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STOCHASTIC PROCESSES WITH VALUE IN EXPONENTIAL TYPE ORLICZ SPACES

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Let (T, Θ) be a compact measurable topological space and $\Psi_q(x) = \exp|x|^q - 1$, $1 \le q < \infty$. Let $X = \{X(\omega, t), \omega \in \Omega, t \in T\}$ be a Θ-measurable stochastic process such that $\|X(s) - X(t)\|_{L^{\Psi_q(\Omega)}} \le d(s, t)$ for every $(s, t) \in T \otimes T$, where $d(\cdot, \cdot)$ is some continuous pseudometric on (T, Θ). We give a sufficient condition expressed in terms of a majorizing measure on (T, d) in order that X take values in the Orlicz space $L^{\Psi_{q'}}(T, \mu)$, where $q \le q' < \infty$ and μ any Borel probability measure on (T, Θ).

1. Introduction and main result. In a recent paper [2], Marcus and Pisier have considered measurable stochastic processes having strongly integrable sample paths. Let (T, Θ) be a compact measurable space and $\Psi_q(x) = \exp|x|^q - 1$, $q \ge 1$. Let μ be any Borel probability on (T, Θ) and introduce the Orlicz space

$$L^{\Psi_q}(T, \mu) = \left\{ f\colon T \to C\colon \exists\, c > 0\colon \int_T \Psi_q\left[\frac{f(t)}{c}\right] d\mu(t) < \infty \right\},$$

and its Orlicz norm,

$$\|f\|_{L^{\Psi_q(T, \mu)}} = \inf\left\{ c > 0\colon \int_T \Psi_q\left[\frac{f(t)}{c}\right] d\mu(t) \le 1 \right\}.$$

Let $d(\cdot, \cdot)$ be a Θ-continuous pseudometric on $T \otimes T$ and consider any Θ-measurable stochastic process $X = \{X(\omega, t), \omega \in \Omega, t \in T\}$ such that

$$\forall\, (s, t) \in T \otimes T, \qquad \|X(s) - X(t)\|_{L^{\Psi_q(T,\mu)}} \le d(s, t). \tag{1.1}$$

Let $q \le q^* \le \infty$ and suppose

$$J_{q, q'}(T, d) = \int_0^{\operatorname{diam}(T, d)} [\log N_d(T, u)]^{1/q - 1/q'}\, du < \infty, \tag{1.2}$$

where as usual, $N_d(T, u)$ denotes the minimal number of d-balls of radius u enough to cover T.

In [2], the authors show that (1.2) implies that X takes value in $L^{\Psi_{q'}}(T, \mu)$, almost surely, for every Borel probability measure μ. We refer the reader to [2] for the proof and other interesting results. Our purpose in this work is to state a sufficient condition similar to (1.2), expressed in terms of majorizing measures on (T, d). This will necessitate a quite different approach than in [2]. Our result can be stated as follows.

THEOREM 1.1. *Let (T, Θ) be a compact measurable space and X a real valued measurable stochastic process $\{X(\omega, t), \omega \subset \Omega, t \subset T\}$ satisfying the*

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condition (1.1) *for some* $1 \le q < \infty$ *and some continuous pseudometric* $d(\cdot,\cdot)$ *on* T. *Assume that*

there exists a Borel probability measure μ *on* (T,Θ) *such that*

$$J_{q,q'}(T,d,\mu) = \sup_{t\in T}\left\{\int_0^{\operatorname{diam}(T,d)}\left[\log\left(1+\frac{1}{\mu\{s: d(s,t)\le u\}}\right)\right]^{1/q-1/q'} du\right\} < \infty, \tag{1.3}$$

for some $q \le q' < \infty$ *and*

$$\int_0^{\operatorname{diam}(T,d)} \frac{\mu\otimes\mu\{(s,t)\in T\otimes T: 0<d(s,t)\le u\}}{u}\,du < \infty. \tag{1.4}$$

Then the process X *has sample paths in* $L^{\Psi_{q'}}(T,v)$ *almost surely, for every Borel probability measure* v *on* T.

In particular, when (T,d) *is a compact ultrametric space, that is, when*

$$\forall\, s,t,u\in T,\qquad d(s,t)\le \sup\{d(s,u), d(u,t)\},$$

the same conclusion holds, without assumption (1.4).

2. Preparation. For each $t\in T$ and each $\varepsilon>0$, we denote $B_d(t,\varepsilon) = \{s\in T: d(s,t)\le\varepsilon\}$. Since (T,Θ) is compact, there is a compact subset

$$K = \overline{\{s\in T: \exists\,\varepsilon>0: \mu\{B_d(s,\varepsilon)\}=0\}},$$

such that $\mu(K)=0$. Thus, there is no loss when assuming $\mu\{B_d(t,\varepsilon)\}>0$, for all $t\in T$ and all $\varepsilon>0$. Let $\{\varepsilon_n, n\ge 1\}$ be a sequence decreasing to zero and let S_n be a subset of T satisfying

$$\bigcup_{s\in S_n} B_d(s,\varepsilon_n) = T,\qquad n\ge 1. \tag{2.1}$$

For every $n\ge1$, let $\Pi_n=\{\pi_n(s), s\in S_n\}$ be the induced partition of T. Let also $X=\{X(\omega,t),\omega\in\Omega, t\in T\}$ be any Θ-measurable stochastic process satisfying

$$\forall\,(s,t)\in T\otimes T,\qquad E|X(s)-X(t)| = \delta(s,t)<\infty. \tag{2.2}$$

We consider two types of approximation. The first one is connected with the sequence $\{\Pi_n, n\ge1\}$ and gives a step process whose sample paths are therefore in any Orlicz space $L^{\Psi_{q'}}(T,v)$ almost surely:

$$\forall\, t\in T, \forall\, n\ge 1,\qquad X_n^{(1)}(t) = \sum_{s\in S_n} I_{\pi_n(s)}(t)\int_{B_d(s,\varepsilon_n)} X(u)\frac{\mu(du)}{\mu_n(s)}, \tag{2.3}$$

where for simplicity we note $\mu_n(s)=\mu\{B_d(s,\varepsilon_n)\}$.

The second approximation is needed to obtain a majorizing measure type condition:

$$\forall\, t\in T, \forall\, n\ge 1,\qquad X_n^{(2)}(t) = \int_{B_d(t,\varepsilon_n)} X(u)\frac{\mu(du)}{\mu_n(t)}. \tag{2.4}$$

In the sequel, we denote $X_n^{(1)}(t)$ and $X_n^{(2)}(t)$ by $X_n(t)$, except when it is necessary to distinguish them. The following lemma is very classical.

LEMMA 2.1. *Assume that the identity map* $i\colon (T,d) \to (T,\delta)$ *is uniformly continuous. Then, if the sequence* $\{\varepsilon_n, n \ge 1\}$ *decreases sufficiently fast to zero, one has*

$$\forall t \in T, \qquad P\Big\{\lim_{n\to\infty} X_n(t) = X(t)\Big\} = 1. \tag{2.5}$$

PROOF. By assumption, $\Delta(\varepsilon) = \sup\{\delta(s,t)\colon d(s,t) \le 2\varepsilon\}$ tends to zero with ε, so that we can choose a sequence $\{\varepsilon_n, n \ge 1\}$ such that $\sum_{n\ge 1}\sqrt{\Delta(\varepsilon_n)} < \infty$. Further,

$$P\Big\{|X(t) - X_n(t)| > \sqrt{\Delta(\varepsilon_n)}\Big\} \le \sqrt{\Delta(\varepsilon_n)}$$

by applying the Tchebycheff inequality. The proof is achieved by applying the Borel–Cantelli lemma. □

Consider now a sequence of functions $b_n\colon T \to R^+$ such that

$$\sup_{t\in T}\left\{\sum_{n=1}^{\infty} b_n(t)\right\} < \infty, \tag{2.6}$$

and put $R_N(t) = \sum_{n=N}^{\infty} b_n(t)$, $R_N = \sup\{R_N(t), t \in T\}$. Let $\{A_n, n \ge 1\}$ be a sequence of events such that $P\{\bigcap_{n\ge1}(A_n)^c\} > 0$ and set $\Omega_1 = \bigcap_{n\ge1}(A_n)^c$. Let $\varphi\colon R \to R^+$ be any convex nondecreasing function. One easily has

$$I_{\Omega_1}\varphi\left[\frac{X(t) - X_0(t)}{R_1(t)}\right] \le \sum_{n\ge1}\frac{b_n(t)}{R_1(t)}\varphi\left[\frac{X_n(t) - X_{n-1}(t)}{b_n(t)}\right] I_{(A_n)^c}. \tag{2.7}$$

Integrating first with respect to dP, then with respect to any Borel probability measure v on (T,Θ), one obtains

$$\begin{aligned} &E\left\{I_{\Omega_1}\int_T \varphi\left[\frac{X(t) - X_0(t)}{R_1}\right] v(dt)\right\} \\ &\quad\le \sum_{n\ge1} E\left\{I_{(A_n)^c}\int_T \varphi\left[\frac{X_n(t) - X_{n-1}(t)}{b_n(t)}\right]\frac{b_n(t)}{R_1(t)} v(dt)\right\}, \\ &\quad= B_1. \end{aligned} \tag{2.8}$$

Thus, if $B_1 < \infty$ and $X_0 = X_0^{(1)}$,

$$\Omega_1 \subset \{\omega\colon X(\omega,\cdot) \in L^{\varphi}(T,v)\}. \tag{2.9}$$

We are therefore in a position to state

LEMMA 2.2. *Let* (T,Θ) *be a compact measurable space and* X *a* Θ*-measurable stochastic process* $\{X(\omega,t), \omega \in \Omega, t \in T\}$ *satisfying* (2.2). *Assume that the identity map* $i\colon (T,d) \to (T,\delta)$ *is uniformly continuous and let* $\{\varepsilon_n, n \ge 1\}$ *be a sequence decreasing to zero such that the conclusion of Lemma* 2.1 *holds.*

Suppose further that there exist a sequence of functions $b_n: T \to R^+$, $n \ge 1$, a convex nondecreasing function $\varphi: R \to R^+$ and a sequence $\{A_n, n \ge 1\}$ of events satisfying (with $X_0 = X_0^{(1)}$)

$$\sup\Big\{ \sum_{n\ge 1} b_n(t) \Big\} < \infty, \tag{2.10}$$

$$P\Big\{ \bigcap_{n\ge 1} (A_n)^c \Big\} \ge \rho > 0, \tag{2.11}$$

$$\sum_{n\ge 1} E\left\{ I_{(A_n)^c} \int_T \varphi\left[\frac{X_n(t) - X_{n-1}(t)}{b_n(t)} \right] \frac{b_n(t)}{R_1(t)} v(dt) \right\} < \infty, \tag{2.12}$$

for some Borel probability measure v on T. Then, with probability greater than ρ, the sample paths of X belong to $L^{\varphi}(T, v)$.

3. Proof of Theorem 1.1. Since we assume (1.1), we can choose $\varepsilon_n = 2^{-n}\operatorname{diam}(T, d)$, $n \ge 0$, in order that (2.5) holds. Let v be any Borel probability on (T, Θ) and let $N \ge 1$ be *fixed*. Set

$$X_N(t) = X_N^{(1)}(t) \quad \text{and} \quad \mu_N(t) = \mu_N(s) \quad \text{if } t \in \pi_N(s),$$

and for all $n > N$,

$$X_n(t) := X_n^{(2)}(t),$$

$$b_n(t) = 3\left[\log\left(1 + \frac{1}{\mu_n(t)\mu_{n-1}(t)} \right) \right]^{1/q - 1/q'} \varepsilon_{n-1},$$

$$k_n(t) = \left[\frac{X_n(t) - X_{n-1}(t)}{b_n(t)} \right]^{(q'/q)-1}, \tag{3.1}$$

$$k_n^*(t) = \left[\log\left(1 + \frac{1}{\mu_n(t)\mu_{n-1}(t)} \right) \right]^{1/q - 1/q'},$$

$$A_n = \{ \exists\, t \in T: k_n(t) > k_n^*(t) \}.$$

We have

$$\begin{aligned} &I_{(A_n)^c} \int_T \Psi_{q'}\left[\frac{X_n(t) - X_{n-1}(t)}{b_n(t)} \right] \frac{b_n(t)}{R_N(t)} v(dt) \\ &\le I_{(A_n)^c} \int_T \Psi_q\left[\frac{X_n(t) - X_{n-1}(t)}{b_n(t)} k_n(t) \right] \frac{b_n(t)}{R_N(t)} v(dt), \\ &\le \int_T \Psi_q\left[\frac{X_n(t) - X_{n-1}(t)}{b_n(t)} k_n^*(t) \right] \frac{b_n(t)}{R_N(t)} v(dt). \end{aligned} \tag{3.2}$$

By integrating with respect to dP, then using Jensen's inequality,

$$E\left\{\Psi_q\left[\frac{X_n(t)-X_{n-1}(t)}{b_n(t)}k_n^*(t)\right]\right\}$$

$$\le E\left\{\iint_{B_n(t)\otimes B_{n-1}(t)}\Psi_q\left[\frac{X(u)-X(v)}{b_n(t)}k_n^*(t)\right]\frac{\mu(du)\mu(dv)}{\mu_n(t)\mu_{n-1}(t)}\right\},$$

where we write $B_N(t)=B(s,\varepsilon_N)$ if $t\in\pi_N(s)$ and $B_n(t)=B(t,\varepsilon_n)$ if $n>N$,

$$\le 1,$$

once

$$\sup\left\{\frac{d(u,v)k_n^*(t)}{b_n(t)},\ u\in B_n(t),\ v\in B_{n-1}(t)\right\}\le 1.$$

But, by (3.1) this quantity is less than $3\varepsilon_{n-1}k_n^*(t)[b_n(t)]^{-1}\le 1$. Therefore, for every $n\ge N$,

$$E\left\{I_{(A_n)^c}\int_T\Psi_{q'}\left[\frac{X_n(t)-X_{n-1}(t)}{b_n(t)}\right]\frac{b_n(t)}{R_N(t)}v(dt)\right\}\le\int\frac{b_n(t)}{R_N(t)}v(dt),$$

and thus,

$$\sum_{n=N}^{\infty}E\left\{I_{(A_n)^c}\int_T\Psi_{q'}\left[\frac{X_n(t)-X_{n-1}(t)}{b_n(t)}\right]\frac{b_n(t)}{R_N(t)}v(dt)\right\}\le 1. \tag{3.3}$$

Further,

$$\sup\left\{\sum_{n=N}^{\infty}b_n(t),\ t\in T\right\}$$

$$\le\sup\{b_N(t),\ t\in T\} \tag{3.4}$$

$$+O(1)\sup\left\{\int_0^{\varepsilon_N}\left[\log\left(1+\frac{1}{\mu\{B_d(t,u)\}}\right)\right]^{1/q-1/q'}\mu(du),\ t\in T\right\},$$

which is finite by (1.3). We now turn to the control of the sequence $\{A_n,\ n\ge 1\}$. First observe

$$\forall n\ge N,$$

$$A_n\subset\left\{\exists\, t\in T:\ |X_n(t)-X_{n-1}(t)|\ \ge 3\varepsilon_{n-1}\left[\log\left(1+\frac{1}{\mu_n(t)\mu_{n-1}(t)}\right)\right]^{1/q}\right\} \tag{3.5}$$

and by applying Jensen's inequality,

$$(3.6)\qquad \begin{aligned}&|X_n(t) - X_{n-1}(t)|\\ &\le 3\varepsilon_{n-1}[\Psi_q]^{-1}\left\{\iint_{\substack{B_n(t)\otimes B_{n-1}(t)\\ d(u,v)\neq 0}} \Psi_q\left[\frac{X(u)-X(v)}{d(u,v)}\right]\frac{\mu(du)\mu(dv)}{\mu_n(t)\mu_{n-1}(t)}\right\}.\end{aligned}$$

Therefore, for every $n \ge N$,

$$(3.7)\qquad \begin{aligned}A_n &\subset \left\{\exists t \in T: \iint_{\substack{B_n(t)\otimes B_{n-1}(t)\\ d(u,v)\neq 0}} \Psi_q\left[\frac{X(u)-X(v)}{d(u,v)}\right]\mu(du)\mu(dv) \ge 1\right\}\\ &\subset \left\{\iint_{0<d(u,v)\le 3\varepsilon_{n-1}} \Psi_q\left[\frac{X(u)-X(v)}{d(u,v)}\right]\mu(du)\mu(dv) \ge 1\right\},\end{aligned}$$

since $d(u, v) \le 3\varepsilon_{n-1}$ when $(u, v) \in B_n(t) \otimes B_{n-1}(t)$. Further, by applying Tchebycheff's inequality, one obtains,

$$(3.8)\qquad \forall\, n \ge N, \qquad P\{A_n\} \le \mu \otimes \mu\{(u, v) \in T \otimes T: 0 < d(u, v) \le 3\varepsilon_{n-1}\}.$$

We finally obtain, by letting $\Omega_N = \bigcap_{n\ge N}(A_n)^c$ and using assumption (1.4),

$$(3.9)\qquad \lim_{N\to\infty} P\{\Omega_N\} = 1.$$

The proof is achieved by applying Lemma 2.2.

When (T, d) is a compact ultrametric space, the two sequences of approximation described in (2.2) and (2.3) are identical, since $\pi(s) = B(t, \varepsilon_n)$ for every $t \in \pi_n(s)$, $s \in S_n$ and $n \ge 1$. The same proof, with the modifications

$$(3.10)\qquad \begin{aligned}\tilde{b}_n(t) &= 3\varepsilon_{n-1}\left[\log\left(1 + \frac{2^n}{\mu_n(t)\mu_{n-1}(t)}\right)\right]^{1/q-1/q'},\\ \tilde{k}_n^*(t) &= \left[\log\left(1 + \frac{2^n}{\mu_n(t)\mu_{n-1}(t)}\right)\right]^{1/q-1/q'},\end{aligned}$$

leads to (3.3) and (3.4), and for every $n \ge N$,

$$(3.11)\qquad A_n \subset \left\{\exists\, t \in T: \iint_{B_n(t)\otimes B_{n-1}(t)} \Psi_q\left[\frac{X(u)-X(v)}{d(u,v)}\right]\mu(du)\mu(dv) \ge 2^n\right\},$$

so that

$$(3.12)\qquad P\{A_n\} \le 2^{-n} \sum_{\substack{s\in S_n\\ s'\in S_{n-1}}} \mu\{\pi_n(s)\}\mu\{\pi_{n-1}(s')\} \le 2^{-n},$$

which easily implies (3.9). □

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