsolution on $\{t > 0\}$ with limit value one at $(0, 0)$, limit $\frac{1}{2}$ at $(x, 0)$, when $|x| \geq \frac{1}{2}$, and limit $1 - |x|$ at $(x, 0)$, when $|x| \leq \frac{1}{2}$. Thus $1 - h$ is a barrier at $(0, 0)$. Similar constructions give barriers at other points. Thus the Dirichlet problem is solvable.

APPENDIX 2

Let $\tilde{L} = a(x, t)\,\partial^2/\partial x^2 - \partial/\partial t$ in $\{x > 0\}$ with a satisfying (0.1) and C^2 in the region. Let

$$K(x, t) = t^{-\Lambda_1/2\Lambda_2} e^{-x^2/4\Lambda_2 t}, \quad \text{for } t > 0,$$

and

$$H(x, t) = \begin{cases} \left(xt^{-3/2}\right)^{\Lambda_1/\Lambda_2} e^{-x^2/4\Lambda_2 t}, & \text{for } x > 0,\ t > 0, \\ 0, & \text{for } x > 0,\ t \leq 0. \end{cases}$$

It can be verified that $LK \leq 0$ in $t > 0$ and $\tilde{L}H \leq 0$ in $x > 0$. Therefore it follows from the maximum principle that sets of Λ_1/Λ_2 dimensional measure 0 on $t = 0$ (or $x = 0$) have zero parabolic measure with respect to L (or $\tilde{L}$).

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