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A MULTIVARIATE EXTENSION OF HOEFFDING'S LEMMA

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Hoeffding's lemma gives an integral representation of the covariance of two random variables in terms of the difference between their joint and marginal probability functions, i.e.,

$$\operatorname{cov}(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{P(X > x, Y > y) - P(X > x)P(Y > y)\}\, dx\, dy.$$

This identity has been found to be a useful tool in studying the dependence structure of various random vectors.

A generalization of this result for more than two random variables is given. This involves an integral representation of the multivariate joint cumulant. Applications of this include characterizations of independence. Relationships with various types of dependence are also given.

1. Introduction. It is well known that if a random variable (r.v.) X has distribution function (d.f.) $F(x)$ with finite expectation, then

$$EX = \int_0^{\infty}(1 - F(x))\, dx - \int_{-\infty}^{0} F(x)\, dx. \tag{1}$$

The extension to high-order moments is straightforward. That is, if $E|X|^n < \infty$,

$$EX^n = n\left[\int_0^{\infty} x^{n-1}(1 - F(x))\, dx - \int_{-\infty}^{0} x^{n-1}F(x)\, dx\right]. \tag{2}$$

Hoeffding (1940) gave a bivariate version of identity (1), which is mentioned in Lehmann (1966). Let $F_{X,Y}(x, y)$, $F_X(x)$, $F_Y(y)$ denote the joint and marginal distributions of random vector (X, Y), where $E|XY|$, $E|X|$, $E|Y|$ are assumed finite. Hoeffding's lemma is

$$EXY - EXEY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{F_{X,Y}(x, y) - F_X(x)F_Y(y)\}\, dx\, dy. \tag{3}$$

Lehmann (1966) used this result to characterize independence, among other things, and Jogdeo (1968) extended Lehmann's bivariate characterization of independence. Jogdeo obtained an extension of formula (3) which we now give. Let (Y_1, Y_2, Y_3) be a triplet independent of (X_1, X_2, X_3) and having the same distribution as $(-X_1, X_2, X_3)$. Then

$$E(X_1 - Y_1)(X_2 - Y_2)(X_3 - Y_3) = \iiint_{-\infty}^{\infty} K(u_1, u_2, u_3)\, du_1\, du_2\, du_3, \tag{4}$$

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where

$$K(u_1, u_2, u_3) = \{P(B_1A_2A_3) + P(B_1)P(A_2A_3) - P(A_2)P(B_1A_3) - P(A_3)P(B_1A_2)\} - \{P(A_1A_2A_3) + P(A_1)P(A_2A_3) - P(A_2)P(A_1A_3) - P(A_3)P(A_1A_2)\}$$

and $A_i = \{X_i \le u_i\}$, $i = 1,2,3$, $B_1 = \{X_1 \ge -u_1\}$. Jogdeo mentioned that a similar result holds for $n \ge 3$. We give a different generalization of Hoeffding's lemma. Notice that expression (3) can be rewritten as

$$\text{Cov}(X, Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{Cov}(\chi_X(x), \chi_Y(y))\, dx\, dy, \tag{5}$$

where $\chi_X(x) = 1$ if $X > x$, 0 otherwise, and that the covariance is the second-order joint cumulant for the random vector (X, Y). In the following we extend the results to the rth-order joint cumulant where $r \ge 3$.

2. Main results. Consider a random vector $(X_1, \dots, X_r)$, where $E|X_i|^r < \infty$, $i = 1, \dots, r$.

DEFINITION 1. The *rth-order joint cumulant* of $(X_1, \dots, X_r)$ denoted by $\text{cum}(X_1, \dots, X_r)$ is defined by

$$\text{cum}(X_1, \dots, X_r) = \sum (-1)^{p-1}(p-1)!\left(E\prod_{j\in\nu_1} X_j\right)\cdots\left(E\prod_{j\in\nu_p} X_j\right), \tag{6}$$

where summation extends over all partitions $(\nu_1, \dots, \nu_p)$, $p = 1, 2, \dots, r$, of $\{1, \dots, r\}$.

It can be shown [see Brillinger (1975)] that $\text{cum}(X_1, \dots, X_r)$ is the coefficient of the term $(i)^r t_1 \cdots t_r$ in the Taylor series expansion of $\log E(\exp i\sum_{j=1}^r t_j X_j)$. Furthermore the following properties are easy to check:

(i) $\text{cum}(a_1X_1, \dots, a_rX_r) = a_1 \cdots a_r \text{cum}(X_1, \dots, X_r)$;
(ii) $\text{cum}(X_1, \dots, X_r)$ is symmetric in its arguments;
(iii) if any group of the X's are independent of the remaining X's, then $\text{cum}(X_1, \dots, X_r) = 0$;
(iv) for the random variable $(Y_1, X_1, \dots, X_r)$, $\text{cum}(X_1 + Y_1, X_2, \dots, X_r) = \text{cum}(X_1, \dots, X_r) + \text{cum}(Y_1, X_2, \dots, X_r)$;
(v) for μ constant, $r \ge 2$, $\text{cum}(X_1 + \mu, X_2, \dots, X_r) = \text{cum}(X_1, \dots, X_r)$;
(vi) for $(X_1, \dots, X_r), (Y_1, \dots, Y_r)$ independent

$$\text{cum}(X_1 + Y_1, \dots, X_r + Y_r) = \text{cum}(X_1, \dots, X_r) + \text{cum}(Y_1, \dots, Y_r);$$

(vii) $\text{cum}\, X_j = EX_j$, $\text{cum}(X_j, X_j) = \text{Var}\, X_j$ and $\text{cum}(X_i, X_j) = \text{cov}(X_i, X_j)$.

To represent certain moments by cumulants, we have the following useful identity.

LEMMA 1. *If* $E|X_i|^m < \infty$,

$$EX_1 \cdots X_m - EX_1 \cdots EX_m = \sum \operatorname{cum}(X_k, k \in \nu_1) \cdots \operatorname{cum}(X_k, k \in \nu_p), \tag{7}$$

where Σ *extends over all partitions* $(\nu_1, \dots, \nu_p)$, $p = 1, \dots, m-1$, *of* $\{1, \dots, m\}$.

PROOF. In the case of $m = 2$, $p = m - 1 = 1$ and (7) reduces to the well known

$$EX_1X_2 - EX_1EX_2 = \operatorname{cum}(X_k, k \in \nu_1) = \operatorname{cov}(X_1, X_2).$$

Notice that

$$\begin{aligned} EX_1 \cdots X_m - EX_1 \cdots EX_m = EX_1 \cdots X_{m-2}X_{m-1}X_m \\ - EX_1 \cdots EX_{m-2}EX_{m-1}X_m \\ + EX_1 \cdots EX_{m-2}\operatorname{cov}(X_{m-1}, X_m). \end{aligned} \tag{8}$$

Introduce the new notation $Y_i = X_i$, $i = 1, \dots, m-2$, $Y_{m-1} = X_{m-1}X_m$. By Theorem 2.3.2 in Brillinger (1975), page 21, and induction we get (7). □

Our main result is the following.

THEOREM 1. *For the random vector* $(X_1, \dots, X_r)$, $r > 1$, *if* $E|X_i|^r < \infty$, $i = 1, 2, \dots, r$, *then*

$$\operatorname{cum}(X_1, \dots, X_r) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \operatorname{cum}(\chi_{X_1}(x_1), \dots, \chi_{X_r}(x_r))\, dx_1 \cdots dx_r, \tag{9}$$

where $\chi_{X_i}(x_i) = 1$ *if* $X_i > x_i$, 0 *otherwise.*

To prove the theorem, we need a lemma which is of some independent interest.

LEMMA 2. *If* $E|X_1 \cdots X_r| < \infty$, *we have*

$$\begin{aligned} EX_1 \cdots X_r = (-1)^r \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big\{ F(\mathbf{x}) - \sum_{j=1}^{r} \varepsilon(x_j) F(\mathbf{x}^{(j)}) \\ + \sum_{i<j} \varepsilon(x_i)\varepsilon(x_j) F(\mathbf{x}^{(i,j)}) \\ + \cdots + (-1)^r \prod_{j=1}^{r} \varepsilon(x_j) \Big\}\, dx_1 \cdots dx_r, \end{aligned} \tag{10}$$

where $\varepsilon(x_i) = 1$ *if* $x_i \geq C$, 0 *otherwise. Here* $\mathbf{x}^{(i_1, \dots, i_k)}$ *represents* $(x_1, \dots, x_{i_1-1}, x_{i_1+1}, \dots, x_{i_2-1}, x_{i_2+1}, \dots, x_{i_k-1}, x_{i_k+1}, \dots, x_r)$. *Also* $F(\mathbf{x}^{(i_1, \dots, i_k)})$ *is the marginal d.f. of* $\mathbf{X}^{(i_1, \dots, i_k)}$. *We omit the subscripts for* F *for simplicity when there is no ambiguity, e.g.,* $F(\mathbf{x}^{(1)})$ *is the marginal of* $(X_2, \dots, X_r)$.

PROOF. First, we have the identity

$$X_i \equiv \int_{-\infty}^{\infty} \left(\varepsilon(x_i) - I_{(-\infty, x_i]}(X_i)\right) dx_i, \tag{11}$$

where $I_{(-\infty, x_i]}(X_i) = 1$ if $X_i \le x_i$, 0 otherwise. Then by Fubini's theorem

$$\begin{aligned}
EX_1 \cdots X_r &= E\left\{\prod_{i=1}^{r} \int_{-\infty}^{\infty} \left[\varepsilon(x_i) - I_{(-\infty, x_i]}(X_i)\right] dx_i\right\} \\
&= E\left\{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{r} \left[\varepsilon(x_i) - I_{(-\infty, x_i]}(X_i)\right] dx_1 \cdots dx_r\right\} \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} E\left\{\prod_{i=1}^{r} \left[\varepsilon(x_i) - I_{(-\infty, x_i]}(X_i)\right]\right\} dx_1 \cdots dx_r \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{\prod_{i=1}^{r} \varepsilon(x_i) - \sum_{j=1}^{r} \prod_{k \ne j} \varepsilon(x_k) F(x_j)\right. \\
&\qquad \left. + \sum_{i<j} \prod_{k \ne i, j} \varepsilon(x_k) F(x_i, x_j) + \cdots + (-1)^r F(\mathbf{x})\right\} dx_1 \cdots dx_r,
\end{aligned}$$

which is just the right side of (10). □

REMARK 1. It is easy to see that (1) can be written as $EX = \int_{-\infty}^{\infty}(\varepsilon(x) - F(x))\, dx$, which is a special case of (10). Thus Lemma 2 is an extension of (1).

REMARK 2. Using the identity

$$X_i^{n_i} \equiv \int_{-\infty}^{\infty} n_i x_i^{n_i - 1} \left[\varepsilon(x_i) - I_{(-\infty, x_i]}(X_i)\right] dx_i,$$

we can also obtain an extension of (2), i.e.,

$$\begin{aligned}
EX_1^{n_1} \cdots X_k^{n_k} &= (-1)^k n_1 \cdots n_k \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{n_1 - 1} \cdots x_k^{n_k - 1} \\
&\quad \times \left\{F(x_1, \ldots, x_k) - \sum_{j=1}^{k} \varepsilon(x_j) F(\mathbf{x}^{(j)})\right. \\
&\quad + \sum_{i<j} \varepsilon(x_i)\varepsilon(x_j) F(\mathbf{x}^{(i,j)}) \\
&\quad \left. + \cdots + (-1)^k \prod_{i=1}^{k} \varepsilon(x_i)\right\} dx_1 \cdots dx_k,
\end{aligned} \tag{12}$$

where $n_i \ge 1$, $n_1 + \cdots + n_k \le r$.

REMARK 3. When the X_i's are nonnegative (12) reduces to

$$EX_1^{n_1} \cdots X_k^{n_k} = \int_{-\infty}^{\infty} \cdots \int_0^{\infty} n_1 \cdots n_k x_1^{n_1-1} \cdots x_k^{n_k-1} \times \bar{F}(x_1, \dots, x_k)\, dx_1 \cdots dx_k, \tag{13}$$

where $\bar{F}(x_1, \dots, x_k)$ is the survival function $P(X_i > x_i,\ i = 1, \dots, k)$. The bivariate case of (13) was mentioned by Barlow and Proschan (1981), page 135.

The proof of the Theorem 1 involves routine algebra and the use of Fubini's theorem and Lemma 2. We have

$$\begin{aligned}
&\operatorname{cum}(X_1, \dots, X_r)\\
&= \sum (-1)^{p-1}(p-1)!\Big(E\prod_{j\in\nu_1} X_j\Big)\cdots\Big(E\prod_{j\in\nu_p} X_j\Big)\\
&= E(X_1, \dots, X_r) - \sum E\Big(\prod_{j\in\nu_1} X_j\Big)E\Big(\prod_{j\in\nu_2} X_j\Big)\\
&\quad + \cdots + (-1)^{r-1}(r-1)!\prod_{j=1} EX_j\\
&= (-1)^r \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big\{ F(\mathbf{x}) - \sum_{j=1}^{r} \varepsilon(x_j) F(\mathbf{x}^{(j)})\\
&\qquad + \sum_{i<j} \varepsilon(x_i)\varepsilon(x_j) F(\mathbf{x}^{(i,j)})\\
&\qquad + \cdots + (-1)^r \prod_{j=1}^{r} \varepsilon(x_j) \Big\}\, dx_1 \cdots dx_r\\
&\quad - (-1)^{n_{\nu_1}+n_{\nu_2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big\{ F(x_j,\ j\in\nu_1)\\
&\qquad - \sum_{k\in\nu_1} \varepsilon(x_k) F(x_j,\ j \in \nu_1 \setminus k)\\
&\qquad + \cdots + (-1)^{n_{\nu_1}} \prod_{j\in\nu_1} \varepsilon(x_j) \Big\}\\
&\qquad \times \Big\{ F(x_i,\ i\in\nu_2) - \sum_{k\in\nu_2} \varepsilon(x_k) F(x_i,\ i\in\nu_2 \setminus k)\\
&\qquad + \cdots + (-1)^{n_{\nu_2}} \prod_{j\in\nu_2} \varepsilon(x_i) \Big\}\, dx_1 \cdots dx_r\\
&\quad + \cdots + (-1)^r (r-1)! \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{r} [\varepsilon(x_i) - F_i(x_i)]\, dx_1 \cdots dx_r,
\end{aligned}$$

where n_{ν_i} is the number of indices in ν_i, $F(x_j,\ j \in \nu_i)$ is the marginal of r.v.'s in

ν_i and $F_i(x)$ is the marginal of X_i. All terms with $\varepsilon(x_i)$ factors cancel and the quantities $\sum_{i=1}^{j} n_{\nu_i}$, $j = 2, \ldots, p$, are all equal to r. Thus

$$\begin{aligned}
\operatorname{cum}&(X_1, \ldots, X_r)\\
&= (-1)^r \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big\{ F(\mathbf{x}) - \sum F(x_j, j \in \nu_1) F(x_i, i \in \nu_2)\\
&\qquad\qquad + \cdots + (-1)^r (r-1)! \prod_{i=1}^{r} F_i(x_i) \Big\}\, dx_1 \cdots dx_r\\
&= (-1)^r \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big\{ \sum (-1)^{p-1} (p-1)! F(x_j, j \in \nu_1)\\
&\qquad\qquad \times \cdots \times F(x_j, j \in \nu_p) \Big\}\, dx_1 \cdots dx_r\\
&= (-1)^r \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \operatorname{cum}\big(1 - \chi_{X_1}(x_1), \ldots, 1 - \chi_{X_r}(x_r)\big)\, dx_1 \cdots dx_r\\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \operatorname{cum}\big(\chi_{X_1}(x_1), \ldots, \chi_{X_r}(x_r)\big)\, dx_1 \cdots dx_r.
\end{aligned}$$

The last equality follows upon using properties (i), (iii) and (iv) of the cumulant. This completes the proof.

REMARK 4. The result of Theorem 1 gives that

$$\operatorname{cum}(X_1, \ldots, X_r) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \operatorname{cum}\big(\chi_{X_1}(x_1), \ldots, \chi_{X_r}(x_r)\big)\, dx_1 \cdots dx_r.$$

The integral can then be expressed in a variety of ways. A general form is

$$(-1)^{\operatorname{card} B} \operatorname{cum}\big(\chi_{X_i}(x_i),\ i \in A;\ 1 - \chi_{X_i}(x_i),\ i \in B\big),$$

where $A \cup B = \{1, 2, \ldots, r\}$. We then have various combinations of the distribution and/or survival function in the integrand. Some examples:

(i) for $A = \phi$, card $B = r$ the integrand is

$$(-1)^r \Big\{ F(\mathbf{x}) - \sum F(x_j, j \in v_1) F(x_i, i \in v_2) + \cdots + (-1)^r (r-1)! \prod_{i=1}^{r} F_i(x_i) \Big\};$$

(ii) for $B = \phi$ the integrand is

$$\bar{F}(\mathbf{x}) - \sum \bar{F}(x_j, j \in v_1) \bar{F}(x_i, i \in v_2) + \cdots + (-1)^r (r-1)! \prod_{i=1}^{r} \bar{F}_i(x_i).$$

3. Applications. In some sense, the cumulant is a measure of the independence of certain classes of r.v.'s.

The following result was shown by Jogdeo (1968). Let $F_{X_1, X_2, X_3}(x_1, x_2, x_3)$ belong to the family $\mathscr{M}(3)$, where $\mathscr{M}(3)$ denotes the class of trivariate distributions such that there exists a choice of α and Δ_i, $i = 1, 2, 3$, such that

$$P(X_1\Delta_1 x_1, X_2\Delta_2 x_2, X_3\Delta_3 x_3)\Delta \prod_{i=1}^{3} P(X_i\Delta_i x_i), \tag{14}$$

for all x_1, x_2, x_3, where the Δ, Δ_i each denote one of the inequalities $\geq$ or $\leq$. Then X_i, X_j for all $i \neq j$ are uncorrelated and $EX_1X_2X_3 = EX_1EX_2EX_3$ if and only if the X_i's are mutually independent.

Using Theorem 1, we get this conclusion directly. The "if" part is trivial. Conversely, since $F \in \mathscr{M}(3)$ we know $F_{X_iX_j}(x_i, x_j) \in \mathscr{M}(2)$ [$\mathscr{M}(n)$ can be defined similarly]. Since X_i and X_j are uncorrelated this implies the X_i's are pairwise independent by Hoeffding's lemma. Thus, using Remark 4, (9) becomes

$$\begin{aligned} &EX_1X_2X_3 - EX_1EX_2EX_3 \\ &\quad = \pm \int\int\int_{-\infty}^{\infty} \{P(X_1\Delta_1 x_1, X_2\Delta_2 x_2, X_3\Delta_3 x_3) \\ &\qquad\qquad - P(X_1\Delta_1 x_1)P(X_2\Delta_2 x_2)P(X_3\Delta_3 x_3)\}\, dx_1\, dx_2\, dx_3. \end{aligned}$$

Now since $F \in \mathscr{M}(3)$ the integrand will not change sign, so that $EX_1X_2X_3 = EX_1EX_2EX_3$ implies

$$P(X_1\Delta_1 x_1, X_2\Delta_2 x_2, X_3\Delta_3 x_3) = P(X_1\Delta_1 x_1)P(X_2\Delta_2 x_2)P(X_3\Delta_3 x_3),$$

for all x_1, x_2, x_3 which means that the X_i's are independent.

The n-dimension extension is straightforward and is given in the following discussion.

THEOREM 2. *If $F_{X_1,\dots,X_n}(x_1,\dots,x_n) \in \mathscr{M}(n)$, then $EX_{i_1} \cdots X_{i_k} = \prod_{j=1}^{k} EX_{i_j}$ for all subsets $\{i_1,\dots,i_k\}$ of $\{1,\dots,n\}$ if and only if $X_1,\dots,X_n$ are independent.*

PROOF. $F_{X_1,\dots,X_n}(x_1,\dots,x_n) \in \mathscr{M}(n)$ means $F_{X_{i_1},\dots,X_{i_k}}(x_{i_1},\dots,x_{i_k}) \in \mathscr{M}(k)$ for any subset $(i_1,\dots,i_k)$. By induction on n, using Theorem 1, we obtain

$$\begin{aligned} &EX_1 \cdots X_n - \prod_{j=1}^{n} EX_j \\ &\quad = \pm \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ P(X_i\Delta_i x_i, i = 1,\dots,n) - \prod_{i=1}^{n} P(X_i\Delta_i x_i) \right\} dx_1 \cdots dx_n. \end{aligned}$$

The integrand will not change sign, so $EX_1 \cdots X_n = \prod_{j=1}^{n} EX_j$ implies that the X_i are mutually independent. □

Several authors have discussed dependence structures in which uncorrelatedness implies independence. Among them are Lehmann (1966), Jogdeo (1968), Joag-Dev (1983) and Chhetry, Kimeldorf and Zahed (1986).

We now give a definition from Joag-Dev (1983). Let $\mathbf{X} = (X_1,\dots,X_n)$ be a random vector, A be a subset of $\{1,\dots,n\}$ and $\mathbf{x} = (x_1,\dots,x_n)$ a vector of constants.

Definition 2. Random vectors are said to be PUOD (positive upper orthant dependent) if (a) (which follows) holds, PLOD (positive lower orthant dependent) if (b) holds and POD (positive orthant dependent) if (a) and (b) hold, where

$$\text{(a)} \qquad P(\mathbf{X} > \mathbf{x}) \geq \prod_{i=1}^{n} P(X_i > x_i),$$

$$\text{(b)} \qquad P(\mathbf{X} \leq \mathbf{x}) \geq \prod_{i=1}^{n} P(X_i \leq x_i).$$

If the reverse inequalities between the probabilities in (a) and (b) hold the three concepts are called NUOD, NLOD and NOD, respectively.

Note. In Definition 2 in Block and Ting (1981), POD is used for what is called PUOD in this paper.

Definition 3. A vector $\mathbf{X}$ is said to be SPOD (strongly positively orthant dependent) if for every set of indices A and for all $\mathbf{x}$ the following three conditions hold:

$$\text{(c)} \qquad P(\mathbf{X} > \mathbf{x}) \geq P(X_i > x_i,\ i \in A) P(X_j > x_j,\ j \in A^c),$$

$$\text{(d)} \qquad P(\mathbf{X} \leq \mathbf{x}) \geq P(X_i \leq x_i,\ i \in A) P(X_j \leq x_j,\ j \in A^c),$$

$$\text{(e)} \qquad P(X_i > x_i,\ i \in A,\ X_j \leq x_j,\ j \in A^c) \leq P(X_i > x_i,\ i \in A) P(X_j \leq x_j,\ j \in A^c).$$

The relationships among these definitions are as follows:

$$(15) \qquad \text{Association} \Rightarrow \text{SPOD} \Rightarrow \text{POD} \begin{array}{c} \nearrow \text{PLOD} \searrow \\ \searrow \text{PUOD} \nearrow \end{array} \mathscr{M}(n).$$

Since association, SPOD, POD, PLOD, PUOD are all subclasses of $\mathscr{M}(n)$, Theorem 2 generalizes some results in Lehmann (1966) and it gives us another proof of Theorem 2 in Joag-Dev (1983) as well as some new characterizations of independence for POD random variables. Corollary 1 is the result of Joag-Dev.

Corollary 1. *Let $X_1, \dots, X_n$ be SPOD and assume* $\operatorname{cov}(X_i, X_j) = 0$ *for all $i \neq j$. Then $X_1, \dots, X_n$ are mutually independent.*

Proof. Since $X_1, \dots, X_n$ SPOD implies $(X_1, \dots, X_n) \in \mathscr{M}(n)$ by Theorem 2 we need only check $EX_{i_1} \cdots X_{i_k} = \prod_{j=1}^{k} EX_{i_j}$ for all subsets $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$. When $n = 2$ SPOD is equivalent to POD and uncorrelatedness implies X_1, X_2 independent. By induction on n we may assume all subsets with

$n-1$ r.v.'s are mutually independent and thus $EX_{i_1} \cdots X_{i_k} = \prod_{j=1}^{k} EX_{i_j}$ for all $1 \le k \le n-1$. Hence $\operatorname{cum}(X_k,\ k \in \nu_p) = 0$, whenever $1 < \operatorname{card}(\nu_p) \le n-1$. So we only need to check $EX_1 \cdots X_n = \prod_{j=1}^{n} EX_j$. By Lemma 1, Theorem 1 and because of the independence of any $(n-1)$ r.v.'s,

$$
\begin{aligned}
EX_1 \cdots X_n - \prod_{j=1}^{n} EX_j
&= \sum \operatorname{cum}(X_k,\ k \in \nu_1) \cdots \operatorname{cum}(X_i,\ k \in \nu_P) \\
&= \operatorname{cum}(X_1, \dots, X_n) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ P(\mathbf{X} > \mathbf{x}) - \prod_{j=1}^{n} P(X_j > x_j) \right\} dx_1 \cdots dx_n \ge 0.
\end{aligned}
\tag{16}
$$

Similarly,

$$
\begin{aligned}
EX_1 \cdots X_n - \prod_{j=1}^{n} EX_j
&= E(-X_1)(-X_2)X_3 \cdots X_n - E(-X_1)E(-X_2)EX_3 \cdots EX_n \\
&= \operatorname{cum}(-X_1, -X_2, X_3, \dots, X_n) \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{ P(-X_1 > x_1, -X_2 > x_2, X_3 > x_3 \cdots X_n > x_n) \\
&\qquad - P(-X_1 > x_1) P(-X_2 > x_2) P(X_3 > x_3) \cdots P(X_n > x_n) \} \, dx_1 \cdots dx_n \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \Big\{ P(X_1 < -x_1, X_2 < -x_2, X_i > x_i, i = 3, \dots, n) \\
&\qquad - P(X_1 < -x_1) P(X_2 < -x_2) \prod_{i=3}^{n} P(X_i > x_i) \Big\} \, dx_1 \cdots dx_n \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{ P(X_j < -x_j,\ j = 1,2,\ X_i > x_i,\ i = 3, \dots, n) \\
&\qquad - P(X_j < -x_j,\ j = 1,2) P(X_i > x_i,\ i = 3, \dots, n) \} \, dx_1 \cdots dx_n \\
&\le 0.
\end{aligned}
\tag{17}
$$

The last equality holds by the induction assumption of mutual independence and the last inequality is due to SPOD. Combining (16) and (17) completes the proof. □

Theorem 3. *Let X_1, X_2, X_3 be POD and assume X_i, X_j for all $i \ne j$ are uncorrelated. Then X_1, X_2, X_3 are mutually independent.*

Proof. The following two summands are nonnegative since X_1, X_2, X_3 are POD. By Lemma 1 we then have

$$\begin{aligned}
&\left[P(X_1 > x_1, X_2 > x_2, X_3 > x_3) - \prod_{i=1}^{3} P(X_i > x_i)\right] \\
&\quad + \left[P(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3) - \prod_{i=1}^{3} P(X_i \le x_i)\right] \\
&\quad = \operatorname{cum}\big(\chi_{X_1}(x_1), \chi_{X_2}(x_2), \chi_{X_3}(x_3)\big) \\
&\qquad + \sum_{i \ne j \ne k} P(X_i > x_i) \operatorname{cov}\big(\chi_{X_j}(x_j), \chi_{X_k}(x_k)\big) \\
&\qquad + \operatorname{cum}\big(1 - \chi_{X_1}(x_1), 1 - \chi_{X_2}(x_2), 1 - \chi_{X_3}(x_3)\big) \\
&\qquad + \sum_{i \ne j \ne k} P(X_i \le x_i) \operatorname{cov}\big(\chi_{X_j}(x_j), \chi_{X_k}(x_k)\big) \\
&\quad = \sum_{i \ne j} \operatorname{cov}\big(\chi_{X_i}(x_i), \chi_{X_j}(x_j)\big).
\end{aligned}$$

Since X_i, X_j POD and $\operatorname{cov}(X_i, X_j) = 0$ we obtain $\operatorname{cov}(\chi_{X_i}(x_i), \chi_{X_j}(x_j)) = 0$. Thus $P(X_i > x_i,\ i = 1,2,3) - \prod_{i=1}^{3} P(X_i > x_i) = 0$, i.e., X_1, X_2, X_3 are mutually independent. □

Remark 5. For three r.v.'s X_1, X_2, X_3 the mixed positive dependence defined in Chhetry, Kimeldorf and Zahed (1986) implies POD but the converse is not true as shown by an example in Joag-Dev (1983). Notice that since the mixed positive dependence implies POD in Corollary 1, the SPOD can be relaxed to this mixed condition.

Theorem 4. *Assume $n = 2l + 1$ is an odd positive integer and $X_1, \dots, X_n$ are POD. Then if $E(X_{i_1} \cdots X_{i_k}) = EX_{i_1} \cdots EX_{i_k}$, where $2 \le k \le 2l$ for any subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, 2l + 1\}$, it follows that $X_1, \dots, X_n$ are mutually independent.*

Proof. By Theorem 2, we need only check $EX_1 \cdots X_n = EX_1 \cdots EX_n$. On the one hand,

$$\begin{aligned}
&EX_1 \cdots X_n - EX_1 \cdots EX_n \\
&\quad = \operatorname{cum}(X_1, \dots, X_n) \\
&\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left\{ P(X_i > x_i, i = 1, \dots, n) - \prod_{j=1}^{n} P(X_j > x_j) \right\} dx_1 \cdots dx_n \\
&\quad \ge 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned} &EX_1 \cdots X_n - EX_1 \cdots EX_n \\ &\quad = (-1)^{2l+1}\{E(-X_1)\cdots(-X_n) - E(-X_1)\cdots E(-X_n)\} \\ &\quad = (-1)^{2l+1}\mathrm{cum}(-X_1,\dots,-X_n) \\ &\quad = (-1)^{2l+1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big\{P(X_i < -x_i,\, i=1,\dots,n) \\ &\qquad\qquad - \prod_{j=1}^{n} P(X_j < -x_j)\Big\}\, dx_1 \cdots dx_n \le 0. \quad \square \end{aligned}$$

REMARK 6. For $n = 4$ we construct in Example 1 POD r.v.'s such that any three X_i's are independent but the X_i's are not mutually independent. This shows that the conditions of Theorem 4 are reasonable. In Example 2 we show that for POD r.v.'s $\mathrm{cov}(X_i, X_j) = 0$ is not enough to give mutual independence when $2l + 1 > 3$.

EXAMPLE 1. Let $X_1,\dots,X_4$ have the following distribution. It is easy to check that for $i \neq j \neq k$, X_i, X_j, X_k are mutually independent and that $X_1,\dots,X_4$ are POD.

X_1	X_2	X_3	X_4	Pr
1	1	1	1	1/8
1	1	0	0	1/8
1	0	1	0	1/8
0	1	1	0	1/8
1	0	0	1	1/8
0	1	0	1	1/8
0	0	1	1	1/8
0	0	0	0	1/8

Since $P(X_i > 1/2,\, i = 1,\dots,4) - \prod_{i=1}^{4} P(X_i > 1/2) = 1/16 > 0$, $X_1,\dots,X_4$ are not mutually independent. Notice also that

$$\begin{aligned} &P(X_1 \le x_1,\, X_2 \le x_2,\, X_3 > x_3,\, X_4 > x_4) \\ &\quad - P(X_1 \le x_1,\, X_2 \le x_2)P(X_3 > x_3,\, X_4 > x_4) \\ &\quad = \mathrm{cum}\big(1 - \chi_{X_1}(x_1), 1 - \chi_{X_2}(x_2), \chi_{X_3}(x_3), \chi_{X_4}(x_4)\big) \\ &\quad = \mathrm{cum}\big(\chi_{X_i}(x_i),\, i = 1,\dots,4\big) \\ &\quad = P(X_i > x_i,\, i = 1,\dots,4) - \prod_{i=1}^{4} P(X_i > x_i) \\ &\quad \ge 0, \end{aligned}$$

so these r.v.'s are not SPOD.

EXAMPLE 2. Let $X_1, \dots, X_5$ have the following distribution:

X_1	X_2	X_3	X_4	X_5	Pr
1	1	1	1	1	1/16
1	1	0	0	1	1/16
1	0	1	0	1	1/16
0	1	1	0	1	1/16
1	0	0	1	1	1/16
0	1	0	1	1	1/16
0	0	1	1	1	1/16
0	0	0	0	1	1/16
1	1	1	1	0	1/16
1	1	0	0	0	1/16
1	0	1	0	0	1/16
0	1	1	0	0	1/16
1	0	0	1	0	1/16
0	1	0	1	0	1/16
0	0	1	1	0	1/16
0	0	0	0	0	1/16

It is easy to check that this is PUOD and PLOD, thus it is POD. However $EX_iX_j = 4/16$ and $EX_i = 1/2$ for all i, j.

In this example we can use Theorem 3 to prove that any X_i, X_j, X_k are mutually independent since subsets of POD r.v.'s are still POD.

Newman and Wright (1981), using an inequality for the ch.f.'s of r.v.'s $X_1, \dots, X_m$, provided another proof for the characterization of the independence of associated r.v.'s. This is Theorem 1 of Newman and Wright (1981). These authors proved that if $X_1, \dots, X_m$ are associated with finite variance, joint and marginal ch.f.'s $\phi(r_1, \dots, r_m)$ and $\phi_j(r_j)$, then

$$\left|\phi(r_1, \dots, r_m) - \prod_{j=1}^{m} \phi_j(r_j)\right| \le \tfrac{1}{2} \sum_{j \ne k} |r_j||r_k| \operatorname{cov}(X_j, X_k). \tag{18}$$

To extend this inequality, we need the following lemma.

LEMMA 3. *For the r.v.* $(X_1, \dots, X_m)$ *with* $E|X_i|^m < \infty$, $m > 1$,

$$\begin{aligned} &\operatorname{cum}(\exp(ir_1X_1), \dots, \exp(ir_mX_m)) \\ &\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} i^m r_1 \cdots r_m \exp\left(i \sum_{j=1}^{m} r_j x_j\right) \\ &\qquad \times \operatorname{cum}(\chi_{X_1}(x_1), \dots, \chi_{X_m}(x_m))\, dx_1 \cdots dx_m, \end{aligned} \tag{19}$$

where $r_1, \dots, r_m$ *are real numbers and* $\chi_{X_j}(x_j) = 1$ *when* $X_j > x_j$, *and* 0 *otherwise.*

PROOF. This proof of the result is similar to that of Lemma 2. Use the identity

$$\exp(ir_kX_k) - 1 \equiv i\int_{-\infty}^{\infty} r_k\exp(ir_kx_k)\big(\varepsilon(x_k) - I_{(-\infty, x_k]}(X_k)\big)\,dx_k.$$

Notice that

$$\varepsilon(x_i) - I_{(-\infty, x_k]}(X_i) = \begin{cases} \chi_{X_k}(x_k) & \text{for } x_k \ge 0, \\ \chi_{X_k}(x_k) - 1 & \text{for } x_k < 0. \end{cases}$$

After doing the obvious calculation, we obtain by property (v) (following Definition 1) of the joint cumulant that

$$\begin{aligned}
&\operatorname{cum}\big(\exp(ir_kX_k),\ k = 1,\dots, m\big)\\
&\quad= \operatorname{cum}\big(\exp(ir_kX_k) - 1,\ k = 1,\dots, m\big)\\
&\quad= \sum(-1)^p(p-1)!\prod_{l=1}^{p}\left[E\prod_{k\in v_l}(\exp(ir_kX_k) - 1)\right]\\
&\quad= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} i^m r_1\cdots r_m\exp\left(i\sum_{j=1}^{m} r_jx_j\right)\\
&\qquad\qquad\times\left\{\sum(-1)^p(p-1)!\prod_{l=1}^{p}\left[E\left(\prod_{k\in v_l}\chi_{X_k}(x_k)\right)\right]\right\}dx_1\cdots dx_m\\
&\quad= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} i^m r_1\cdots r_m\exp\left(i\sum_{j=1}^{m} r_jx_j\right)\\
&\qquad\qquad\times\operatorname{cum}\big(\chi_{X_1}(x_1),\dots,\chi_{X_m}(x_m)\big)\,dx_1\cdots dx_m. \qquad\square
\end{aligned}$$

Using Lemma 3, we can obtain a result parallel to (18) for certain classes of r.v.'s.

THEOREM 5. *If $X_1,\dots, X_m$ are r.v.'s such that $E|X_j|^m < \infty$, $j = 1,\dots, m$, and $\operatorname{cum}(\chi_{X_{i_1}}(x_1),\dots,\chi_{X_{i_k}}(x_k))$ has the same sign for all subsets $\{i_1,\dots, i_k\}$ of $\{1,\dots, m\}$ and all $x_1,\dots, x_k$. Then*

$$\left|\phi(r_1,\dots, r_m) - \prod_{j=1}^{m}\phi_j(r_j)\right| \le \prod_{j=1}^{m}|r_j|\sum|\operatorname{cum}(X_k,\ k\in\nu_1)|\cdots|\operatorname{cum}(X_k,\ k\in\nu_p)|. \tag{20}$$

Here $\phi(r_1,\dots, r_m)$ and $\phi_j(r_j)$ are the joint and marginal ch.f.'s of $(X_1,\dots, X_m)$,

Σ *extends over all partitions* $(\nu_1,\dots,\nu_p)$, $p = 1,\dots,m-1$, *and whenever* $\mathrm{card}(\nu_l) = 1$, $|r_k|\cdot|\mathrm{cum}(X_k,\ k\in\nu_l)|$ *is replaced by* 1. *Here* $k \in \nu_l$.

PROOF. From Lemma 3, Theorem 1 and the fact that the $\mathrm{cum}(\chi_{X_1}(x_1),\dots,\chi_{X_m}(x_m))$ all have the same sign we have for $m > 1$,

$$\begin{aligned}
&|\mathrm{cum}(\exp(ir_1X_1),\dots,\exp(ir_mX_m))|\\
&\quad= \left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} i^m r_1\cdots r_m \exp\left(i\sum_{j=1}^{m} r_j x_j\right)\right.\\
&\qquad\qquad \left.\times\,\mathrm{cum}\big(\chi_{X_1}(x_1),\dots,\chi_{X_m}(x_m)\big)\,dx_1\cdots dx_m\right|\\
&\quad\le |r_1|\cdots|r_m|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}|\mathrm{cum}\big(\chi_{X_1}(x_1),\dots,\chi_{X_m}(x_m)\big)|\,dx_1\cdots dx_m\\
&\quad\le |r_1|\cdots|r_m|\left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{cum}\big(\chi_{X_1}(x_1),\dots,\chi_{X_m}(x_m)\big)\,dx_1\cdots dx_m\right|\\
&\quad= |r_1|\cdots|r_m|\,|\mathrm{cum}(X_1,\dots,X_m)|.
\end{aligned}\tag{21}$$

For $m = 1$, $\mathrm{cum}(\exp(ir_1X_1)) = E(\exp ir_1X_1) = \phi_1(r_1)$ which is bounded by 1. Combining (21) and Lemma 1, we get

$$\begin{aligned}
&\left|\phi(r_1,\dots,r_m) - \prod_{j=1}^{m}\phi_j(r_j)\right|\\
&\quad= \left|E\prod_{k=1}^{m}\exp(ir_kX_k) - \prod_{k=1}^{m}E\exp(ir_kX_k)\right|\\
&\quad= \left|\sum \mathrm{cum}\big(\exp ir_jX_j,\ j\in\nu_1\big)\cdots\mathrm{cum}\big(\exp ir_jX_j,\ j\in\nu_p\big)\right|\\
&\quad\le \sum|\mathrm{cum}(\exp ir_jX_j,\ j\in\nu_1)|\cdots|\mathrm{cum}(\exp ir_jX_j,\ j\in\nu_p)|\\
&\quad\le |r_1|\cdots|r_m|\sum|\mathrm{cum}(X_k,\ k\in\nu_1)|\cdots|\mathrm{cum}(X_k,\ k\in\nu_p)|.
\end{aligned}$$

Whenever $\mathrm{card}\,\nu_l = 1$, $|r_k|\,|\mathrm{cum}(X_k,\ k\in\nu_l)|$ is replaced by 1, $k\in\nu_l$. □

REMARK 7. In Example 3 we define r.v.'s which are uncorrelated but not mutually independent. By Corollary 1 they cannot be associated so that Theorem 1 of Newman and Wright (1981) does not apply. However, Theorem 4 gives an upper bound for the difference of ch.f.'s, since it is easy to check $\mathrm{cum}(\chi_{X_i}(x_i), \chi_{X_j}(x_j)) = 0$, $i \ne j$, and $\mathrm{cum}(\chi_{X_1}(x_1), \chi_{X_2}(x_2), \chi_{X_3}(x_3) \ge 0$ for all x_1, x_2, x_3.

Example 3. Consider the r.v.'s X_1, X_2, X_3 with the following distribution:

X_1	X_2	X_3	Pr
1	1	1	1/4
1	0	0	1/4
0	1	0	1/4
0	0	1	1/4

These are PUOD but not POD.

For nonnegative r.v.'s we can go further.

Theorem 6. *If the r.v.'s $X_1, \dots, X_m$ are nonnegative (nonpositive) and PUOD (PLOD) with finite mth moments, then*

$$\left|\phi(r_1, \dots, r_m) - \prod_{j=1}^{m} \phi_j(r_j)\right| \le |r_1| \cdots |r_m|\, |EX_1 \cdots X_m - EX_1 \cdots EX_m|. \tag{22}$$

Proof. We prove the PUOD case only. Using Lemmas 1 and 3 and Remark 3,

$$\begin{aligned}
&\left|\phi(r_1, \dots, r_m) - \prod_{j=1}^{m} \phi_j(r_j)\right| \\
&\quad = \left|E \exp\left(i \sum_{j=1}^{m} r_j X_j\right) - \prod_{j=1}^{m} E \exp(i r_j X_j)\right| \\
&\quad = \left|\int_{-\infty}^{\infty} \cdots \int_{0}^{\infty} i^m r_1 \cdots r_m \exp\left(i \sum_{j=1}^{m} r_i x_j\right)\right. \\
&\qquad\qquad \left. \times [\bar{F}(x_1, \dots, x_m) - \bar{F}_1(x_1) \cdots \bar{F}_m(x_m)]\, dx_1 \cdots dx_m \right| \qquad (23) \\
&\quad \le |r_1| \cdots |r_m| \int_{-\infty}^{\infty} \cdots \int_{0}^{\infty} |\bar{F}(x_1, \dots, x_m) \\
&\qquad\qquad - \bar{F}_1(x_1) \cdots \bar{F}_m(x_m)|\, dx_1 \cdots dx_m \\
&\quad = |r_1| \cdots |r_m| \left|\int_{-\infty}^{\infty} \cdots \int_{0}^{\infty} [\bar{F}(x_1, \dots, x_m) \right. \\
&\qquad\qquad \left. - \bar{F}_1(x_1) \cdots \bar{F}_m(X_m)]\, dx_1 \cdots dx_m \right| \\
&\quad = |r_1| \cdots |r_m|\, |EX_1 \cdots X_m - EX_1 \cdots EX_m|. \qquad \square
\end{aligned}$$

COROLLARY 2. *Under the conditions of Theorem* 5, *if* $EX_1 \cdots X_n = EX_1 \cdots EX_m$, *then* $X_1, \ldots, X_n$ *are independent.*

4. Cumulants and dependence. Cumulants provide us with useful measures of the joint statistical dependence of random variables. However, the relationships with positive and negative dependence are not similar to those in the bivariate (covariance) case. We give some examples to illustrate the relationship between the sign of the cumulant and dependence in the trivariate case.

REMARK 8. By property (iii) of cumulants if any group of X's is independent of the remaining X's, then $\mathrm{cum}(X_1, \ldots, X_r) = 0$. The converse is true for normal distributions when $r = 2$ but not for $r > 2$. For the trivariate normal, we can have $\mathrm{cum}(X_1, X_2, X_3) = 0$, where X_1, X_2, X_3 are not necessarily independent.

REMARK 9. Assume $EX_i \geq 0$ for $i = 1, 2, 3$. Also assume that $\mathrm{cov}(X_i, X_j) \geq 0$ for $i, j = 1, 2, 3$ [or the even stronger conditions $\mathrm{cov}(\chi_{X_j}(x_j), \chi_{X_j}(x_j)) \geq 0$ and $\mathrm{cum}(X_1, X_2, X_3) \geq 0$]. These do not imply PUOD as is shown in the following example.

EXAMPLE 4. Let X_1, X_2, X_3 take the values $0, \pm 1$ with: $P(X_1 = x_1, X_2 = x_2, X_3 = x_3, x_1x_2x_3 \neq 0) = 0$; $P(X_1 = X_2 = X_3 = 0) = 0$; $P(X_i = 0, X_j = x_j, X_k = x_k, x_jx_k > 0) = 1/9$, $i, j, k = 1, 2, 3$, $x_j = x_k = 1$ or $x_j = x_k = -1$; and $P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = 1/36$ for the remaining cases. It is easy to check that $EX_i = EX_1X_2X_3 = 0$, $EX_iX_j > 0$ and $\mathrm{cum}(X_1, X_2, X_3) = 0$ but

$$P(X_1 > 0, X_2 > 0, X_3 > 0) - P(X_1 > 0)P(X_2 > 0)P(X_3 > 0)$$
$$= -(11/36)^3 < 0.$$

REMARK 10. Let $EX_i \geq 0$ and assume (X_1, X_2, X_3) PUOD. This does not imply $\mathrm{cum}(X_1, X_2, X_3) \geq 0$ as is shown in Example 5.

EXAMPLE 5. Let (X_1, X_2, X_3) have the following distribution. It is easy to check that (X_1, X_2, X_3) is PUOD and that $EX_i = 0$, but $\mathrm{cum}(X_1, X_2, X_3) = -0.15 < 0$.

X_1	X_2	X_3	Pr
1	1	1	0.35
1	1	−1	0.05
1	−1	1	0.05
−1	1	1	0.05
0	0	−1	0.05
−1	0	0	0.05
0	−1	0	0.05
−1	−1	−1	0.35

REMARK 11. Let (X_1, X_2, X_3) be associated. It need not be true that $\text{cum}(X_1, X_2, X_3) \geq 0$ as is shown in Example 6.

EXAMPLE 6. Assume (X_1, X_2, X_3) are binary r.v.'s with distribution $P(X_1 = X_2 = X_3 = 0) = 0.3$; $P(X_1 = x_1,\ X_2 = x_2,\ X_3 = x_3) = 0.1$ for all other $\{x_1, x_2, x_3\} \in \{0,1\}^3$.

Checking all binary nondecreasing functions $\Gamma(X_1, X_2, X_3)$ and $\Delta(X_1, X_2, X_3)$ we have $\text{cov}(\Gamma, \Delta) \geq 0$. Thus (X_1, X_2, X_3) are associated but $\text{cum}(X_1, X_2, X_3) = -0.012 < 0$.

REMARK 12. If (X, Y) are binary and $\text{cov}(X, Y) \geq 0$, then (X, Y) is associated as was shown in Barlow and Proschan (1981). However, if (X_1, X_2, X_3) are binary, then $\text{cov}(X_i, X_j) \geq 0$, $i, j = 1,2,3$, and $\text{cum}(X_1, X_2, X_3) \geq 0$ do not imply (X_1, X_2, X_3) associated as is seen in Example 7.

EXAMPLE 7. Assume (X_1, X_2, X_3) are binary r.v.'s with the following distribution. Then $\text{cov}(X_i, X_j) = 1/180 > 0$ and $\text{cum}(X_1, X_2, X_3) = 1/135 > 0$. However, for the increasing functions $\max(X_1, X_2)$ and $\max(X_1, X_3)$,

$$\text{cov}(\max(X_1, X_2), \max(X_1, X_3)) = -1/900 < 0,$$

so (X_1, X_2, X_3) are not associated.

X_1	X_2	X_3	Pr
0	0	0	0
0	0	1	1/30
0	1	0	1/30
1	0	0	1/30
1	1	0	1/10
1	0	1	1/10
0	1	1	1/10
1	1	1	6/10

If we add some restrictions, some results can be obtained. We will give these and omit the easy proofs.

PROPOSITION 1. *If* $\text{cov}(X_i, X_j) = Q$ *for* $i, j = 1,2,3$, *then* (X_1, X_2, X_3) *PUOD implies* $\text{cum}(X_1, X_2, X_3) \geq 0$ *and* (X_1, X_2, X_3) *PLOD implies* $\text{cum}(X_1, X_2, X_3) \leq 0$.

REMARK 13. Notice that under the preceding assumptions we have the peculiar situation that PUOD $\Leftrightarrow$ NLOD and PLOD $\Leftrightarrow$ NUOD.

PROPOSITION 2. *Let* (X_1, X_2, X_3) *be a binary trivariate r.v. If* $\operatorname{cov}(X_i, X_j) \geq 0$, $\operatorname{cum}(X_1, X_2, X_3) > 0$ *and additionally condition* (M) *holds, then* (X_1, X_2, X_3) *is associated for* $i, j, k = 1, 2, 3$.

$$(\mathrm{M}) \qquad \begin{cases} \operatorname{cov}\left(X_i \amalg X_j X_k, X_j \amalg X_k\right) \geq 0, \\ \operatorname{cov}\left(X_i \amalg X_j, X_i \amalg X_k\right) \geq 0, \end{cases}$$

where

$$X_i \amalg X_j = 1 - (1 - X_i)(1 - X_j) = \max(X_i, X_j).$$

To prove Proposition 2, we need to check for all binary increasing functions Γ and Δ that $\operatorname{cov}(\Gamma(X_1, X_2, X_3), \Delta(X_1, X_2, X_3)) \geq 0$. We leave this to the reader.

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