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# THE ASYMPTOTIC DISTRIBUTION OF TRIMMED SUMS

BY SÁNDOR CSÖRGŐ[1], ERICH HAEUSLER AND DAVID M. MASON[2]

*Szeged University, University of Munich and University of Munich*

Let $X_{1,n} \le \cdots \le X_{n,n}$ be the order statistics of $n$ independent and identically distributed random variables and let $m_n$ and $k_n$ be positive integers such that $m_n \to \infty$, $k_n \to \infty$, $m_n/n \to 0$ and $k_n/n \to 0$ as $n \to \infty$. We find a necessary and sufficient condition for the existence of normalizing and centering constants $A_n > 0$ and $B_n$ such that the sequence

$$T_n = A_n^{-1}\left\{ \sum_{i=m_n+1}^{n-k_n} X_{i,n} - B_n \right\}$$

is stochastically compact and completely describe the possible subsequential limiting distributions. We also give a necessary and sufficient condition for the existence of $A_n$ and $B_n$ such that $T_n$ is asymptotically normal. A variant of Stigler's theorem when $m_n/n \to \alpha$ and $k_n/n \to 1 - \beta$, where $0 < \alpha < \beta < 1$, is also obtained as a by-product.

**1. Introduction and statements of results.** Let $X, X_1, X_2, \ldots$ be a sequence of independent nondegenerate random variables with a common (right-continuous) distribution function $F$ and for each integer $n \ge 1$ let $X_{1,n} \le \cdots \le X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. The asymptotic behavior of trimmed sums of $X_1, \ldots, X_n$ has been investigated by a large number of authors. A trimmed sum may be obtained by excluding either a fixed or a growing number of the smallest and largest order statistics from the sum $X_1 + \cdots + X_n$ at each stage $n$, or by discarding either a fixed or a growing number of extreme terms largest in absolute value. Sometimes extreme values of truncated summands are thrown away. Results on the asymptotic distribution of trimmed sums have been obtained by Darling (1952), Arov and Bobrov (1960), Stigler (1973), Hall (1978), Teugels (1981), Maller (1982), Mori (1984), M. Csörgő, S. Csörgő, Horváth and Mason (CsCsHM) (1986b), S. Csörgő, Horváth and Mason (1986), Kuelbs and Ledoux (1987), Pruitt (1985) and Hahn and Kuelbs (1985).

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Introduce the inverse or quantile function $Q$ of $F$ defined as

$$Q(s) = \inf\{x\colon F(x) \ge s\}, \qquad 0 < s \le 1; \qquad Q(0) = Q(0+),$$

and for $0 < s < 1 - t < 1$ consider the truncated mean and variance functions

$$M(s, 1-t) = \int_s^{1-t} Q(u)\, du,$$

$$\sigma^2(s, 1-t) = \int_s^{1-t}\int_s^{1-t} (u \wedge v - uv)\, dQ(u)\, dQ(v),$$

where $u \wedge v = \min(u, v)$, $u \vee v = \max(u, v)$. Let $m_n$ and $k_n$ be any sequences of integers satisfying

(1.1) $0 \le m_n < n - k_n \le n$, $m_n \to \infty$, $k_n \to \infty$, $m_n/n \to 0$ and $k_n/n \to 0$.

One of the problems that we are going to consider is when the following central limit theorem (1.2) holds:

$$\frac{1}{a_n(m_n, k_n)}\left\{\sum_{i=m_n+1}^{n-k_n} X_{i,n} - \mu_n(m_n, k_n)\right\} \to_{\mathscr{D}} N(0,1), \tag{1.2}$$

where $\mu_n(m_n, k_n) = nM(m_n/n, 1 - k_n/n)$, $a_n(m_n, k_n) = n^{1/2}\sigma(m_n/n, 1 - k_n/n)$ and where $\to_{\mathscr{D}}$ denotes convergence in distribution and $N(0,1)$ is a standard normal random variable. (Here and elsewhere in this paper, unless otherwise specified, all limit relations are assumed to hold as $n \to \infty$, or an unspecified convergence statement will hold along a subsequence of the positive integers.) When $F$ is in the domain of attraction of a nondegenerate stable or normal law, (1.2) was shown by S. Csörgő, Horváth and Mason (1986) when $m_n \equiv k_n$. It was remarked in that paper that the same methods could be used to show (1.2) in this case when $m_n \not\equiv k_n$ (perhaps by assuming some additional regularity conditions on $F$). The aim of the present paper is to give an exhaustive study of the problem of asymptotic distribution for arbitrary $F$ of sums of the form

$$\sum_{i=m_n+1}^{n-k_n} X_{i,n}.$$

With the exception of Theorem 5 and its proof, throughout the present section and Section 2, which contains the proofs, *two sequences of positive integers* $m_n$ *and* $k_n$ *will be fixed so that they satisfy* (1.1). Therefore in the notation we may drop the dependence on $m_n$ and $k_n$ for convenience. Besides the already introduced quantities $a_n = a_n(m_n, k_n)$, $\mu_n = \mu_n(m_n, k_n)$, we need the following notation:

$$b_n = \sigma\left(\frac{m_n}{n}, 1 - \frac{k_n}{n}\right) = n^{-1/2} a_n,$$

$$\Psi_{1,n}(c) = \begin{cases} \left(\frac{m_n}{n}\right)^{1/2} \left\{ Q\left(\frac{m_n}{n} + c\frac{m_n^{1/2}}{n}\right) - Q\left(\frac{m_n}{n}\right) \right\} \Big/ b_n, \\ \qquad\qquad -\frac{m_n^{1/2}}{2} \le c \le \frac{m_n^{1/2}}{2}, \\ \Psi_{1,n}\left(-\frac{m_n^{1/2}}{2}\right), \quad -\infty < c < -\frac{m_n^{1/2}}{2}, \\ \Psi_{1,n}\left(\frac{m_n^{1/2}}{2}\right), \quad \frac{m_n^{1/2}}{2} < c < \infty, \end{cases}$$

$$\Psi_{2,n}(c) = \begin{cases} \left(\frac{k_n}{n}\right)^{1/2} \left\{ Q\left(1 - \frac{k_n}{n} + c\frac{k_n^{1/2}}{n}\right) - Q\left(1 - \frac{k_n}{n}\right) \right\} \Big/ b_n, \\ \qquad\qquad -\frac{k_n^{1/2}}{2} \le c \le \frac{k_n^{1/2}}{2}, \\ \Psi_{2,n}\left(\frac{k_n^{1/2}}{2}\right), \quad \frac{k_n^{1/2}}{2} < c < \infty, \\ \Psi_{2,n}\left(-\frac{k_n^{1/2}}{2}\right), \quad -\infty < c < -\frac{k_n^{1/2}}{2}, \end{cases}$$

$$r_{1,n} = -\left(\frac{m_n}{n}\right)^{1/2} \int_{m_n/n}^{1-k_n/n} (1-s)\, dQ(s)/b_n,$$

$$r_{2,n} = -\left(\frac{k_n}{n}\right)^{1/2} \int_{m_n/n}^{1-k_n/n} s\, dQ(s)/b_n,$$

$$\sigma_{1,n} = \sigma\left(\frac{m_n}{n}, \frac{1}{2}\right)\Big/ b_n = \sigma\left(\frac{m_n}{n}, \frac{1}{2}\right)\Big/ \sigma\left(\frac{m_n}{n}, 1 - \frac{k_n}{n}\right),$$

$$\sigma_{2,n} = \sigma\left(\frac{1}{2}, 1 - \frac{k_n}{n}\right)\Big/ b_n = \sigma\left(\frac{1}{2}, 1 - \frac{k_n}{n}\right)\Big/ \sigma\left(\frac{m_n}{n}, 1 - \frac{k_n}{n}\right).$$

For large enough $n$ the functions $\Psi_{1,n}$ and $\Psi_{2,n}$ are obviously well defined, nondecreasing and left-continuous on $\mathbb{R}$.

From the technical point of view, our most important results are contained in the following two theorems.

**Theorem 1.** *Assume that there exists a subsequence $\{n'\}$ of the positive integers such that for two nondecreasing left-continuous real-valued functions*

*$\Psi_1$ and $\Psi_2$ with $\Psi_1(0) = \Psi_2(0) = 0$, we have*

$$\Psi_{i,n'}(c) \to \Psi_i(c), \tag{1.3}$$

*at every continuity point $c \in \mathbb{R}$ of $\Psi_i$ for $i = 1,2$. Assume further that there exist (necessarily) nonpositive constants $r_1$ and $r_2$ such that*

$$r_{1,n'} \to r_i, \quad \text{for } i = 1,2. \tag{1.4}$$

*Then, necessarily, $\Psi_1(c) \le -r_1$ and $\Psi_2(c) \ge r_2$ for all $c$, and*

$$\frac{1}{a_{n'}(m_{n'}, k_{n'})}\left\{\sum_{i=m_{n'}+1}^{n'-k_{n'}} X_{i,n'} - \mu_{n'}(m_{n'}, k_{n'})\right\} \to_{\mathscr{D}} V_1 = V_1(\Psi_1, \Psi_2),$$

*where*

$$V_1 = V_1(\Psi_1, \Psi_2) = \int_0^{-Z_1}(Z_1 + x)\, d\Psi_1(x) + Z + \int_{-Z_2}^{0}(Z_2 + x)\, d\Psi_2(x),$$

*where $(Z_1, Z, Z_2)$ is a trivariate normal random vector with mean vector zero and covariance matrix*

$$\Sigma = \begin{pmatrix} 1 & r_1 & 0 \\ r_1 & 1 & r_2 \\ 0 & r_2 & 1 \end{pmatrix}. \tag{1.5}$$

*Moreover, if $\operatorname{Var}(X) = \infty$, and for some positive constants $\sigma_1$ and $\sigma_2$ (necessarily satisfying $\sigma_1^2 + \sigma_2^2 = 1$), we have*

$$\sigma_{1,n'} \to \sigma_i, \quad \text{for } i = 1,2, \tag{1.6}$$

*then $Z$ can be written as the sum of two independent normal random variables $W_1$ and $W_2$, $W_i =_{\mathscr{D}} N(0, \sigma_i^2)$ for $i = 1,2$, such that the vectors $(Z_1, W_1)$ and $(Z_2, W_2)$ are independent.*

THEOREM 2. *Assume that there exists a subsequence $\{n'\}$ of the positive integers such that for some $c$,*

$$|\Psi_{1,n'}(c)| \to \infty \tag{1.7}$$

*or*

$$|\Psi_{2,n'}(c)| \to \infty. \tag{1.8}$$

*Furthermore, assume that there exists a sequence of positive constants $A_n$ such that for two nondecreasing left-continuous real-valued functions $\Psi_1^*$ and $\Psi_2^*$ on $\mathbb{R}$ with $\Psi_1^*(0) = \Psi_2^*(0) = 0$, we have*

$$(a_{n'}/A_{n'})\Psi_{i,n'}(t) \to \Psi_i^*(t), \tag{1.9}$$

*at every continuity point $t$ of $\Psi_i^*$ for $i = 1,2$. Then, necessarily, $\Psi_1^*(c) = 0$ and $\Psi_2^*(-c) = 0$ for all $c > 0$, and*

$$\frac{1}{A_{n'}}\left\{\sum_{i=m_{n'}+1}^{n'-k_{n'}} X_{1,n'} - \mu_{n'}(m_{n'}, k_{n'})\right\} \to_{\mathscr{D}} V_2 = V_2(\Psi_1^*, \Psi_2^*),$$

*where, with independent standard normal random variables* $Z_1$ *and* $Z_2$,

$$V_2 = V_2(\Psi_1^*, \Psi_2^*) = \int_0^{-Z_1} (Z_1 + x)\, d\Psi_1^*(x) + \int_{-Z_2}^{0} (Z_2 + x)\, d\Psi_2^*(x).$$

In Section 3, it is remarked that when $\mathrm{Var}(X) < \infty$, the functions $\Psi_1$ and $\Psi_2$ appearing in $V_1$ are identically equal to zero. Also by Proposition 1 in the Appendix, the limiting random variable $V_1$ in the second part of the statement of Theorem 1 is a normal random variable if and only if both $\Psi_1$ and $\Psi_2$ are identically equal to zero, and $V_2$ is never a nondegenerate normal random variable. Examples will be given in Section 3 showing that, in fact, it is possible for the limiting random variables $V_1$ and $V_2$ to be nondegenerate and nonnormal along subsequences of the positive integers.

We can now state our two main results concerning the stochastic compactness and asymptotic normality of trimmed sums. The sufficiency parts of these results are more or less straightforward corollaries to the previous technical theorems.

We say that a sequence of random variables is stochastically compact if every subsequence contains a subsequence that converges in distribution to a nondegenerate random variable.

THEOREM 3. *Let* $\{m_n\}$ *and* $\{k_n\}$ *be any two sequences of positive integers satisfying* (1.1). *Then there exist sequences of normalizing and centering constants* $A_n > 0$ *and* $B_n$ *such that the sequence of random variables*

$$A_n^{-1}\left\{ \sum_{i=m_n+1}^{n-k_n} X_{1,n} - B_n \right\} \tag{1.10}$$

*is stochastically compact if and only if there exists a sequence of positive constants* $C_n$ *such that for every subsequence* $\{n'\}$ *of the positive integers there exists a further subsequence* $\{n''\}$ *of* $\{n'\}$ *satisfying*

$$0 \le \lim_{n''\to\infty} \frac{a_{n''}}{C_{n''}} < \infty \tag{1.11}$$

*and that for appropriate nondecreasing left-continuous real-valued functions* $\Psi_1^*$ *and* $\Psi_2^*$ *defined on* $\mathbb{R}$ *and depending on* $\{n''\}$,

$$\lim_{n''\to\infty} \frac{a_{n''}}{C_{n''}} \Psi_{1,n''}(c) = \Psi_i^*(c), \tag{1.12}$$

*at every continuity point* $c$ *of* $\Psi_i^*$ *for* $i = 1,2$, *where at least one of the* $\Psi_i^*$, $i = 1,2$, *is not identically equal to zero when the limit in* (1.11) *is equal to zero. In this case,* $A_n$ *can be chosen to be* $C_n$ *and* $B_n$ *to be* $\mu_n$, $n \ge 1$. *Furthermore, when the sequence of random variables given in* (1.10) *is stochastically compact, all subsequential limiting random variables are affine transforms of random variables of the form of* $V_1(\Psi_1^*, \Psi_2^*)$ *or* $V_2(\Psi_1^*, \Psi_2^*)$ *of Theorems* 1 *and* 2.

The nature of Theorem 3 being rather exhaustive, the given necessary and sufficient conditions may seem somewhat complicated. It will be clear from the proof of this theorem that a simple sufficient condition for the existence of $\{A_n\}$ and $\{B_n\}$ to make the sequence in (1.10) stochastically compact is

$$\limsup_{n\to\infty}|\Psi_{1,n}(c)| < \infty, \quad \text{for all } c, \qquad i = 1,2. \tag{1.13}$$

In this case $A_n$ can be chosen to be $a_n$ and $B_n$ to be $\mu_n$, $n \geq 1$. In fact, if we restrict ourselves to these "natural" normalizing and centering constants $a_n = a_n(m_n, k_n)$ and $\mu_n = \mu_n(m_n, k_n)$, then the sequence of random variables on the left-hand side of (1.2) is stochastically compact *if and only if* condition (1.13) holds.

However, turning to the problem of the characterization of the asymptotic normality of trimmed sums, the result is simply formulated.

THEOREM 4. *Let $\{m_n\}$ and $\{k_n\}$ be any two sequences of positive integers satisfying* (1.1). *There exist sequences of normalizing and centering constants $A_n > 0$ and $B_n$ such that*

$$A_n^{-1}\left\{\sum_{i=m_n+1}^{n-k_n} X_{i,n} - B_n\right\} \to_{\mathscr{D}} N(0,1) \tag{1.14}$$

*if and only if*

$$\lim_{n\to\infty} \Psi_{1,n}(c) = 0, \quad \textit{for all } c, \qquad i = 1,2. \tag{1.15}$$

*In this case, $A_n$ can be chosen to be $a_n$ and $B_n$ to be $\mu_n$, $n \geq 1$.*

For clarity Theorem 4 is stated along the whole sequence of positive integers, but the proof will show that it also holds along subsequences.

To round off the present study, we finally discuss the extreme case of heavy trimming when, for example, $m_n = [\alpha n]$ and $k_n = n - [\beta n]$, where $0 < \alpha < \beta < 1$ and $[\cdot]$ denotes the integer part function. [The other extreme case of very light trimming when $m_n \equiv m$ and $k_n \equiv k$ with some fixed positive integers $m$ and $k$ is completely solved in S. Csörgő, Haeusler and Mason (1989). Some results for this case are contained in S. Csörgő, Horváth and Mason (1986) and in CsCsHM (1986b).] We obtain the following variant of Stigler's (1973) theorem.

THEOREM 5. *Assume*

$$m_n = [\alpha n] \quad \textit{and} \quad k_n = n - [\beta n], \tag{1.16}$$

*where $0 < \alpha < \beta < 1$. Then for any underlying quantile function $Q$, we have when $\sigma(\alpha, \beta) > 0$*

$$\frac{1}{a_n(m_n, k_n)}\left\{\sum_{i=m_n+1}^{n-k_n} X_{1,n} - \mu_n(m_n, k_n)\right\} \to_{\mathscr{D}} V_1,$$

*where*

$$V_1 = \frac{\alpha^{1/2}}{\sigma(\alpha,\beta)}(Q(\alpha+) - Q(\alpha))\min(0, Z_1) + Z + \frac{(1-\beta)^{1/2}}{\sigma(\alpha,\beta)}(Q(\beta+) - Q(\beta))\max(0, -Z_2),$$

*where* $(Z_1, Z, Z_2)$ *is a trivariate normal random vector with mean zero and covariance matrix*

$$\begin{pmatrix} 1-\alpha & r_1 & (\alpha(1-\beta))^{1/2} \\ r_1 & 1 & r_2 \\ (\alpha(1-\beta))^{1/2} & r_2 & \beta \end{pmatrix},$$

*where*

$$r_1 = -\alpha^{1/2}\int_\alpha^\beta (1-s)\,dQ(s) \quad \text{and} \quad r_2 = -(1-\beta)^{1/2}\int_\alpha^\beta s\,dQ(s).$$

Note that the limiting random variable $V_1$ here is also of the form of $V_1(\Psi_1, \Psi_2)$ of Theorem 1 with

(1.17)
$$\Psi_1(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \dfrac{\alpha^{1/2}}{\sigma(\alpha,\beta)}(Q(\alpha+) - Q(\alpha)), & \text{if } x > 0, \end{cases}$$
$$\Psi_2(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \dfrac{(1-\beta)^{1/2}}{\sigma(\alpha,\beta)}(Q(\beta+) - Q(\beta)), & \text{if } x > 0. \end{cases}$$

This fact puts Stigler's theorem into a broader picture. Indeed, under (1.16) we have

(1.18)
$$\Psi_{i,n}(x) \to \Psi_i(x), \qquad i = 1, 2,$$

for any $x \in \mathbb{R}$, with the $\Psi_i$ functions in (1.17), and

(1.19)
$$r_{1,n} \to r_i, \qquad i = 1, 2,$$

with the $r_i$ constants of Theorem 5. Hence Theorem 5 will, in fact, follow from the proof of Theorem 1. Note that in Theorem 5, $V_1 = Z$, a standard normal variable, if and only if $Q$ is continuous at $\alpha$ and $\beta$ and $\sigma(\alpha, \beta) > 0$.

The proofs are given in the next section. Section 3 contains a discussion of the conditions given previously, with special reference to the case when the whole sum $X_1 + \cdots + X_n$, when properly centered and normalized, is stochastically compact. This section also contains a number of examples. Finally, in the Appendix some technical results required in Section 2 are proved.

Our methods of proof are probabilistic in nature and independent of the use of the characteristic function. After submission of this paper we received a preprint entitled "Asymptotic normality and subsequential limits of trimmed sums" by P. Griffin and W. Pruitt. They obtain by characteristic function methodology mathematically equivalent versions of our Theorems 1–4.

**2. Proofs.** Let $\{U_n\}_{n\geq 1}$ be a sequence of independent random variables uniformly distributed on $(0,1)$. For any integer $n \geq 1$, let $U_{1,n} \leq \cdots \leq U_{n,n}$ denote the order statistics of $U_1, \dots, U_n$ and let $G_n$ be the (right-continuous) empirical distribution function of $U_1, \dots, U_n$. The two sequences $\{X_n\}_{n\geq 1}$ and $\{Q(U_n)\}_{n\geq 1}$ are equal in law, and, consequently, the two processes $\{X_{i,n}: 1 \leq i \leq n,\ n \geq 1\}$ and $\{Q(U_{i,n}): 1 \leq i \leq n,\ n \geq 1\}$ are equal in law as well. Therefore, without loss of generality, we may assume $X_{i,n} = Q(U_{i,n})$ for all $1 \leq i \leq n$ and $n \geq 1$. Moreover, CsCsHM (1986a) (cf. Theorem 2.1 and Corollary 2.1) have constructed a probability space $(\Omega, \mathscr{A}, P)$ carrying a sequence $U_1, U_2, \dots$ of independent random variables uniformly distributed on $(0,1)$ and a sequence of Brownian bridges $B_n(s)$, $0 \leq s \leq 1$, $n = 1, 2, \dots$, such that for all $0 \leq \nu < 1/4$,

$$\sup_{1/n \leq s \leq 1-1/n} |n^{1/2}\{G_n(s) - s\} - B_n(s)|/(s(1-s))^{1/2-\nu} = O_P(n^{-\nu}), \tag{2.1}$$

$$\sup_{1/n \leq s \leq 1-1/n} |n^{1/2}\{s - U_n(s)\} - B_n(s)|/(s(1-s))^{1/2-\nu} = O_P(n^{-\nu}), \tag{2.2}$$

where $U_n(s) = U_{i,n}$ for $(i-1)/n < s \leq i/n$ and $i = 1, \dots, n$ and $U_n(0) = U_{1,n}$ is the uniform empirical quantile function. Again, without loss of generality, we may and do assume throughout the proofs of Theorems 1 and 2 that we are on this space $(\Omega, \mathscr{A}, P)$. The only exception will be the proof of Lemma 2.4, where we use a Skorohod construction. We shall be using the integral convention of S. Csörgő, Horváth and Mason (1986).

The following identity will be crucial to the proofs:

$$\begin{aligned} a_n^{-1}\left\{\sum_{i=m_n+1}^{n-k_n} X_{i,n} - \mu_n\right\} &= a_n^{-1} n \int_{m_n/n}^{1-k_n/n} (s - G_n(s))\, dQ(s) \\ &\quad + a_n^{-1} n \int_{m_n/n}^{U_{m_n,n}} \left(G_n(s) - \frac{m_n}{n}\right) dQ(s) \\ &\quad + a_n^{-1} n \int_{U_{n-k_n,n}}^{1-k_n/n} \left(G_n(s) - \frac{n-k_n}{n}\right) dQ(s) \\ &\equiv Y_n + R_{1,n} + R_{2,n}. \end{aligned} \tag{2.3}$$

It will become clear in Lemma 2.2 that $Y_n$ is *always* asymptotically standard normal and the "remainders" $R_{1,n} \leq 0$ and $R_{2,n} \geq 0$ contribute the rest. To prove the normality of $Y_n$, we need the following technical step.

LEMMA 2.1. *For any quantile function $Q$,*

$$\limsup_{s,t \downarrow 0} \frac{tQ^2(1-t) + sQ^2(s)}{\sigma^2(s, 1-t)} < \infty.$$

PROOF. We only have to consider the infinite variance case in which

$$S^2(s,1-t) = \int_s^{1-t} Q^2(u)\,du \to \infty, \quad \text{as } s,t \downarrow 0.$$

Then, exactly as in Gnedenko and Kolmogorov (1954), page 173, one can show that

$$R(s,t) = \int_s^{1-t} Q(u)\,du / S(s,1-t) \to 0, \quad \text{as } s,t \downarrow 0. \tag{2.4}$$

Hence, using the well-known decomposition

$$\begin{aligned} \sigma^2(s,1-t) &= tQ^2(1-t) + sQ^2(s) + S^2(s,1-t) \\ &\quad - \left\{ tQ(1-t) + sQ(s) + \int_s^{1-t} Q(u)\,du \right\}^2, \end{aligned} \tag{2.5}$$

writing $f(s,t) \sim g(s,t)$ when their ratio goes to one as $s,t \downarrow 0$ and introducing $r(s,t) = \{tQ^2(1-t) + sQ^2(s)\}/S^2(s,1-t)$, we have

$$\begin{aligned} &\frac{tQ^2(1-t) + sQ^2(s)}{\sigma^2(s,1-t)} \\ &= \frac{r(s,t)}{r(s,t) + 1 - \{[(tQ(1-t) + sQ(s))/S(s,1-t)] + R(s,t)\}^2} \\ &\sim \frac{r(s,t)}{r(s,t) + 1 - \left\{(tQ(1-t) + sQ(s))^2/S^2(s,1-t)\right\}} \\ &\le \frac{r(s,t)}{r(s,t) + 1 - (s \vee t)\left\{\left(t^{1/2}|Q(1-t)| + s^{1/2}|Q(s)|\right)^2/S^2(s,1-t)\right\}} \\ &\le \frac{r(s,t)}{r(s,t) + 1 - 2(s \vee t) r(s,t)} \sim \frac{r(s,t)}{r(s,t)+1}. \end{aligned}$$

□

LEMMA 2.2. *For any underlying distribution,* $Y_n = Z_n + o_P(1)$, *where*

$$Z_n = -b_n^{-1} \int_{m_n/n}^{1-k_n/n} B_n(s)\,dQ(s)$$

*is a standard normal random variable for each* $n \ge 1$.

PROOF. Observe that by (2.1) we have for any $0 < \nu < 1/4$ that

$$\begin{aligned} |Y_n - Z_n| &= O_P(n^{-\nu}) b_n^{-1} \int_{m_n/n}^{1-k_n/n} (u(1-u))^{1/2-\nu}\,dQ(u) \\ &\le O_P(n^{-\nu}) b_n^{-1} \left\{ \int_{m_n/n}^{1/2} u^{1/2-\nu}\,dQ(u) + \int_{1/2}^{1-k_n/n} (1-u)^{1/2-\nu}\,dQ(u) \right\}. \end{aligned}$$

Applying integration by parts, it is easy to see by Lemma 2.1 that

$$\begin{aligned}|Y_n - Z_n| &= O_p(m_n^{-\nu}) b_n^{-1} |Q(m_n/n)| (m_n/n)^{1/2} + O_p(m_n^{-\nu}) \\ &\quad + O_p(k_n^{-\nu}) b_n^{-1} |Q(1 - k_n/n)| (k_n/n)^{1/2} + O_p(k_n^{-\nu}) \\ &= O_p\big((m_n \wedge k_n)^{-\nu}\big).\end{aligned}$$

Since $E\{Z_n\} = 0$ and $\operatorname{Var}\{Z_n\} = 1$, $Z_n$ is a standard normal random variable. □

PROOF OF THEOREM 1. For notational convenience the primes will be dropped from the $\{n'\}$ sequence in the proofs of Theorems 1 and 2. Any convergence and stochastic order relations should be understood as taking place along $\{n'\}$.

Set $Z_{1,n} = (n/m_n)^{1/2} B_n(m_n/n)$ and $Z_{2,n} = (n/k_n)^{1/2} B_n(1 - k_n/n)$. Note that the claimed bounds on $\Psi_1$ and $\Psi_2$ follow by an elementary argument based on the definition of $\Psi_{i,n}$ and $r_{i,n}$, $i = 1, 2$. For any $M > 0$, with $I(\cdot)$ being the indicator function, write

$$I_{1,n}(M) = I(|Z_{1,n}| < M), \qquad I_{2,n}(M) = I(|Z_{2,n}| < M),$$

$$Y_{1,n}(M) = I_{1,n}(M) \int_0^{-Z_{1,n}} (Z_{1,n} + x)\, d\Psi_{1,n}(x),$$

$$Y_{2,n}(M) = I_{2,n}(M) \int_{-Z_{2,n}}^{0} (Z_{2,n} + x)\, d\Psi_{2,n}(x).$$

We separate the essential steps of the proof as lemmas.

LEMMA 2.3. *For $M > 0$, under the assumptions of the first part of Theorem* 1,

$$I_{1,n}(M) R_{1,n} = Y_{1,n}(M) + o_P(1), \tag{2.6}$$

$$I_{2,n}(M) R_{2,n} = Y_{2,n}(M) + o_P(1). \tag{2.7}$$

PROOF. First consider (2.6). Choose any $0 < \nu < 1/4$. From (2.2) it is easy to conclude that

$$\big(n/m_n^{1/2}\big)\big(U_{m_n,n} - m_n/n\big) + Z_{1,n} = O_P(m_n^{-\nu}). \tag{2.8}$$

Introduce the indicators

$$I_{1,n}^*(M) = I\Big(|Z_{1,n}| < M \text{ and } \big(n/m_n^{1/2}\big)|U_{m_n,n} - m_n/n| < M\Big).$$

In view of (2.8) it is obvious that

$$P\{I_{1,n}^*(M) = I_{1,n}(M)\} \to 1, \tag{2.9}$$

from which it follows that $I_{1,n}(M) R_{1,n} = I_{1,n}^*(M) R_{1,n} + o_P(1)$. Observe that for any $0 < \nu < 1/4$ and all $n$ sufficiently large,

$$I_{1,n}^*(M)\left|R_{1,n} - b_n^{-1}\int_{m_n/n}^{U_{m_n,n}}\left\{B_n(s) + \left(s - \frac{m_n}{n}\right)n^{1/2}\right\}dQ(s)\right|$$
$$\le \sup_{m_n/2n \le s \le 2m_n/n}\left(\frac{n}{m_n}\right)^{1/2}|n^{1/2}\{G_n(s) - s\} - B_n(s)|$$
$$\times b_n^{-1}\left(\frac{m_n}{n}\right)^{1/2}\int_{m_n/n - Mm_n^{1/2}/n}^{m_n/n + Mm_n^{1/2}/n} dQ(s),$$

which by (2.1) and (1.3) equals $(\Psi_{1,n}(M) - \Psi_{1,n}(-M))O_p(m_n^{-\nu}) = o_p(1)$. Next we have

$$I_{1,n}^*(M)\left|b_n^{-1}\int_{m_n/n}^{U_{m_n,n}}\left\{B_n(s) + \left(s - \frac{m_n}{n}\right)n^{1/2}\right\}dQ(s)\right.$$
$$\left. - b_n^{-1}\int_{m_n/n}^{m_n/n - B_n(m_n/n)/n^{1/2}}\left\{B_n(s) + \left(s - \frac{m_n}{n}\right)n^{1/2}\right\}dQ(s)\right|$$
$$\le \left\{\sup_{s \in J_n}\left(\frac{n}{m_n}\right)^{1/2}\left|B_n(s) + \left(s - \frac{m_n}{n}\right)n^{1/2}\right|\right\} \tag{2.10}$$
$$\times\left\{|\Psi_{1,n}(\tilde{U}_{m_n,n}) - \Psi_{1,n}(-Z_{1,n})|I_{1,n}^*(M)\right\},$$

where $J_n = [m_n/n - Mm_n^{1/2}/n, m_n/n + Mm_n^{1/2}/n]$ and $\tilde{U}_{m_n,n} = n(U_{m_n,n} - m_n/n)/m_n^{1/2}$. The first factor on the right-hand side of inequality (2.10) can easily be shown to be a $O_P(1)$ random variable. Now we show the following.

CLAIM. *Under the assumptions of the first part of Theorem* 1,

$$D_n \equiv I_{1,n}^*(M)|\Psi_{1,n}(\tilde{U}_{m_n,n}) - \Psi_{1,n}(-Z_{1,n})| \to_P 0.$$

PROOF. Since $\Psi_1$ is a left-continuous function on $[-M, M]$, given any $\varepsilon > 0$, we can find divisory points $-M = t_0 < t_1 < \cdots < t_{k+1} = M$ such that

$$|\Psi_1(s) - \Psi_1(t)| < \varepsilon/2, \quad \text{whenever } s, t \in (t_i, t_{i+1}], \qquad i = 0, \dots, k. \tag{2.11}$$

[Cf. the left-continuous version of Lemma 1 on page 110 of Billingsley (1968).] For any sufficiently small $\delta > 0$ choose continuity points $l_i < r_i$ of $\Psi_1$, $i = 0, \dots, k$, such that $[l_i, r_i] \subset (t_i, t_{i+1}]$, $l_i - t_i < \delta$ and $t_{i+1} - r_i < \delta$ for each $i = 0, \dots, k$. Let $I(\delta, n)$ denote the indicator function of the event that $I_{1,n}^*(M) = 1$ and both $-Z_{1,n}$ and $\tilde{U}_{m_n,n}$ lie in $[l_i, r_i]$ for some $i = 0, \dots, k$. Since $\Psi_{1,n}$ is a nondecreasing function, we see that on the event indicated by $I(\delta, n)$,

$$|\Psi_{1,n}(\tilde{U}_{m_n,n}) - \Psi_{1,n}(-Z_{1,n})| \le \Psi_{1,n}(r_i) - \Psi_{1,n}(l_i),$$

for some $i = 0, \dots, k$. Hence, using (1.3), (2.11) and the triangle inequality, we see that for all $n$ sufficiently large $I(\delta, n)D_n \le \varepsilon$. But it is straightforward to establish by (2.8) that

$$\lim_{\delta \downarrow 0} \liminf_{n \to \infty} P\{I(\delta, n) = I^*_{1,n}(M)\} = 1.$$

Thus we conclude that $P\{D_n \le \varepsilon\} \to 1$ for all $\varepsilon > 0$, proving the claim. □

Now we know that the right-hand side of (2.10) goes to zero in probability. Observe next that by the change of variables $x = (s - m_n/n)(n/m_n^{1/2})$,

$$I^*_{1,n}(M) b_n^{-1} \int_{m_n/n}^{m_n/n - B_n(m_n/n)/n^{1/2}} \left\{ B_n(s) + \left(s - \frac{m_n}{n}\right) n^{1/2} \right\} dQ(s)$$

$$= I^*_{1,n}(M) \int_0^{-Z_{1,n}} \left\{ \left(\frac{n}{m_n}\right)^{1/2} B_n\left(\frac{m_n}{n} + x\frac{m_n^{1/2}}{n}\right) + x \right\} d\Psi_{1,n}(x),$$

which by (2.9) equals

$$I_{1,n}(M) \int_0^{-Z_{1,n}} \left\{ \left(\frac{n}{m_n}\right)^{1/2} B_n\left(\frac{m_n}{n} + x\frac{m_n^{1/2}}{n}\right) + x \right\} d\Psi_{1,n}(x) + o_P(1).$$

Notice further that

$$I_{1,n}(M) \left| \int_0^{-Z_{1,n}} \left\{ \left[ \left(\frac{n}{m_n}\right)^{1/2} B_n\left(\frac{m_n}{n} + x\frac{m_n^{1/2}}{n}\right) + x \right] - Z_{1,n} - x \right\} d\Psi_{1,n}(x) \right|$$

$$\le \{\Psi_{1,n}(M) - \Psi_{1,n}(-M)\} \sup_{-M \le x \le M} \left(\frac{n}{m_n}\right)^{1/2}$$

$$\times \left| B_n\left(\frac{m_n}{n} + x\frac{m_n^{1/2}}{n}\right) - B_n\left(\frac{m_n}{n}\right) \right|,$$

where the first factor is bounded. Using now the representation $B_n(t) = W_n(t) - tW_n(1)$, $0 \le t \le 1$, where $W_n(\cdot)$ is a standard Wiener process, and applying standard bounds on the distribution of the absolute supremum of a Wiener process, it can be seen that the second factor converges to zero in probability. This completes the proof of (2.6). Since the proof of (2.7) follows in the same way, Lemma 2.3 is completely proved. □

For each $M > 0$ set

$$V_1(M) = I(|Z_1| < M) \int_0^{-Z_1} (Z_1 + x)\, d\Psi_1(x)$$

$$+ Z + I(|Z_2| < M) \int_{-Z_2}^{0} (Z_2 + x)\, d\Psi_2(x)$$

$$\equiv Y_1(M) + Z + Y_2(M),$$

where $(Z_1, Z, Z_2)$ is the trivariate normal random vector as in the statement of Theorem 1. The next step in the proof of the first part of Theorem 1 is as follows.

LEMMA 2.4. *Under the assumptions of the first part of Theorem* 1,

$$Y_{1,n}(M) + Z_n + Y_{2,n}(M) \to_{\mathscr{D}} V_1(M),$$

*for each* $M > 0$, *where* $Z_n$ *is defined in Lemma* 2.2.

PROOF. Elementary calculations show that for each $n \geq 1$, $(Z_{1,n}, Z_n, Z_{2,n})$ is a trivariate normal random vector with mean vector zero and covariance matrix

$$\begin{pmatrix} 1 - m_n/n & r_{1,n} & \left(m_n k_n/n^2\right)^{1/2} \\ r_{1,n} & 1 & r_{2,n} \\ \left(m_n k_n/n^2\right)^{1/2} & r_{2,n} & 1 - k_n/n \end{pmatrix},$$

which by assumption (1.4) and by (1.1) converges to $\Sigma$ in (1.5). Thus

$$(Z_{1,n}, Z_n, Z_{2,n}) \to_{\mathscr{D}} (Z_1, Z, Z_2).$$

Applying the Skorohod representation theorem, there exists a probability space carrying a copy of $(Z_1, Z, Z_2)$ and a sequence of random vectors $(Z_{1,n}^*, Z_n^*, Z_{2,n}^*) =_{\mathscr{D}} (Z_{1,n}, Z_n, Z_{2,n})$ for each $n \geq 1$ such that

$$(Z_{1,n}^*, Z_n^*, Z_{2,n}^*) \to (Z_1, Z, Z_2) \quad \text{a.s.} \tag{2.12}$$

Let $Y_{1,n}^*(M)$ and $Y_{2,n}^*(M)$ denote the corresponding versions of $Y_{1,n}(M)$ and $Y_{2,n}(M)$. To finish the proof of Lemma 2.4, it is enough to show that

$$Y_{i,n}^*(M) - Y_i(M) \to 0 \quad \text{a.s.,} \qquad i = 1, 2. \tag{2.13}$$

Notice that

$$\left| Y_{1,n}^*(M) - I(|Z_{1,n}^*| < M) \int_0^{-Z_{1,n}^*} (Z_1 + x)\, d\Psi_{1,n}(x) \right| \leq \{\Psi_{1,n}(M) - \Psi_{1,n}(-M)\} |Z_{1,n}^* - Z_1|,$$

which by (2.12) and (1.3) converges almost surely to zero. Next we have

$$\begin{aligned} I(|Z_{1,n}^*| < M, |Z_1| < M) \bigg| \int_0^{-Z_{1,n}^*} (Z_1 + x)\, d\Psi_{1,n}(x) \\ - \int_0^{-Z_1} (Z_1 + x)\, d\Psi_{1,n}(x) \bigg| \\ \leq 2MI(|Z_{1,n}^*| < M, |Z_1| < M) |\Psi_{1,n}(-Z_1) - \Psi_{1,n}(-Z_{1,n}^*)|, \end{aligned} \tag{2.14}$$

which, since $-Z_1$ is almost surely equal to a continuity point of $\Psi_1$ and each $\Psi_{1,n}$ is nondecreasing, converges to zero almost surely by (2.12) and (1.3). It is easy to see by (2.12) that

$$I(|Z_{1,n}^*| < M) - I(|Z_{1,n}^*| < M, |Z_1| < M) \to 0 \quad \text{a.s.,}$$

$$I(|Z_1| < M) - I(|Z_{1,n}^*| < M, |Z_1| < M) \to 0 \quad \text{a.s.}$$

Therefore the previous relations and a final application of (1.3) give (2.13) for $i = 1$. The case $i = 2$ of (2.13) follows in the same way. □

It is now simple to finish the proof of the first part of Theorem 1. From (2.3) and Lemma 2.2 we have

$$T_n \equiv a_n^{-1}\left\{\sum_{i=m_n+1}^{n-k_n} X_{1,n} - \mu_n\right\} = R_{1,n} + Z_n + R_{2,n} + o_P(1).$$

For $M > 0$ set $T_n(M) = I_{1,n}(M)R_{1,n} + Z_n + I_{2,n}(M)R_{2,n}$. Observe, trivially,

$$\lim_{M\to\infty}\limsup_{n\to\infty} P\{|T_n - T_n(M)| > \varepsilon\} = 0,$$

for all $\varepsilon > 0$, and by combining Lemmas 2.3 and 2.4, for each $M > 0$,

$$T_n(M) \to_{\mathscr{D}} V_1(M).$$

Since, obviously,

$$V_1(M) \to_{\mathscr{D}} V_1, \quad \text{as } M \to \infty,$$

the first part of Theorem 1 follows from Theorem 4.2 in Billingsley (1968).

In order to establish the second part of Theorem 1, we need the following lemma, whose proof follows by integration by parts and an application of Lemma 2.1 and (2.4). The routine details are omitted.

LEMMA 2.5. *Whenever* $\operatorname{Var}(X) = \infty$,

$$\lim_{s,t\downarrow 0}\left\{\int_s^{1/2} u\,dQ(u) + \int_{1/2}^{1-t}(1-u)\,dQ(u)\right\}\Big/\sigma(s,1-t) = 0.$$

Set

$$W_{1,n} = -b_n^{-1}\int_{m_n/n}^{1/2} B_n(s)\,dQ(s) \quad\text{and}\quad W_{2,n} = -b_n^{-1}\int_{1/2}^{1-k_n/n} B_n(s)\,dQ(s).$$

Clearly, $Z_n = W_{1,n} + W_{2,n}$ and $(Z_{1,n}, W_{1,n}, W_{2,n}, Z_{2,n})$ is a quadrivariate normal random vector with mean vector zero,

$$\operatorname{Cov}(Z_{1,n}, W_{2,n}) = -\left(\frac{m_n}{n}\right)^{1/2}\frac{1}{b_n}\int_{1/2}^{1-k_n/n}(1-s)\,dQ(s),$$

$$\operatorname{Cov}(W_{1,n}, W_{2,n}) = \frac{1}{b_n^2}\int_{m_n/n}^{1/2} s\,dQ(s)\int_{1/2}^{1-k_n/n}(1-t)\,dQ(t),$$

$$\operatorname{Cov}(Z_{2,n}, W_{1,n}) = -\left(\frac{k_n}{n}\right)^{1/2}\frac{1}{b_n}\int_{m_n/n}^{1/2} s\,dQ(s).$$

Application of Lemma 2.5 shows that each of these covariances converges to zero. It is easy now to see by (1.6) and the previous considerations that

$$(Z_{1,n}, W_{1,n}, W_{2,n}, Z_{2,n}) \to_{\mathscr{D}} (Z_1, W_1, W_2, Z_2).$$

This relation combined with the proof of the first part of the theorem completes the proof of the second part of Theorem 1. □

PROOF OF THEOREM 2. Assumptions (1.7) and (1.8) combined with (1.9) imply that $a_n/A_n \to 0$, which when combined with Lemma 2.7 gives $\Psi_1^*(c) = 0$ and $\Psi_2^*(-c) = 0$ for all $c > 0$. Notice that by (2.3),

$$A_n^{-1}\left\{\sum_{i=m_n+1}^{n-k_n} X_{1,n} - \mu_n\right\} = \frac{a_n}{A_n}R_{1,n} + \frac{a_n}{A_n}Y_n + \frac{a_n}{A_n}R_{2,n},$$

which by Lemma 2.2 equals $a_n R_{1,n}/A_n + a_n R_{n,2}/A_n + o_p(1)$. That this random variable converges in distribution to $V_2$ follows, after obvious changes of notation, by the same method used to prove the first part of Theorem 1. □

PROOF OF THEOREM 3. To keep track of all subsequences and sub-subsequences of the positive integers occurring in the proof, the following notation will be used: $\{n_1\}$ denotes a subsequence of the positive integers, $\{n_2\}$ denotes a subsequence of $\{n_1\}$, etc. For convenient reference later on we state the following version of the Helly–Bray theorem.

LEMMA 2.6. *Let $f_n$ be a sequence of nondecreasing left-continuous functions defined on $\mathbb{R}$ such that $f_n(0) = 0$ for all $n \geq 1$. If for all $x$,*

$$\limsup_{n\to\infty} |f_n(x)| < \infty,$$

*then there exists a subsequence $\{n_1\}$ of $\{n\}$ and a nondecreasing left-continuous function $f$ with $f(0) = 0$ such that $f_{n_1}(x) \to f(x)$ as $n_1 \to \infty$ at every continuity point $x$ of $f$.*

First assume the existence of a sequence $C_n$ having the stated properties. Choose any subsequence $\{n_1\}$ of $\{n\}$ and a further subsequence $\{n_2\}$ of $\{n_1\}$ such that (1.11) and (1.12) hold. Consider the case when the limit in (1.11) is equal to zero. Then, by assumption, at least one of the two $\Psi_i$, $i = 1, 2$, functions appearing in (1.12) is not identically equal to zero. It is apparent that this entails that for some $c$ and $i = 1$ or 2,

$$|\Psi_{1,n_2}(c)| \to \infty, \quad \text{as } n_2 \to \infty.$$

Applying Theorem 2, with $V_2$ of the form given there, we have

$$C_{n_2}^{-1}\left\{\sum_{i=m_{n_2}+1}^{n_2-k_{n_2}} X_{i,n_2} - \mu_{n_2}\right\} \to_{\mathscr{D}} V_2, \quad \text{as } n_2 \to \infty.$$

Next suppose that the limit in (1.11) is equal to a positive constant $0 < \gamma < \infty$. On account of (1.12), this implies that for $i = 1, 2$,

$$\Psi_{1,n_2}(c) \to \Psi_i(c), \quad \text{as } n_2 \to \infty, \tag{2.15}$$

at every continuity point $c$ of $\Psi_i \equiv \Psi_i^*/\gamma$. Observing that the sequences $r_{1,n_2}$ and $r_{2,n_2}$ are uniformly bounded with all terms being nonpositive, we can find a further subsequence $\{n_3\}$ of $\{n_2\}$ and finite nonpositive constants $r_1$ and $r_2$ such that for $i = 1, 2$,

$$r_{1, n_3} \to r_i, \quad \text{as } n_3 \to \infty.$$

Since (2.15) also holds along the sequence $\{n_3\}$, Theorem 1 yields

$$C_{n_3}^{-1}\left\{\sum_{i=m_{n_3}+1}^{n_3-k_{n_3}} X_{1, n_3} - \mu_{n_3}\right\} \to_{\mathscr{D}} \gamma V_1, \quad \text{as } n_3 \to \infty,$$

where $V_1$ is of the form given in Theorem 1. Proposition 2 in the Appendix says that neither of these random variables $V_1$ and $V_2$ is degenerate. Thus by setting $A_n = C_n$ and $B_n = \mu_n$, we conclude that the resulting sequence of random variables in (1.10) is stochastically compact.

Now assume that for appropriate sequences of normalizing and centering constants $A_n > 0$ and $B_n$ the sequence of random variables in (1.10) is stochastically compact. Choose any subsequence $\{n_1\}$ of $\{n\}$. Suppose first that for all $c \in \mathbb{R}$ and $i = 1, 2$,

$$\limsup_{n_1 \to \infty} |\Psi_{i, n_1}(c)| < \infty.$$

By Lemma 2.6 along with the boundedness of the sequences $r_{1, n_1}$ and $r_{2, n_2}$, we can choose a further subsequence $\{n_2\}$ of $\{n_1\}$ such that for $i = 1, 2$,

$$\Psi_{i, n_2}(c) \to \Psi_i(c), \quad \text{as } n_2 \to \infty,$$

at every continuity point $c$ of $\Psi_i$, where $\Psi_1$ and $\Psi_2$ are nondecreasing left-continuous functions equal to zero at zero, and for nonpositive finite constants $r_1$ and $r_2$ for $i = 1, 2$,

$$r_{i, n_2} \to r_i, \quad \text{as } n_2 \to \infty.$$

Applying Theorem 1, we have

$$a_{n_2}^{-1}\left\{\sum_{i=m_{n_2}+1}^{n_2-k_{n_2}} X_{i, n_2} - \mu_{n_2}\right\} \to_{\mathscr{D}} V_1.$$

Since, by assumption, the sequence in (1.10) is stochastically compact, there exists a further subsequence $\{n_3\}$ of $\{n_2\}$ such that

$$A_{n_3}^{-1}\left\{\sum_{i=m_{n_3}+1}^{n_3-k_{n_3}} X_{i, n_3} - B_{n_3}\right\}$$

converges in distribution to a nondegenerate random variable $V$. Therefore, by the convergence-of-types theorem [Gnedenko and Kolmogorov (1954), pages 40–42], there exist $0 < \gamma < \infty$ and $-\infty < \delta < \infty$, such that

$$A_{n_3}^{-1} a_{n_3} \to \gamma \quad \text{and} \quad A_{n_3}^{-1}(\mu_{n_3} - B_{n_3}) \to \delta, \quad \text{as } n_3 \to \infty,$$

and hence for $i = 1, 2$,

$$A_{n_3}^{-1} a_{n_3} \Psi_{i, n_3}(c) \to \gamma \Psi_i(c) \equiv \Psi_i^*(c),$$

at every continuity point $c$ of $\Psi_i^*$ and $V = \gamma V_1 + \delta$.

Next assume that for some $c \in \mathbb{R}$ and $i = 1$ or 2,

$$\limsup_{n_1 \to \infty} |\Psi_{i,n_1}(c)| = \infty. \tag{2.16}$$

Obviously, (2.16), holding for some $c$ and $i = 1$ or 2, allows us to choose a further subsequence $\{n_2\}$ of $\{n_1\}$ such that for the same $c$ and $i$ for which (2.16) is true, we have

$$\lim_{n_2 \to \infty} |\Psi_{i,n_2}(c)| = \infty. \tag{2.17}$$

At this stage, in order to complete the proof of Theorem 3, we require a number of technical lemmas.

LEMMA 2.7. *For all $c > 0$,*

$$\limsup_{n \to \infty} |\Psi_{1,n}(c)| < \infty, \tag{2.18}$$

$$\limsup_{n \to \infty} |\Psi_{2,n}(-c)| < \infty. \tag{2.19}$$

PROOF. First consider (2.18). We shall assume that $Q(s) \to -\infty$ as $s \downarrow 0$, since otherwise the assertion is trivial. Choose any $c > 0$. The following inequality for large $n$ when combined with Lemma 2.1 gives (2.18):

$$|\Psi_{1,n}(c)| \le b_n^{-1} 2\left(\frac{m_n}{n}\right)^{1/2} \left|Q\left(\frac{m_n}{n}\right)\right|.$$

The second assertion is proven similarly. □

LEMMA 2.8. *For $i = 1, 2$,*

$$\lim_{M \to \infty} \liminf_{n \to \infty} P\{|R_{i,n}| < M\} > 0. \tag{2.20}$$

PROOF. First consider the case $i = 1$. Observe that

$$|R_{1,n}| \le b_n^{-1} n^{1/2} \left|G_n\left(\frac{m_n}{n}\right) - \frac{m_n}{n}\right| \left|Q(U_{m_n,n}) - Q\left(\frac{m_n}{n}\right)\right|.$$

Choose any $c > 0$. On the event $A_n(c) = \{0 \le U_{m_n,n} - m_n/n \le c m_n^{1/2}/n\}$, we have

$$|R_{1,n}| \le m_n^{-1/2} n \left|G_n\left(\frac{m_n}{n}\right) - \frac{m_n}{n}\right| |\Psi_{1,n}(c)|. \tag{2.21}$$

By (2.1) (or the classical central limit theorem)

$$m_n^{-1/2} n \left\{G_n\left(\frac{m_n}{n}\right) - \frac{m_n}{n}\right\} \to_{\mathscr{D}} N(0,1) \tag{2.22}$$

and by (2.2) [or cf. Balkema and de Haan (1975)]

$$m_n^{-1/2} n \left\{U_{m_n,n} - m_n/n\right\} \to_{\mathscr{D}} N(0,1), \tag{2.23}$$

which in turn implies that

$$P\{A_n(c)\} \to P\{0 \le N(0,1) \le c\}. \tag{2.24}$$

That (2.20) holds for the case $i = 1$ is now a direct consequence of (2.18), (2.21), (2.22) and (2.24). The case $i = 2$ is proven analogously. □

We shall say that two sequences of random variables $L_n$ and $M_n$ are asymptotically independent if

$$\sup_{x,y} |P\{L_n \le x,\, M_n \le y\} - P\{L_n \le x\}\, P\{M_n \le y\}| \to 0.$$

The next lemma follows from a special case of Satz 4 of Rossberg (1967) by an elementary method similar to the one used in the proof of Theorem 2 of Mason (1985). The simple details are omitted.

LEMMA 2.9. *The two sequences of random variables* $|R_{1,n}|$ *and* $|R_{2,n}|$ *are asymptotically independent.*

LEMMA 2.10. *Let* $\Delta_{1,n}$ *and* $\Delta_{2,n}$ *be two sequences of random variables such that* $\Delta_{1,n} + \Delta_{2,n}$ *is stochastically bounded, the sequences* $|\Delta_{1,n}|$ *and* $|\Delta_{2,n}|$ *are asymptotically independent and for* $i = 1,2$,

$$\lim_{M\to\infty} \liminf_{n\to\infty} P\{|\Delta_{i,n}| < M\} > 0. \tag{2.25}$$

*Then both* $\Delta_{1,n}$ *and* $\Delta_{2,n}$ *are stochastically bounded.*

PROOF. Choose any $M > 0$. We see that

$$P\{|\Delta_{1,n} + \Delta_{2,n}| > M\} \ge P\{|\Delta_{1,n}| > 2M,\, |\Delta_{2,n}| < M\} + P\{|\Delta_{2,n}| > 2M,\, |\Delta_{1,n}| < M\},$$

which by the asymptotic independence assumption equals

$$P\{|\Delta_{1,n}| > 2M\}P\{|\Delta_{2,n}| < M\} + P\{|\Delta_{2,n}| > 2M\}P\{|\Delta_{1,n}| < M\} + o(1).$$

By assumption (2.25) for some $\varepsilon > 0$ this last expression is for large enough $M$ and $n$ greater than or equal to

$$\varepsilon\big(P\{|\Delta_{1,n}| > 2M\} + P\{|\Delta_{2,n}| > 2M\}\big) + o(1).$$

Thus, since necessarily,

$$\lim_{M\to\infty} \limsup_{n\to\infty} P\{|\Delta_{1,n} + \Delta_{2,n}| > M\} = 0,$$

$$\lim_{M\to\infty} \limsup_{n\to\infty} P\{|\Delta_{i,n}| > 2M\} = 0, \qquad i = 1,2,$$

implying that both $\Delta_{1,n}$ and $\Delta_{2,n}$ are stochastically bounded. □

LEMMA 2.11. *Whenever there exists a subsequence* $\{n_1\}$ *of* $\{n\}$ *with accompanying sequences of normalizing and centering constants* $A_{n_1} > 0$ *and* $B_{n_1}$ *such* *that the sequence*

$$A_{n_1}^{-1}\left\{\sum_{i=m_{n_1}+1}^{n_1-k_{n_1}} X_{i,n_1} - B_{n_1}\right\} \tag{2.26}$$

*is stochastically bounded, then both sequences* $a_{n_1}R_{i,n_1}/(A_{n_1} \vee a_{n_1})$, $i = 1, 2$, *are stochastically bounded.*

PROOF. Notice that by (2.3) we can write the expression in (2.26) as

$$T_{n_1} \equiv A_{n_1}^{-1}a_{n_1}\left(Y_{n_1} + R_{1,n_1} + R_{2,n_1}\right) - A_{n_1}^{-1}\left(B_{n_1} - \mu_n\right).$$

Consider an independent copy,

$$T'_{n_1} \equiv A_{n_1}^{-1}a_{n_1}\left(Y'_{n_1} + R'_{1,n_1} + R'_{2,n_1}\right) - A_{n_1}^{-1}\left(B_{n_1} - \mu_n\right),$$

of $T_{n_1}$. Since both $T_{n_1}$ and $T'_{n_1}$ are stochastically bounded,

$$T_{n_1} - T'_{n_1} = A_{n_1}^{-1}a_{n_1}\left(Y_{n_1} + R_{1,n_1} + R_{2,n_1} - Y'_{n_1} - R'_{1,n_1} - R'_{2,n_1}\right)$$

is also stochastically bounded, and from $A_{n_1}/(A_{n_1} \vee a_{n_1}) \le 1$ we see that

$$a_{n_1}\left(Y_{n_1} + R_{1,n_1} + R_{2,n_1} - Y'_{n_1} - R'_{1,n_1} - R'_{2,n_1}\right)/\left(A_{n_1} \vee a_{n_1}\right)$$

is stochastically bounded. Lemma 2.2 implies that $Y_{n_1}$ and $Y'_{n_1}$ are stochastically bounded, from which it is easy to infer that

$$a_{n_1}\left(R_{1,n_1} + R_{2,n_1} - R'_{1,n_1} - R'_{2,n_1}\right)/\left(A_{n_1} \vee a_{n_1}\right)$$

is stochastically bounded. Observe that for any $M > 0$,

$$P\left\{a_{n_1}|R_{1,n_1} + R_{2,n_1}|/\left(A_{n_1} \vee a_{n_1}\right) < 2M\right\} \ge P\left\{|R_{1,n_1}| < M, |R_{2,n_1}| < M\right\}.$$

By Lemma 2.9 the right-hand side of this last inequality equals

$$P\left\{|R_{1,n_1}| < M\right\}P\left\{|R_{2,n_1}| < M\right\} + o(1).$$

Hence by Lemma 2.8, we have

$$\lim_{M\to\infty}\liminf_{n\to\infty} P\left\{a_{n_1}|R_{1,n_1} + R_{2,n_1}|/\left(A_{n_1} \vee a_{n_1}\right) < 2M\right\} > 0.$$

Since $R'_{1,n_1} + R'_{2,n_1} =_{\mathscr{D}} R_{1,n_1} + R_{2,n_1}$ and $R_{1,n_1} + R_{2,n_1}$ and $R'_{1,n_1} + R'_{2,n_1}$ are independent, we conclude from Lemma 2.10 that the sequence $a_{n_1}|R_{1,n_1} + R_{2,n_1}|/(A_{n_1} \vee a_{n_1})$ is stochastically bounded. A second application of Lemma 2.10 yields that both $a_{n_1}R_{i,n_1}/(A_{n_1} \vee a_{n_1})$, $i = 1, 2$, are stochastically bounded. □

LEMMA 2.12. *Whenever there exists a subsequence* $\{n_1\}$ *of* $\{n\}$ *such that for some* $c$ *and* $i = 1$ *or* 2,

$$\left|\Psi_{i,n_1}(c)\right| \to \infty, \quad \text{as } n_1 \to \infty, \tag{2.27}$$

*and a sequence of positive constants* $A_{n_1}$ *such that both* $a_{n_1}R_{i,n_1}/(A_{n_1} \vee a_{n_1})$,

$i = 1, 2$, *are stochastically bounded, then*

$$A_{n_1}^{-1} a_{n_1} \to 0, \quad \text{as } n_1 \to \infty, \tag{2.28}$$

$$\limsup_{n_1 \to \infty} A_{n_1}^{-1} a_{n_1} \left| \Psi_{i, n_1}(c) \right| < \infty, \qquad i = 1, 2, \quad \text{for all } c. \tag{2.29}$$

PROOF. First assume that (2.27) holds for $i = 2$. By Lemma 2.7 necessarily $c > 0$. Choose any $d \ge c$. We get on the event

$$A_{n_1}(c, d) = \left\{ 1 - \frac{k_{n_1}}{n_1} + d \frac{k_{n_1}^{1/2}}{n_1} \le U_{n_1 - k_{n_1} - [c k_{n_1}^{1/2}] - 1, n_1} \right\},$$

$$\begin{aligned}
\frac{a_{n_1} R_{2, n_1}}{A_{n_1} \vee a_{n_1}} &\ge \frac{n_1}{(A_{n_1} \vee a_{n_1})} \\
&\quad \times \int_{1 - k_{n_1}/n_1}^{1 - k_{n_1}/n_1 + d k_{n_1}^{1/2}/n_1} \left\{ 1 - \frac{k_{n_1}}{n_1} - G_{n_1}\left( 1 - \frac{k_{n_1}}{n_1} + d \frac{k_{n_1}^{1/2}}{n_1} \right) \right\} dQ(s) \\
&\ge \frac{n_1}{(A_{n_1} \vee a_{n_1})} \int_{1 - k_{n_1}/n_1}^{1 - k_{n_1}/n_1 + d k_{n_1}^{1/2}/n_1} \left\{ 1 - \frac{k_{n_1}}{n_1} \right. \\
&\qquad \left. - G_{n_1}\left( U_{n_1 - k_{n_1} - [c k_{n_1}^{1/2}] - 1, n_1} \right) \right\} dQ(s) \\
&\ge (A_{n_1} \vee a_{n_1})^{-1} c k_{n_1}^{1/2} \left\{ Q\left( 1 - \frac{k_{n_1}}{n_1} + d \frac{k_{n_1}^{1/2}}{n_1} \right) - Q\left( 1 - \frac{k_{n_1}}{n_1} \right) \right\} \\
&= c a_{n_1} (A_{n_1} \vee a_{n_1})^{-1} \Psi_{2, n_1}(d) \ge 0.
\end{aligned}$$

Notice that by (2.2) we have

$$P\{A_{n_1}(c, d)\} \to P\{N(0,1) > d + c\} > 0, \quad \text{as } n_1 \to \infty.$$

This along with the fact that necessarily $\Psi_{2, n_1}(d) \to \infty$ and the assumption that $a_{n_1} R_{2, n_1} / (A_{n_1} \vee a_{n_1})$ is stochastically bounded forces (2.28) to hold. We also see immediately that for all $d \ge 0$,

$$0 \le \limsup_{n_1 \to \infty} A_{n_1}^{-1} a_{n_1} \Psi_{2, n_1}(d) < \infty,$$

which when combined with Lemma 2.7 establishes (2.29) for the case $i = 2$. The assertion that (2.29) also holds for the case $i = 1$ follows by a similar argument. □

We are now prepared to complete the proof of Theorem 3. By assumption, we can choose a further sequence $\{n_3\}$ of $\{n_2\}$ such that the sequence

$$A_{n_3}^{-1} \left\{ \sum_{i = m_{n_3} + 1}^{n_3 - k_{n_3}} X_{i, n_3} - B_{n_3} \right\}$$

converges in distribution to a nondegenerate random variable, and since necessarily (2.17) holds for the same $c$ and $i = 1$ or 2 along the sequence $\{n_3\}$, a straightforward application of Lemmas 2.11 and 2.12 yields

$$\limsup_{n_3 \to \infty} A_{n_3}^{-1} a_{n_3} |\Psi_{i, n_3}(c)| < \infty, \qquad i = 1, 2, \quad \text{for all } c,$$

and $A_{n_3}^{-1} a_{n_3} \to 0$ as $n_3 \to \infty$. Applying Lemma 2.6, we can choose a further subsequence $\{n_4\}$ of $\{n_3\}$ such that for nondecreasing left-continuous functions $\Psi_1^*$ and $\Psi_2^*$ equal to zero at zero, for $i = 1, 2$,

$$A_{n_4}^{-1} a_{n_4} \Psi_{i, n_4}(c) \to \Psi_i^*(c), \quad \text{as } n_4 \to \infty,$$

at every continuity point $c$ of $\Psi_i^*$. Thus by Theorem 2

$$A_{n_4}^{-1} \left\{ \sum_{i = m_{n_4} + 1}^{n_4 - k_{n_4}} X_{i, n_4} - \mu_{n_4} \right\} \to_{\mathscr{D}} V_2,$$

where $V_2$ is of the form given in Theorem 2, but since $\{n_4\} \subset \{n_3\}$,

$$A_{n_4}^{-1} \left\{ \sum_{i = m_{n_4} + 1}^{n_4 - k_{n_4}} X_{i, n_4} - B_{n_4} \right\}$$

converges in distribution to a nondegenerate random variable $V$. Therefore by the convergence-of-types theorem $V_2 =_{\mathscr{D}} V + \delta$ for some $-\infty < \delta < \infty$. This implies that $V_2$ is nondegenerate and, hence, at least one of the two functions $\Psi_1^*$ and $\Psi_2^*$ is not identically equal to zero. Setting $C_n = A_n$ completes the proof of Theorem 3. □

Proof of Theorem 4. First assume that (1.15) holds. Then by Theorem 1 we have (1.14) with $A_n = a_n$ and $B_n = \mu_n$, since for every subsequence $\{n_1\}$ of $\{n\}$ there exists a further subsequence $\{n_2\}$ of $\{n_1\}$ such that $r_{i, n_2} \to r_i$ for appropriate nonpositive constants $r_i$, $i = 1, 2$.

Next assume that there exist sequences of normalizing and centering constants $A_n > 0$ and $B_n$ such that (1.14) is true, but there exists a subsequence $\{n_1\}$ of $\{n\}$, a $c$, and $i = 1$ or 2 such that

$$\lim_{n_1 \to \infty} |\Psi_{i, n_1}(c)| > 0. \tag{2.30}$$

Since the sequence in (1.14) is obviously stochastically compact, Theorem 3 implies that there exists a further subsequence $\{n_2\}$ of $\{n_1\}$ such that

$$A_{n_2}^{-1} \left\{ \sum_{i = m_{n_2} + 1}^{n_2 - k_{n_2}} X_{i, n_2} - B_{n_2} \right\} \tag{2.31}$$

converges in distribution to an affine transform of either a $V_1$ or a $V_2$ random variable. [Note that if $V_1$ is the case, due to (2.30), at least one of the functions $\Psi_1$ or $\Psi_2$ is not identically equal to zero. Moreover, by the discussion in the next section, we must have $\mathrm{Var}(X) = \infty$. So we can also assume that $\{n_2\}$ is such that $\sigma_{i,n_2} \to \sigma_i$ as $n_2 \to \infty$ for $i = 1,2$ with $\sigma_1^2 + \sigma_2^2 = 1$, implying that the random variable $Z$ in $V_1$ has the representation given in the second part of the statement of Theorem 1.] However, since we are assuming that (1.14) holds, the sequence of random variables in (2.31) must converge in distribution to a standard normal random variable, forcing the $V_1$ or $V_2$ random variable to be a nondegenerate normal random variable. In either case, Proposition 1 in the Appendix says that this cannot happen. Thus (1.15) must be true. This completes the proof of Theorem 4. □

PROOF OF THEOREM 5. Note that with $\{m_n\}$ and $\{k_n\}$ as in (1.16) we have (1.18) and (1.19). Now the proof of the first part of Theorem 1 remains valid word for word along the *whole* sequence $\{n\}$ with the present $\{m_n\}$ and $\{k_n\}$. The only difference is, of course, that the matrix given in the proof of Lemma 2.4 now converges to the covariance matrix in the formulation of the theorem. Noticing also that

$$\int_0^{-Z_1}(Z_1 + x)\, d\Psi_1(x) = \frac{\alpha^{1/2}}{\sigma(\alpha,\beta)}(Q(\alpha+) - Q(\alpha))\min(0, Z_1),$$

$$\int_{Z_2}^{0}(Z_2 + x)\, d\Psi_2(x) = \frac{(1-\beta)^{1/2}}{\sigma(\alpha,\beta)}(Q(\beta+) - Q(\beta))\max(0, -Z_2),$$

with the $\Psi$ functions of (1.17), the theorem is proved. □

**3. Discussion and examples.** We begin with a brief discussion of what Theorem 4 implies when the distribution function $F$ has a finite positive variance. In this case it is easy to show that the lim sup appearing in Lemma 2.1 is equal to zero. This implies (1.15) for all sequences $m_n$ and $k_n$ satisfying (1.1) so that the functions $\Psi_1$ and $\Psi_2$ appearing in $V_1$ are identically equal to zero, as already remarked in Section 1. Thus Theorem 4 yields the central limit theorem (1.2) if $F$ has a finite variance. Theorem 4 also contains (1.2) when $m_n \equiv k_n$ for all $F$ being in the domain of attraction of a nondegenerate stable or normal law, since the technical lemmas given in S. Csörgő, Horváth and Mason (1986) can be used to verify (1.15).

Next we consider distribution functions for which the entire partial sum $X_1 + \cdots + X_n$, when properly centered and normalized, is stochastically compact. For the sake of simplicity of the exposition we restrict ourselves from now on to sequences $m_n \equiv k_n$ so that, if we refer to a sequence $k_n$ satisfying (1.1), we always mean a pair $m_n$, $k_n$ with $m_n \equiv k_n$. It is shown in S. Csörgő, Haeusler and Mason (1989) that there exist sequences $A_n$ and $B_n$ of normalizing and centering constants such that $A_n^{-1}(X_1 + \cdots + X_n - B_n)$ is stochastically compact if and only if

(3.1) $$\limsup_{t\downarrow 0} t\big(Q^2(t+) + Q^2(1-t)\big)/S^2(t, 1-t) < \infty,$$

which in turn is equivalent to

(3.2) $$\limsup_{t\downarrow 0} t\big(Q^2(t/\lambda) + Q^2(1-t/\lambda)\big)/\sigma^2(t, 1-t) < \infty, \quad \text{for all } \lambda > 0.$$

It is also proved in that paper that these conditions are equivalent to the Feller (1967) condition for stochastic compactness. Obviously, (3.2) entails uniform boundedness of the functions $\Psi_{i,n}$, $i = 1,2$, $n = 1,2,\dots$, so that (1.13) is satisfied, which means that for any stochastically compact $F$ and any sequence $k_n$ of positive integers satisfying (1.1) the sequence in (1.10) is stochastically compact with $A_n = a_n(k_n, k_n)$ and $B_n = \mu_n(k_n, k_n)$. Moreover, only subsequential limits of the form $V_1$ are possible with bounded functions $\Psi_i$, $i = 1,2$. The following example shows that nonnormal *subsequential* limits are possible when $F$ is stochastically compact.

EXAMPLE 1. Fix any sequence $r_1, r_2, \dots$ of strictly positive constants such that $\sum_{k=1}^{\infty} r_k = 1$. On $[1/2, 1)$ let the quantile function $Q$ be inductively defined by $Q(1/2) = -1$, $Q(1/2+) = 1$, $Q(3/4) = 1 + r_1$, $Q$ linear on $(1/2, 3/4]$, and for $k = 1,2,\dots$,

$$Q\big((1-2^{-2k})+\big) = 1 + \sum_{i=1}^{k} r_i + \sum_{i=1}^{k} 2^i S(1/2, 1-2^{-2i}),$$

$$Q(1-2^{-2(k+1)}) = 1 + \sum_{i=1}^{k+1} r_i + \sum_{i=1}^{k} 2^i S(1/2, 1-2^{-2i}),$$

$$Q \text{ linear on } (1-2^{-2k}, 1-2^{-2(k+1)}].$$

For $0 < t < 1/2$ set $Q(t) = -Q((1-t)+)$. Observe that $Q$ is strictly increasing so that the corresponding distribution function is continuous. If $k = 1,2,\dots$ and $1 - 2^{-2k} < 1 - t \le 1 - 2^{-2(k+1)}$, then computations show

$$\frac{tQ^2(1-t)}{S^2(1/2, 1-t)} \le 2^{-2k}\left(4 + \sum_{i=1}^{k} 2^i\right)^2 \le 9,$$

so that the stochastic compactness criterion (3.1) is satisfied. It is straightforward to verify that $Q$ has infinite variance. Since $Q$ is a symmetric quantile function with infinite variance it is easy to infer from (2.5) that

$$\sigma^2(t, 1-t) \sim 2\sigma^2(1/2, 1-t), \quad \text{as } t \downarrow 0. \tag{3.3}$$

For $n = 2,3,\dots$ let $k_n$ be defined for $2^{3k} < n \le 2^{3(k+1)}$, $k = 1,2,\dots$, to be $k_n = 2^{k+1}$. Then the sequence $\{k_n\}$ is increasing. Consider the subsequence $n(l) = 2^{3(l+1)}$, $l = 1,2,\dots$, of the sequence $\{n\}$ of positive integers. Obviously, $k_{n(l)} = 2^{l+1}$ and, for $0 < c < \infty$, routine bounds give

$$10^{-1/2} \le \liminf_{l\to\infty} \Psi_{2,n(l)}(c) \le \limsup_{l\to\infty} \Psi_{2,n(l)}(c) < \infty,$$

where finiteness of the lim sup follows from stochastic compactness, as remarked previously. For $-\infty < c \le 0$ it is easy to see that

$$\Psi_{2,n(l)}(c) \to 0, \quad \text{as } l \to \infty.$$

By Lemma 2.6 there exist a subsequence $\{n'\}$ of $\{n(l)\}$ and a nondecreasing left-continuous function $\Psi_2$ such that $\Psi_{2,n'}(c) \to \Psi_2(c)$ as $n' \to \infty$ at every continuity point $c$ of $\Psi_2$. Obviously, $\Psi_2(c) = 0$ for $-\infty < c \leq 0$ and $10^{-1/2} \leq \Psi_2(c)$ for $0 < c < \infty$. It is also clear that along a further subsequence $\{n''\}$ of $\{n'\}$ the assumptions of the second part of Theorem 1 concerning $\Psi_{1,n}$ and $r_{i,n}$, $\sigma_{i,n}$, $i = 1, 2$, are satisfied. Consequently, on account of Theorem 1, the sequence in (1.10) converges along $\{n''\}$ to a limit $V_1$, which in view of Proposition 2 in the Appendix is nonnormal since $\Psi_2$ is not identically equal to zero.

Theorem 2.1 of Pruitt (1985) states that if $F$ is a symmetric stochastically compact, continuous distribution function and if the $k_n$ summands, which are *largest in absolute value*, are discarded from $X_1 + \cdots + X_n$ at each stage $n$, where $k_n$ statisfies (1.1), then the remaining sums, suitably normalized, are asymptotically normal. This result and Example 1 together say that (perhaps contrary to intuition) trimming the partial sums from a symmetric distribution symmetrically on both sides and trimming such sums by absolute value are two different problems.

We have just seen that for a stochastically compact distribution only the assumptions of Theorem 1 can be satisfied. In our next example we provide a symmetric distribution, necessarily not stochastically compact, for which the assumptions of Theorem 2 are satisfied along an appropriate subsequence for a certain sequence $k_n$.

EXAMPLE 2. On $[1/2, 1)$ let the quantile function $Q$ be inductively defined by $Q(1/2) = -1$, $Q(t) = 1$ for $1/2 < t \leq 3/4$ and for $k = 1, 2, \ldots$, set

$$Q(t) = 1 + \sum_{i=1}^{k} 2^{2i}\sigma(1/2, 1 - 2^{-2i}), \quad \text{for } 1 - 2^{-2k} < t \leq 1 - 2^{-2(k+1)}.$$

To obtain a symmetric quantile function, set $Q(t) = Q((1 - t)+)$ for $0 < t < 1/2$. This $Q$ has infinite variance and (3.3) holds. Let the sequences $m_n$ and $n(l)$ be defined as in Example 1. Then for $0 < c < \infty$ in view of (3.3) it can be shown that

$$\Psi_{2,n(l)}(c) \sim 2^{l+1/2} \to \infty, \quad \text{as } l \to \infty.$$

For $-\infty < c \leq 0$ it is easy to see that

$$\Psi_{2,n(l)}(c) \to 0, \quad \text{as } l \to \infty.$$

So by putting $A_{n(l)} = 2^{l+1/2}$ for $l = 1, 2, \ldots$, and recalling that $F$ is symmetric, we have constructed an example for which the assumptions of Theorem 2 hold.

A prime example of a symmetric distribution function $F$, which is not stochastically compact, is a distribution function with a slowly varying tail, i.e.,

$$1 - F(x) = L(x), \quad x \text{ large}, \tag{3.4}$$

where $L$ is a slowly varying function at infinity. The function $Q(1 - t)$ is then rapidly varying at zero, i.e., for all $\lambda > 1$,

$$Q(1 - t)/Q(1 - t\lambda) \to \infty, \quad \text{as } t \downarrow 0,$$

cf. Corollary 1.2.1.5 in de Haan (1975). Since the variance of $Q$ is infinite, an application of Theorem 1.3.2 in de Haan (1975) together with (3.3) yields,

$$\sigma^2(t, 1-t) \sim 2tQ^2(1-t), \quad \text{as } t \downarrow 0. \tag{3.5}$$

This behavior of the variance enables us to infer the following central limit theorem from Theorem 4.

COROLLARY. *Let $F$ be any symmetric distribution function satisfying* (3.4), *and let $k_n$ be any sequence of integers for which* (1.1) *holds. Then the following three statements are equivalent*:

(i) *There exist normalizing and centering constants $A_n > 0$ and $B_n$ such that*

$$A_n^{-1}\left\{\sum_{i=k_n+1}^{n-k_n} X_{i,n} - B_n\right\} \to_{\mathscr{D}} N(0,1).$$

(ii) $Q(1 - k_n/n + ck_n^{1/2}/n)/Q(1 - k_n/n) \to 1$ *for all $c$.*
(iii) $X_{n-k_n,n}/Q(1 - k_n/n) \to_P 1$.

*If* (i) *holds, then one can choose $A_n = a_n$ and $B_n = 0$ for $n \geq 1$.*

PROOF. By (3.5) we have for $-\infty < c < \infty$,

$$\Psi_{2,n}(c) \sim 2^{-1/2}\left\{\frac{Q(1 - k_n/n + ck_n^{1/2}/n)}{Q(1 - k_n/n)} - 1\right\},$$

so that equivalence of (i) and (ii) is immediate from Theorem 4 combined with the symmetry of $F$. Equivalence of (ii) and (iii) follows from the representation $X_{n-k_n,n} = Q(U_{n-k_n,n})$ and the fact that $n\{U_{n-k_n,n} - (1 - k_n/n)\}/k_n^{1/2}$ is $O_p(1)$. □

EXAMPLE 3. We now demonstrate how this result is applicable in a concrete example. The function $Q(1-s) = e^{1/s}$ is, for $s > 0$ small enough, the quantile function pertaining to a distribution function $F$ with slowly varying upper tail. Assume that $F$ is symmetric. For $-\infty < c < \infty$ and any sequence $k_n$ of integers satisfying (1.1), we have

$$Q(1 - k_n/n + ck_n^{1/2}/n)/Q(1 - k_n/n) = \exp\{cnk_n^{-3/2}(1 + o(1))\}.$$

So we have $\Psi_{2,n}(c) \to 0$ for all $-\infty < c < \infty$ if and only if $n/k_n^{3/2} \to 0$. Since $\Psi_{1,n}(c) = -\psi_{2,n}(-c)$ in the present situation, this is exactly the condition on $k_n$ for asymptotic normality of $a_n^{-1}(X_{k_n+1,n} + \cdots + X_{n-k_n,n})$ to hold.

Consider now the case $k_n \sim n^{2/3}$. In this case it can be shown that the assumptions of the second part of Theorem 1 are satisfied along the whole sequence $\{n\}$. Simple evaluations give that, with $Z_1$ and $Z_2$ being independent standard normal random variables,

$$V_1 =_{\mathscr{D}} -2^{-1/2}e^{Z_1} + 2^{-1/2}e^{-Z_2},$$

so that the distribution of $V_1$ is the convolution of two log-normal distributions.

When $n/k_n^{3/2} \to \infty$ it can be shown that it is impossible to center and normalize the trimmed sums in such a way that they converge in distribution to a nondegenerate limit.

## APPENDIX

Let $\varphi_1$ and $\varphi_2$ be two nondecreasing left-continuous functions on $\mathbb{R}$ with $\varphi_1(0) = \varphi_2(0) = 0$ such that for $r_1 \le 0$ and $r_2 \le 0$, $\varphi_1(c) \le -r_1$ and $\varphi_2(c) \ge r_2$ for all $c$. Also, let $(Z_1, W_1, W_2, Z_2)$ be a quadrivariate normal random vector with mean vector zero and covariance matrix

$$\begin{pmatrix} 1 & r_1 & 0 & 0 \\ r_1 & \sigma_1^2 & 0 & 0 \\ 0 & 0 & \sigma_2^2 & r_2 \\ 0 & 0 & r_2 & 1 \end{pmatrix},$$

with $\sigma_1^2 + \sigma_2^2 = 1$ and set

$$\begin{aligned} N &= \left\{ \int_0^{-Z_1} (Z_1 + x)\, d\varphi_1(x) + W_1 \right\} + \left\{ W_2 + \int_{-Z_2}^0 (Z_2 + x)\, d\varphi_2(x) \right\} \\ &\equiv \{N_1 + W_1\} + \{W_2 + N_2\} \end{aligned}$$

and $M = N_1 + N_2$.

**Proposition 1.** *With the notation given previously, $N =_{\mathscr{D}} N(\mu, \sigma^2)$ if and only if $\varphi_1 \equiv \varphi_2 \equiv 0$, in which case $\mu = 0$ and $\sigma^2 = 1$, and $M$ is never a nondegenerate normal random variable.*

**Proof.** First, obviously, when $\varphi_1 \equiv \varphi_2 \equiv 0$, then $N =_{\mathscr{D}} N(0,1)$. Assume that $N =_{\mathscr{D}} N(\mu, \sigma^2)$. The assumptions imply that $N_1 + W_1$ and $W_2 + N_2$ are independent and this by the Cramér characterization of the normal distribution implies that both $N_1 + W_1$ and $W_2 + N_2$ must be normally distributed. We shall show that $N_i + W_i$ can be normally distributed only if $\varphi_i \equiv 0$, $i = 1, 2$.

Suppose that $N_1 + W_1$ is normally distributed but $\varphi_1$ is not identically equal to zero. We can write

$$N_1 + W_1 =_{\mathscr{D}} \int_0^{-Z_1} (Z_1 + x)\, d\varphi_1(x) + r_1 Z_1 + Y_1,$$

where $Y_1$ is a normally distributed random variable independent of $Z_1$. This forces the random variable

$$\int_0^{-Z_1} (Z_1 + x)\, d\varphi_1(x) + r_1 Z_1 = \int_{-Z_1}^0 (\varphi_1(x) + r_1)\, dx \equiv f(Z_1)$$

to be normal, but $\varphi_1 \le -r_1$ implies that $f$ is a nondecreasing function. This forces $f(z)$ to be linear in $z$, which can only happen if $\varphi_1 \equiv 0$. Similarly, it can be shown that if $N_2 + W_2$ is a normal random variable, then $\varphi_2 \equiv 0$. By inspection, we must have $\mu = 0$ and $\sigma^2 = 1$. This completes the proof of the first part of the proposition. For the proof of the second part, we note that if $M$ were a nondegenerate normal random variable, by independence of $N_1$ and $N_2$ this would force at least one of these two random variables to be nondegenerate normal, which is clearly impossible. □

Let $(Z_1, Z, Z_2)$ be a trivariate normal random vector with mean vector zero and covariance matrix as in (1.5).

PROPOSITION 2. *With the notation given previously, $M + Z$ is always a nondegenerate random variable and $M$ is a nondegenerate random variable if and only if $\varphi_1$ or $\varphi_2$ is not identically equal to zero.*

PROOF. The first part follows from Proposition 1 and the second by a simple conditioning argument. □

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SÁNDOR CSÖRGŐ
BOLYAI INSTITUTE
SZEGED UNIVERSITY
ARADI VÉRTANÚK TERE 1
H-6720 SZEGED
HUNGARY

ERICH HAEUSLER
MATHEMATICAL INSTITUTE
UNIVERSITY OF MUNICH
THERESIENSTRASSE 39
D-8000 MUNICH 2
WEST GERMANY

DAVID M. MASON
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE 19716