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HITTING DISTRIBUTIONS OF SMALL GEODESIC SPHERES

BY MING LIAO

Nankai University

Let M be an n-dimensional Riemannian manifold, $m \in M$ and T be the hitting time of an r-sphere around m by Brownian motion X_t. We have, for any smooth function g on the unit sphere S, under normal coordinates, $E^m[g(X_T/r)] = Ig + r^2 I(v_2 g) + r^3 I(v_3 g) + O(r^4)$ and $E^m[Tg(X_T/r)] = E^m[T]E^m[g(X_T/r)] + r^5 c \sum_i \partial_i s I(z_i g) + O(r^6)$, where I is the uniform probability distribution on S, v_2 and v_3 are smooth functions on S whose expressions involve scalar curvature, Ricci curvature and their derivatives at m, c is a constant and s is the scalar curvature. $v_2 = 0$ if and only if either $n = 2$ or M is an Einstein manifold.

1. Introduction and main results. Let M be an n-dimensional Riemannian manifold and $m \in M$. Consider Brownian motion X_t on M, whose infinitesimal generator is the usual Laplace–Beltrami operator Δ. Let S_r be the geodesic sphere of radius r around m and T_r be the first hitting time of S_r by X_t starting from m. The hitting distribution of S_r, which is just the harmonic measure of S_r (with respect to the center m), is given by $E^m[f(X(T_r))]$ for any smooth function f on S_r. This induces a probability measure H_r on the unit sphere S in M_m which is identified with R^n by the exponential map $\exp_m$. To be precise, put $g(x) = f(\exp_m(rx))$, then $H_r g = E^m[f(X(T_r))]$.

To simplify the notation, we fix a normal coordinate system $(x_1, x_2, \dots, x_n)$ around m throughout. Identify S with $\{x: \sum_{i=1}^n x_i^2 = 1\}$ and m with 0. Then for any smooth function g on S,

$$H_r g = E^m\left[g\left(X(T_r)/r\right)\right]. \tag{1}$$

If $M = R^n$, then $H_r = I$, the uniform probability distribution on S. For any two-dimensional Riemannian manifold M, Pinsky [7] obtained the expansion

$$H_r g = Ig + r^3 I(v_3 g) + O(r^4),$$

where v_3 is a smooth function on S whose explicit expression involves the derivatives of Gauss curvature. $v_3 = 0$ if and only if M is a surface of constant curvature.

In the present paper, we extend the above result to higher dimension. We will see that, in general, an r^2-term appears in the expansion and this term vanishes if and only if either $n = 2$ or M is an Einstein manifold.

To state our results, let R_{ijkl}, R_{jk}, s and $\partial_h R_{ijkl}$, $\partial_h R_{jk}$, $\partial_h s$ be curvature tensor, Ricci curvature tensor, scalar curvature and their derivatives, all evaluated at m. We will adopt the convention to omit the summation sign over repeated indices, e.g., $R_{jk} = R_{jhkh}$ and $s = R_{hh}$.

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For any nonnegative integer k, let $C^k(S)$ be the class of functions on S having continuous k-order derivatives. By choosing local coordinate systems on S, we can define $|g|_k$ for any $g \in C^k(S)$ by

$$|g|_k = \sum_{j=0}^{k} \sum_{i_1, i_2, \dots, i_j} \sup_{z \in S} |\partial_{i_1} \partial_{i_2} \cdots \partial_{i_j} g(z)|.$$

For any open subset G of R^n, we can similarly define $C^k(G)$ and $|f|_k$ for $f \in C^k(G)$.

THEOREM 1. *For $g \in C^2(S)$, we have*

$$E^m[g(X(T_r)/r)] = I + r^2 I(v_2 g) + r^3 I(v_3 g) + O(r^4), \tag{2}$$

where v_2 and v_3 are functions on S defined by, for $z \in S$,

$$v_2(z) = (1/12n)\left[s - nR_{jk} z_j z_k\right], \tag{3}$$

$$v_3(z) = (1/24(n+2))\left[\partial_i s\, z_i - (n+2)\,\partial_i R_{jk}\, z_i z_j z_k\right]. \tag{4}$$

Moreover, $O(r^4)$ in (2) is actually dominated by $Kr^4|g|_2$ for some constant K independent of g.

A necessary and sufficient condition for v_2 to vanish is $R_{jk} = c\delta_{jk}$ for some constant c which may depend on m. If $n = 2$, this condition is automatically satisfied. For $n > 2$, this condition satisfied at all $m \in M$ implies that c is constant on M, so M is an Einstein manifold.

COROLLARY. *$v_2 = 0$ if and only if either $n = 2$ or M is an Einstein manifold.*

Theorem 1 is related to the theory of harmonic manifolds. M is said to be harmonic if $H_r = I$ for all $m \in M$ and all sufficiently small $r > 0$, see [8]. Kozaki and Ogura proved in [4] that on a harmonic manifold, T_r and $X(T_r)$ are independent. See also [5] for a different proof. So it is an interesting problem to characterize those manifolds on which T_r and $X(T_r)$ are independent. It was also proved in [4] that if M has this independence property, then it must have constant scalar curvature. In this paper, we will compute the first term in the expansion of

$$E^m[T_r g(X(T_r)/r)] - E^m[T_r]E^m[g(X(T_r)/r)], \quad \text{for } g \in C^2(S).$$

We can obtain the above result by letting this term be zero.

THEOREM 2. *For $g \in C^2(S)$, we have, with $c = (1/24(n+2)^2(n+4))$,*

$$\begin{aligned} E^m[T_r g(X(T_r)/r)] &= E^m[T_r]E^m[g(X(T_r)/r)] \\ &\quad + r^5 c\, \partial_i s\, I(z_i g) + O(r^6) \end{aligned} \tag{5}$$

and for any $b > 0$,

$$\begin{aligned} E^m[\exp(-bT_r) g(X(T_r)/r)] &= E^m[\exp(-bT_r)]\, E^m[g(X(T_r)/r)] \\ &\quad - br^5 c\, \partial_i s\, I(z_i g) + O(r^6). \end{aligned} \tag{6}$$

Moreover, $O(r^6)$ in (5) and (6) is actually dominated by $Kr^6|g|_2$ for some constant K independent of g.

COROLLARY. *If T_r and $X(T_r)$ are independent for all $m \in M$ and all sufficiently small $r > 0$, then s is constant on M.*

2. Perturbation method. By [2], for any smooth function f defined in a neighborhood of m, we have

$$\Delta f = \Delta_{-2} f + \sum_{j=0}^{\infty} \Delta_j f, \tag{7}$$

where each Δ_j is a second-order linear differential operator which maps a polynomial of degree k into one of degree $k + j$. The first few Δ_j are

$$\Delta_{-2} f = \partial_h \partial_h f, \tag{8}$$

$$\Delta_0 f = \tfrac{1}{3} R_{jakb} x_a x_b \partial_j \partial_k f - \tfrac{2}{3} R_{ia} x_a \partial_i f, \tag{9}$$

$$\Delta_1 f = \tfrac{1}{6} \partial_a R_{jbkc} x_a x_b x_c \partial_j \partial_k f + \tfrac{1}{12}[\partial_i R_{ab} - 6 \partial_a R_{ib}] x_a x_b \partial_i f. \tag{10}$$

Let

$$\Delta'_k = \sum_{j=k}^{\infty} \Delta_j. \tag{11}$$

This is a differential operator of the form $a_{jk}(x)\partial_j \partial_k + b_i(x)\partial_i$ with $a_{jk} = O(r^{k+2})$, $b_i = O(r^{k+1})$.

Let $D = \{x\colon \sum x_i^2 < 1\}$. As in [7], for fixed $g \in C^2(S)$, let u, u_0 and u_1 be defined by

$$\Delta_{-2} u = 0, \quad \text{in } D \text{ and } u = g \text{ on } S, \tag{12}$$

$$\Delta_{-2} u_0 + \Delta_0 u = 0, \quad \text{in } D \text{ and } u_0 = 0 \text{ on } S, \tag{13}$$

$$\Delta_{-2} u_1 + \Delta_1 u = 0, \quad \text{in } D \text{ and } u_1 = 0 \text{ on } S. \tag{14}$$

For simplicity, we will write T for T_r.

LEMMA 1. *We have*

$$E^m[g(X_T/r)] = u(0) + r^2 u_0(0) + r^3 u_1(0) + O(r^4). \tag{15}$$

Moreover, $O(r^4)$ above is dominated by $Kr^4|g|_2$ for some constant K independent of g.

PROOF. Set $U(x) = u(x/r)$, $U_0(x) = u_0(x/r)$ and $U_1(x) = u_1(x/r)$. By Dynkin's formula and (12),

$$\begin{aligned} E^m[g(X_T/r)] &= E^m[U(X_T)] = U(0) + E^m\left[\int_0^T \Delta U(X_t)\,dt\right] \\ &= u(0) + E^m\left[\int_0^T (\Delta_0 u)(X_t/r)\,dt\right] \\ &\quad + rE^m\left[\int_0^T (\Delta_1 u)(X_t/r)\,dt\right] + R_1, \end{aligned}$$

where $R_1 = E^m[\int_0^T \Delta'_2 U(X_t)\,dt]$.

By (13) and (14),

$$\begin{aligned} E^m[g(X_T/r)] &= u(0) - r^2E^m\left[\int_0^T \Delta_{-2}U_0(X_t)\,dt\right] \\ &\quad - r^3E^m\left[\int_0^T \Delta_{-2}U_1(X_t)\,dt\right] + R_1 \\ &= u(0) - r^2E^m\left[\int_0^T \Delta U_0(X_t)\,dt\right] - r^3E^m\left[\int_0^T \Delta U_1(X_t)\,dt\right] \\ &\quad + R_1 + R_2 + R_3 \\ &= u(0) + r^2u_0(0) + r^3u_1(0) + R_1 + R_2 + R_3, \end{aligned}$$

where $R_2 = r^2E^m[\int_0^T \Delta'_0U_0(X_t)\,dt]$ and $R_3 = r^3E^m[\int_0^T \Delta'_0U_1(X_t)\,dt]$.

Now $\Delta'_2U(x) = r^2(\Delta'u)(x/r)$, where Δ' is a second-order linear differential operator with bounded coefficients. By Schauder's estimate [1],

$$|\Delta'_2U|_0 \le r^2|\Delta'u|_0 \le K_1r^2|g|_2, \quad \text{for some constant } K_1.$$

By [2], we have

$$E^m[T] = r^2(1/2n) + r^4(1/12n^2(n+2))s + O(r^6). \tag{16}$$

It follows from above that $|R_1| \le Kr^4|g|_2$ for some constant K. Similarly, we can show that R_2 and R_3 are also dominated by $Kr^4|g|_2$ when K is sufficiently large. □

By (12), $u(0) = Ig$, so in order to prove Theorem 1 it remains to show

$$u_0(0) = I(v_2g) \quad \text{and} \quad u_1(0) = I(v_3g). \tag{17}$$

3. Poisson equation. Let $C_r = \{x: \sum_{i=1}^n x_i^2 = r^2\}$, and $D_r = \{x: \sum_{i=1}^n x_i^2 < r^2\}$ and I_r be the uniform probability distribution on C_r. Let a be the surface area of $S = C_1$ and dv be the volume element of R^n.

LEMMA 2. *Suppose f is a smooth function and satisfies $\Delta_{-2}^{k+1}f = 0$ in D_r for some nonnegative integer k. Then*

$$I_rf = f(0) + \sum_{h=1}^{k} (1/2^hh!\,n(n+2)\cdots(n+2h-2))\Delta_{-2}^hf(0)r^{2h}. \tag{18}$$

Here, by convention, $\sum_{h=1}^0 a_h = 0$.

PROOF. The assertion is clearly true for $k = 0$. Suppose it is proved for some k and assume $\Delta_{-2}^{k+2}f = 0$. Replace f by $\Delta_{-2}f$ in (18) and then multiply both sides by $ar^{n-1}\,dr$ and integrate to obtain

$$\begin{aligned} \int_{D_r} \Delta_{-2}f\,dv &= ar^n(1/n)\Delta_{-2}f(0) \\ &\quad + \sum_{h=1}^{k} ar^{2h+n}(1/2^hh!\,n(n+2)\cdots(n+2h))\,\Delta_{-2}^{h+1}f(0). \end{aligned}$$

By the divergence theorem,

$$\int_{D_r} \Delta_{-2}f\,dv = ar^{n-1}\int_{C_r} \partial_rf\,dI_r = ar^{n-1}\partial_r\int_{C_r} f\,dI_r,$$

so

$$I_r f = f(0) + r^2(1/2n)\,\Delta_{-2} f(0) + \sum_{h=1}^{k} r^{2h+2}\big(1/2^{h+1}(h+1)!\,n(n+2)\cdots(n+2h)\big)\,\Delta_{-2}^{h+1} f(0). \qquad \square$$

COROLLARY. *Suppose p satisfies $\Delta_{-2}^k p = 0$ in $D = D_1$ for some integer $k > 0$. If f is the solution of the Poisson equation*

$$\Delta_{-2} f + p = 0, \quad \text{in } D \text{ and } f = 0 \text{ on } S, \tag{19}$$

then

$$f(0) = \sum_{h=1}^{k} \big(1/2^h h!\,n(n+2)\cdots(n+2h-2)\big)\,\Delta_{-2}^{h-1} p(0). \tag{20}$$

By convention, $\Delta_{-2}^0 p = p$.

4. Some elementary computations. In this section, we establish the following formulas. Let f be a smooth function. Then

$$\Delta_{-2}\Delta_0 f = -\tfrac{2}{3}R_{jk}\partial_j\partial_k f + \Delta_0\Delta_{-2} f, \tag{21}$$

$$\Delta_{-2}^2\Delta_0 f = -\tfrac{4}{3}R_{jk}\partial_j\partial_k\Delta_{-2} f + \Delta_0\Delta_{-2}^2 f, \tag{22}$$

$$\begin{aligned}\Delta_{-2}\Delta_1 f = &-\tfrac{4}{3}\,\partial_j R_{ak}x_a\partial_j\partial_k f + \tfrac{1}{3}\,\partial_i R_{jakb}x_a x_b\partial_i\partial_j\partial_k f\\ &-\tfrac{1}{3}\partial_i s\,\partial_i f + \Delta_1\Delta_{-2} f,\end{aligned} \tag{23}$$

$$\begin{aligned}\Delta_{-2}^2\Delta_1 f = &-2\partial_i R_{jk}\partial_i\partial_j\partial_k f - \tfrac{8}{3}\,\partial_j R_{ak}x_a\partial_j\partial_k\Delta_{-2} f\\ &+\tfrac{2}{3}\,\partial_i R_{jakb}x_a x_b\partial_i\partial_j\partial_k\Delta_{-2} f - \tfrac{2}{3}\,\partial_i s\,\partial_i\Delta_{-2} f + \Delta_1\Delta_{-2}^2 f.\end{aligned} \tag{24}$$

If $\Delta_{-2}^2 f = 0$, then

$$\Delta_{-2}^3\Delta_1 f = -6\,\partial_i R_{jk}\,\partial_i\partial_j\partial_k\Delta_{-2} f. \tag{25}$$

Observe that $\Delta_{-2}(fg) = g\Delta_{-2} f + f\Delta_{-2} g + 2\,\partial_i f\,\partial_i g$. By (9),

$$\Delta_{-2}\Delta_0 f = \tfrac{2}{3}R_{jk}\partial_j\partial_k f + \tfrac{2}{3}(R_{jika} + R_{jaki})x_a\partial_i\partial_j\partial_k f - \tfrac{4}{3}R_{jk}\partial_j\partial_k f + \Delta_0\Delta_{-2} f.$$

PROOF. By Bianchi's first identity, $\Sigma'_{ijk} R_{ijka} = 0$, where Σ'_{ijk} denotes the sum of cyclic permutations of (ijk), i.e.,

$$\sum_{ijk}{}' R_{ijka} = R_{ijka} + R_{kija} + R_{jkia},$$

and $\Sigma'_{ijk} R_{jaki} = \Sigma' R_{ajik} = -\Sigma' R_{kjia} = 0$. So

$$(R_{jika} + R_{jaki})\partial_i\partial_j\partial_k f = 0.$$

This proves (21).

$$\begin{aligned}\Delta_{-2}^2\Delta_0 f &= \Delta_{-2}\big[-\tfrac{2}{3}R_{jk}\partial_j\partial_k f + \Delta_0\Delta_{-2} f\big]\\ &= -\tfrac{2}{3}R_{jk}\partial_j\partial_k\Delta_{-2} f + \Delta_{-2}\Delta_0\Delta_{-2} f\\ &= -\tfrac{4}{3}R_{jk}\partial_j\partial_k\Delta_{-2} f + \Delta_0\Delta_{-2}^2 f.\end{aligned}$$

This is (22).

By (10),

$$\begin{aligned}\Delta_{-2}\Delta_1 f = &\tfrac{1}{3}(\partial_a R_{jhkn} + \partial_h R_{jakh} + \partial_h R_{jhka})x_a \partial_j \partial_k f \\ &+ \tfrac{1}{3}(\partial_i R_{jakb} + \partial_a R_{jikb} + \partial_a R_{jbki})x_a x_b \partial_i \partial_j \partial_k f \\ &+ \tfrac{1}{6}(2\,\partial_j R_{ka} - 6\,\partial_k R_{ja} - 6\,\partial_a R_{jk})x_a \partial_j \partial_k f \\ &+ \tfrac{1}{6}(\partial_i R_{hh} - 6\,\partial_h R_{ih})\,\partial_i f + \Delta_1 \Delta_{-2} f.\end{aligned}$$

By Bianchi's first identity,

$$\sum_{ijk}{}' \partial_a R_{jikb} = 0 \quad \text{and} \quad \sum_{ijk}{}' \partial_a R_{jbki} = 0.$$

By Bianchi's second identity,

$$\partial_h R_{jakh} = -\partial_k R_{jhah} - \partial_a R_{jkhh} = -\partial_k R_{ja} + \partial_a R_{jk},$$

$$\partial_h R_{jhka} = \partial_h R_{hjak} = -\partial_a R_{hhjk} - \partial_j R_{hahk} = \partial_a R_{jk} - \partial_j R_{ak},$$

$$\partial_i s = \partial_i R_{ahah} = -\partial_a R_{aihh} - \partial_h R_{aaih} = \partial_a R_{ai} + \partial_h R_{ih} = 2\,\partial_h R_{ih}.$$

We obtain (23).

$$\begin{aligned}\Delta^2_{-2}\Delta_1 f &= \Delta_{-2}(\Delta_{-2}\Delta_1 f) \\ &= -\tfrac{8}{3}\,\partial_i R_{jk} \partial_i \partial_j \partial_k f + \tfrac{2}{3}\partial_i R_{jhkh} \partial_i \partial_j \partial_k f \\ &\quad + \tfrac{2}{3}(\partial_i R_{jhka} + \partial_i R_{jakh})x_a \partial_i \partial_j \partial_k \partial_h f \\ &\quad - \tfrac{8}{3}\,\partial_j R_{ak} x_a \partial_j \partial_k \Delta_{-2} f + \tfrac{2}{3}\,\partial_i R_{jakb}\, x_a x_b \partial_i \partial_j \partial_k \Delta_{-2} f \\ &\quad - \tfrac{2}{3}\partial_i s\,\partial_i \Delta_{-2} f + \Delta_1 \Delta^2_{-2} f.\end{aligned}$$

By Bianchi's second identity, the coefficients of $\partial_i \partial_j \partial_k \partial_h f$ vanish, and we obtain (24).

Finally, assume $\Delta^2_{-2} f = 0$.

$$\begin{aligned}\Delta^3_{-2}\Delta_1 f &= \Delta_{-2}(\Delta^2_{-2}\Delta_1 f) \\ &= -2\,\partial_i R_{jk}\,\partial_i \partial_j \partial_k \Delta_{-2} f - \tfrac{16}{3}\,\partial_j R_{ik}\,\partial_i \partial_j \partial_k \Delta_{-2} f \\ &\quad + \tfrac{4}{3}\,\partial_i R_{jhkh}\,\partial_i \partial_j \partial_k \Delta_{-2} f \\ &\quad + \tfrac{4}{3}(\partial_i R_{jakh} + \partial_i R_{jhka})x_a \partial_i \partial_j \partial_k \partial_h \Delta_{-2} f.\end{aligned}$$

By Bianchi's first identity, the coefficients of $\partial_i \partial_j \partial_k \partial_h \Delta_{-2} f$ vanish. We obtain (25). □

5. Proof of Theorem 1. A smooth function f is said to be harmonic if $\Delta_{-2} f = 0$. It is clear that the derivatives of a harmonic function are also harmonic.

Since u is harmonic in D, by the formulas in Section 4, we see that $\Delta_{-2}\Delta_0 u$ and $\Delta^2_{-2}\Delta_1 u$ are harmonic in D. We also have

$$\Delta_0 u(0) = 0, \qquad \Delta_{-2}\Delta_0 u(0) = -\tfrac{2}{3} R_{jk} \partial_j \partial_k u(0),$$

$$\Delta_1 u(0) = 0, \qquad \Delta_{-2}\Delta_1 u(0) = -\tfrac{1}{3}\,\partial_i s\,\partial_i u(0),$$

$$\Delta^2_{-2}\Delta_1 u(0) = -2\,\partial_i R_{jk}\,\partial_i \partial_j \partial_k u(0).$$

It follows from (13), (14) and (20) that

$$u_0(0) = -(1/12n(n+2))R_{jk}\partial_j\,\partial_k u(0), \tag{26}$$

$$\begin{aligned} u_1(0) = &-(1/24n(n+2))\,\partial_i s\,\partial_i u(0) \\ &-(1/24n(n+2)(n+4))\,\partial_i R_{jk}\,\partial_i\partial_j\,\partial_k u(0). \end{aligned} \tag{27}$$

By (12) and the classical Poisson formula, we have, for $|x| < 1$,

$$u(x) = \int_S K(x,z)g(z)\,dI(z), \quad \text{where } K(x,z) = (1-|x|^2)/|x-z|^n. \tag{28}$$

Fix $z \in S$, let $K(x) = K(x,z)$. Direct computation shows

$$\partial_i K(0) = nz_i, \tag{29}$$

$$\partial_j\,\partial_k K(0) = n(n+2)z_jz_k - (n+2)\delta_{jk}, \tag{30}$$

$$\partial_i\partial_j\,\partial_k K(0) = n(n+2)(n+4)z_iz_jz_k - n(n+4)\sum_{ijk}{}' z_i\delta_{jk}. \tag{31}$$

Recall Σ'_{ijk} denotes the sum of cyclic permutations of (ijk).

By (3), (4), (26), (27) and (28), we obtain (17). In view of Lemma 1, this proves (2), hence Theorem 1. □

6. Perturbation method (continued). Now we begin the preparation for the proof of Theorem 2.

Define w_0, w_1, v, v_0 and v_1 by

$$\Delta_{-2}w_0 + u_0 = 0, \quad \text{in } D \text{ and } w_0 = 0 \text{ on } S, \tag{32}$$

$$\Delta_{-2}w_1 + u_1 = 0, \quad \text{in } D \text{ and } w_1 = 0 \text{ on } S, \tag{33}$$

$$\Delta_{-2}v + u = 0, \quad \text{in } D \text{ and } v = 0 \text{ on } S, \tag{34}$$

$$\Delta_{-2}v_0 + \Delta_0 v = 0, \quad \text{in } D \text{ and } v_0 = 0 \text{ on } S, \tag{35}$$

$$\Delta_{-2}v_1 + \Delta_1 v = 0, \quad \text{in } D \text{ and } v_1 = 0 \text{ on } S. \tag{36}$$

LEMMA 3. *We have*

$$\begin{aligned} E^m[Tg(X_T/r)] = {}& r^2(1/2n)u(0) + r^4(v_0(0) + w_0(0)) \\ &+ r^5(v_1(0) + w_1(0)) + O(r^6). \end{aligned} \tag{37}$$

Moreover, $O(r^6)$ above is dominated by $Kr^6|g|_2$ for some constant K independent of g.

PROOF. Define W_0, W_1, V, V_0 and V_1 as we defined U, U_0 and U_1 in the proof of Lemma 1. By Itô's formula,

$$\begin{aligned} E^m[Tg(X_T/r)] &= E^m[TU(X_T)] \\ &= E^m\left[\int_0^T U(X_t)\,dt\right] + E^m\left[\int_0^T t\,\Delta U(X_t)\,dt\right]. \end{aligned}$$

In the following computation, $O(r^6)$ is actually controlled by $Kr^6|g|_2$ for some constant K. This can be proved as in the proof of Lemma 1 by keeping track of each term added into $O(r^6)$ and using Schauder's estimate, (16) and the fact that $E^m[T^2] = O(r^4)$ [see (42)].

$$
\begin{aligned}
E^m\left[\int_0^T U(X_t)\,dt\right] &= -r^2E^m\left[\int_0^T \Delta_{-2}V(X_t)\,dt\right] \\
&= -r^2E^m\left[\int_0^T \Delta V(X_t)\,dt\right] + r^2E^m\left[\int_0^T \Delta_0 V(X_t)\,dt\right] \\
&\quad + r^2E^m\left[\int_0^T \Delta_1 V(X_t)\,dt\right] + O(r^6) \\
&= r^2v(0) - r^4E^m\left[\int_0^T \Delta_{-2}V_0(X_t)\,dt\right] \\
&\quad - r^5E^m\left[\int_0^T \Delta_{-2}V_1(X_t)\,dt\right] + O(r^6) \\
&= r^2v(0) - r^4E^m\left[\int_0^T \Delta V_0(X_t)\,dt\right] \\
&\quad - r^5E^m\left[\int_0^T \Delta V_1(X_t)\,dt\right] + O(r^6) \\
&= r^2v(0) + r^4v_0(0) + r^5v_1(0) + O(r^6),
\end{aligned}
$$

$$
\begin{aligned}
E^m\left[\int_0^T t\,\Delta U(X_t)\,dt\right] &= E^m\left[\int_0^T t(\Delta_0 U + \Delta_1 U)(X_t)\,dt\right] + O(r^6) \\
&= -r^2E^m\left[\int_0^T t\,\Delta_{-2}U_0(X_t)\,dt\right] \\
&\quad - r^3E^m\left[\int_0^T t\,\Delta_{-2}U_1(X_t)\,dt\right] + O(r^6) \\
&= -r^2E^m\left[\int_0^T t\,\Delta U_0(X_t)\,dt\right] \\
&\quad - r^3E^m\left[\int_0^T t\,\Delta U_1(X_t)\,dt\right] + O(r^6) \\
&= r^2E^m\left[\int_0^T U_0(X_t)\,dt\right] + r^3E^m\left[\int_0^T U_1(X_t)\,dt\right] + O(r^6) \\
&= -r^4E^m\left[\int_0^T \Delta_{-2}W_0(X_t)\,dt\right] \\
&\quad - r^5E^m\left[\int_0^T \Delta_{-2}W_1(X_t)\,dt\right] + O(r^6) \\
&= -r^4E^m\left[\int_0^T \Delta W_0(X_t)\,dt\right] \\
&\quad - r^5E^m\left[\int_0^T \Delta W_1(X_t)\,dt\right] + O(r^6) \\
&= r^4w_0(0) + r^5w_1(0) + O(r^6).
\end{aligned}
$$

Therefore

$$E^m[Tg(X_T/r)] = r^2 v(0) + r^4(v_0(0) + w_0(0)) + r^5(v_1(0) + w_1(0)) + O(r^6).$$

By (20) and (34), $v(0) = (1/2n)u(0)$. This proves (37). □

LEMMA 4. *For $g \in C^2(S)$, we have*

$$E^m[T^2 g(X_T/r)] = E^m[T^2]E^m[g(X_T/r)] + O(r^6). \tag{38}$$

Moreover, $O(r^6)$ above is dominated by $Kr^6|g|_2$ for some constant K independent of g.

PROOF. Define q by

$$\Delta_{-2}q + v = 0, \quad \text{in } D \text{ and } q = 0 \text{ on } S. \tag{39}$$

By (20) and (34), we see that

$$q(0) = (n + 4/8n^2(n+2))u(0). \tag{40}$$

Let $Q(x) = q(x/r)$ for $x \in D_r$. In the following computation, by using Schauder's estimate as in the proofs of Lemmas 1 and 3, we can show that each $O(r^k)$ term is actually dominated by $Kr^k|g|_2$ for some constant K.

$$\begin{aligned} E^m[T^2 g(X_T/r)] &= E^m[T^2 U(X_T)] \\ &= 2E^m\left[\int_0^T tU(X_t)\,dt\right] + O(r^6) \\ &= -2r^2E^m\left[\int_0^T t\Delta_{-2}V(X_t)\,dt\right] + O(r^6) \\ &= 2r^4E^m\left[\int_0^T t\Delta_{-2}^2 Q(X_t)\,dt\right] + O(r^6) \\ &= 2r^4E^m\left[\int_0^T t\Delta^2 Q(X_t)\,dt\right] + O(r^6). \end{aligned}$$

On the other hand, by Dynkin's formula and Itô's formula,

$$\begin{aligned} E^m[Q(X_T)] &= Q(0) + E^m\left[\int_0^T \Delta Q(X_t)\,dt\right] \\ &= q(0) + E^m[T\Delta Q(X_T)] - E^m\left[\int_0^T t\Delta^2 Q(X_t)\,dt\right]. \end{aligned}$$

Since $q = v = 0$ on S and

$$E^m[T\Delta Q(X_T)] = E^m[T\Delta_{-2}Q(X_T)] + O(r^2) = 0 + O(r^2),$$

we have

$$E^m[T^2 g(X_T/r)] = (n + 4/4n^2(n+2))r^4 u(0) + O(r^6). \tag{41}$$

Put $g = 1$. Then $u(0) = 1$, so

$$E^m[T^2] = (n + 4/4n^2(n+2))r^4 + O(r^6). \tag{42}$$

Now (38) follows from (15) and the above two formulas. □

7. Poisson equation (continued). In Section 3, we obtained a formula to express $f(0)$ when f is the solution of the Poisson equation

(43) $$\Delta_{-2} f + p = 0, \quad \text{in } D \text{ and } f = 0 \text{ on } S$$

and p satisfies $\Delta^k_{-2} p = 0$ for some positive integer k. In the present section, we look for formulas to express the derivatives of f at 0.

LEMMA 5. *Suppose p is harmonic, i.e., $\Delta_{-2} p = 0$. Let f be defined by* (43). *Then*

(44) $$\partial_i f(0) = (1/2(n+2))\, \partial_i p(0),$$

(45) $$\partial_j \partial_k f(0) = (1/2(n+4)) \partial_j \partial_k p(0) - \delta_{jk}(1/n) p(0),$$

(46) $$\partial_i \partial_j \partial_k f(0) = (1/2(n+6)) \partial_i \partial_j \partial_k p(0) - (1/(n+2)) \sum_{ijk}{}' \delta_{jk}\, \partial_i p(0).$$

PROOF. Define

(47) $$Gp(x) = \left(\Gamma\left(\tfrac{1}{2}n - 1\right)/2\pi^{n/2}\right) \int_D |x-y|^{2-n} p(y)\, dv.$$

By classical analysis we have

(48) $$\Delta_{-2} Gp + p = 0, \quad \text{in } D,$$

(49) $$f(x) = Gp(x) - \int_S K(x,z) Gp(z)\, dI(z), \quad \text{for } x \in D.$$

By (29), the harmonicity of p and Lemma 2, we have

$$\left[\partial_i \int_S K(x,z) Gp(z)\, dI(z)\right]_{x=0} = n \int_S z_i Gp(z)\, dI(z)$$
$$= \partial_i Gp(0) - (1/2(n+2))\, \partial_i p(0).$$

Now (44) follows from the above expression and (49).

Direct computation shows

$$\Delta_{-2}(x_j x_k Gp) = 2\delta_{jk} Gp + 2x_j\, \partial_k Gp + 2x_k\, \partial_j Gp - x_j x_k p,$$
$$\Delta^2_{-2}(x_j x_k Gp) = -4\delta_{jk} p + 8\partial_j \partial_k Gp - 4x_j\, \partial_k p - 4x_k\, \partial_j p,$$
$$\Delta^3_{-2}(x_j x_k Gp) = -24 \partial_j \partial_k p.$$

By (30) and Lemma 2, we have

$$\left[\partial_j \partial_k \int_S K(x,z) Gp(z)\, dI(z)\right]_{x=0}$$
$$= n(n+2) I(z_j z_k Gp) - (n+2)\, \delta_{jk} I(Gp)$$
$$= (n+2)\, \delta_{jk} Gp(0) - \tfrac{1}{2}\, \delta_{jk} p(0) + \partial_j \partial_k Gp(0)$$
$$- (1/2(n+4)) \partial_j \partial_k p(0) - (n+2)\, \delta_{jk} Gp(0) + (n+2/2n)\, \delta_{jk} p(0)$$
$$= \delta_{jk}(1/n) p(0) + \partial_j \partial_k Gp(0) - (1/2(n+4)) \partial_j \partial_k p(0).$$

By (49), we obtain (45).

Direct computation shows

$$\Delta_{-2}(x_i x_j x_k Gp) = 2\sum{}'\delta_{jk}x_i Gp + 2\sum{}' x_j x_k \,\partial_i Gp - x_i x_j x_k p,$$

$$\Delta^2_{-2}(x_i x_j x_k Gp) = 8\sum{}'\delta_{jk}\,\partial_i Gp - 4\sum{}'\delta_{jk}x_i\, p + 8\sum{}' x_i \partial_j\,\partial_k Gp - 4\sum{}' x_j x_k\,\partial_i p,$$

$$\Delta^3_{-2}(x_i x_j x_k Gp) = -24\sum{}'\delta_{jk}\,\partial_i p + 48\partial_i\partial_j\,\partial_k Gp - 24\sum{}' x_i\partial_j\,\partial_k p,$$

$$\Delta^4_{-2}(x_i x_j x_k Gp) = -192\,\partial_i\partial_j\,\partial_k p,$$

where Σ' denotes the sum of cyclic permutations of (ijk). By (31) and Lemma 2,

$$\left[\partial_i\partial_j\partial_k\int_S K(x,z)Gp(z)\,dI(z)\right]_{x=0}$$
$$= n(n+2)(n+4)I(z_i z_j z_k Gp) - n(n+4)\sum{}' I(z_i Gp)\delta_{jk}$$
$$= \partial_i\partial_j\,\partial_k Gp(0) - (1/2(n+6))\partial_i\partial_j\,\partial_k p(0) + (1/n+2)\sum{}'\delta_{jk}\,\partial_i p(0).$$

By (49), we obtain (46). □

8. Proof of Theorem 2. Now we are ready to compute $w_0(0)$, $v_0(0)$, $w_1(0)$ and $v_1(0)$.

By (13) and (21),

$$\Delta_{-2}u_0(0) = -\Delta_0 u(0) = 0,$$
$$\Delta^2_{-2}u_0 = -\Delta_{-2}\,\Delta_0 u = \tfrac{2}{3}R_{jk}\partial_j\,\partial_k u.$$

Hence, by (20) and (32), we obtain

$$w_0(0) = (1/2n)u_0(0) + (1/72n(n+2)(n+4))R_{jk}\partial_j\,\partial_k u(0).$$

Similarly, by (20), (34), (35) and formulas in Section 4, we obtain

$$v_0(0) = -(1/12n(n+2))R_{jk}\partial_j\,\partial_k v(0) + (1/36n(n+2)(n+4))R_{jk}\partial_j\,\partial_k u(0).$$

It follows from Lemma 5 and the above two expressions that

(50) $$w_0(0) + v_0(0) = (1/12n)u_0(0) + (1/12n^2(n+2))su(0).$$

By (14), (20), (33) and the formulas in Section 4,

$$w_1(0) = (1/2n)u_1(0) + (1/144n(n+2)(n+4))\,\partial_i s\,\partial_i u(0) + (1/192n(n+2)(n+4)(n+6))\,\partial_i R_{jk}\,\partial_i\partial_j\,\partial_k u(0).$$

Similarly, by (20), (34), (36) and the formulas in Section 4, we obtain

$$\begin{aligned} v_1(0) = &-(1/24n(n+2))\,\partial_i s\,\partial_i v(0) \\ &-(1/24n(n+2)(n+4))\,\partial_i R_{jk}\,\partial_i\partial_j\,\partial_k v(0) \\ &+(1/72n(n+2)(n+4))\,\partial_i s\,\partial_i u(0) \\ &+(1/64n(n+2)(n+4)(n+6))\,\partial_i R_{jk}\,\partial_i\partial_j\,\partial_k u(0). \end{aligned}$$

It follows from the above two expressions, Lemma 5 and the fact that $\partial_i s = 2\,\partial_h R_{ih}$, which we have seen in Section 4, that

$$(51)\quad w_1(0) + v_1(0) = (1/2n)u_1(0) + \left(1/24n(n+2)^2(n+4)\right)\partial_i s\,\partial_i u(0).$$

Now with the help of (15), (16), (50) and (51), we can prove (5) by comparing (37) with

$$E^m[T]E^m[g(X_T/r)].$$

Finally, (6) follows from (5) and (38). □

REMARK. M. Pinsky recently proposed an expansion of

$$E^x\left(\exp(-bT_r/r^2)g(X_{T_r}/r)\right)$$

which can be computed in a way similar to Theorem 1. Our major results can be obtained as consequences of this expansion.

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DEPARTMENT OF MATHEMATICS
NANKAI UNIVERSITY
TIANJIN
PEOPLE'S REPUBLIC OF CHINA