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THE STRUCTURE OF SIGN-INVARIANT GB-SETS AND OF CERTAIN GAUSSIAN MEASURES[1]

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Let $(g_i)_{i\geq 1}$ be an i.i.d. sequence of standard normal r.v.'s. Let A be a family of sequences $a = (a_i)_{i\geq 1}$, $a_i \geq 0$. We relate the quantity $E\,\mathrm{Sup}_{a\in A}\sum_{i\geq 1} a_i|g_i|$ and the geometry of A.

1. Introduction. Consider a separable Hilbert space H. We fix once and for all an orthonormal basis $(e_i)_{i\geq 1}$ of H. An element t of H is often identified with the sequence $(t_i)_{i\geq 1}$, where $t_i = \langle t, e_i\rangle$.

On H is defined a canonical (isonormal) Gaussian process, which we denote $(X_t)_{t\in H}$, such that $E(X_t X_u) = \langle t, u\rangle$. If (g_i) denotes a sequence of independent $N(0,1)$ random variables, a version of X is given by $X_t = \sum_{i=1}^{\infty}\langle t, e_i\rangle g_i$ (where the series converges a.e.). Consider now a set $A \subset H$. Following [1], we say that A is a GB-set if $E\,\mathrm{Sup}_{t\in A} X_t < \infty$. Consider a set $B \subset H$ that satisfies

$$(1)\qquad \forall\, u > 0, \qquad \mathrm{card}\{b \in B;\ \|b\| \geq u\} \leq \exp(1/u^2).$$

A simple computation shows that B is a GB-set, and that actually $E\,\mathrm{Sup}_{t\in B} X_t \leq K$ for some universal constant K. (In the sequel, we denote by K a universal constant, not necessarily the same at each line.) It follows that the closed convex hull C of $B \cup \{0\}$ still satisfies $E\,\mathrm{Sup}_{t\in C} X_t \leq K$, so any subset A of C is a GB-set. A rather remarkable fact is that this method generates all the GB-sets.

THEOREM 1 [2]. *Consider a GB-set A that contains zero, and let $a = E\,\mathrm{Sup}_{t\in A} X_t$. Then one can find a set B that satisfies* (1) *and such that each t in A is the sum of a series $t = \sum_{b\in B}\alpha_b(t)b$, where $\alpha_b(t) \geq 0$, $\sum_{b\in B}\alpha_b(t) \leq Ka$.*

COMMENTS. (1) The restriction "A contains zero" is inessential, since $E\,\mathrm{Sup}_{t\in A} X_t$ is invariant under translation of A.

(2) As is explained in [2], Theorem 1 allows a complete description of all bounded (or continuous) Gaussian processes, and of all Gaussian measures on Banach spaces.

We set $L = \{-1, 1\}^{\mathbb{N}}$. For $\varepsilon = (\varepsilon_i)$ in L, we denote by M_ε the operator on H given by $M_\varepsilon(t) = (\varepsilon_i t_i)$. We say that a subset A of H is *sign-invariant* if $A = M_\varepsilon A$ for ε in L. For a sign-invariant set A, we denote by $|A|$ the set of

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sequences $(|t_i|)$ for $t = (t_i)$ in A. Then

$$E \operatorname*{Sup}_{A} X_t = E \operatorname*{Sup}_{|A|} \sum t_i |g_i|$$

(by this latter expression, we mean $E\ \mathrm{Sup}_{|A|} Y_t$, where Y_t is a separable version of the process $t \to \Sigma t_i |g_i|$.) In the case where A is sign-invariant, it is natural to expect a description of A more precise than that in Theorem 1; but it is not immediately clear how to achieve that goal using the results and the methods of [2]. The purpose of the present note is to introduce the necessary adaptation of the techniques. Our main result is as follows.

THEOREM 2. *If A is a GB-set that is sign-invariant, it can be represented as in Theorem 1, where B is also sign-invariant.*

While it may not be clear at once that this is interesting, the point is that condition (1) is rather restrictive for a sign-invariant set B and so there are unexpectedly few sign-invariant GB-sets. This will be more apparent in the following formulation of Theorem 2, where for a finite subset I of $\mathbb{N}$, and $\eta > 0$, we consider the set

$$C(I, \eta) = \left\{ t \in H;\ \sum_{i \in I} t_i^2 \leq \eta^2;\ \forall\, i \notin I,\ t_i = 0 \right\},$$

which is a ball of radius η and dimension card I.

THEOREM 3. *Let A be a sign-invariant subset of H and let $a = E\ \mathrm{Sup}_A X_t$. Then there exists a sequence (C_n) of sets $C_n = C(I_n, \eta_n)$ such that each t in A is the sum of a series, $t = \Sigma a_n(t) c_n$, where $c_n \in C_n$, $a_n(t) \geq 0$, $\Sigma a_n(t) \leq 1$ and that*

$$\text{(2)} \qquad \textit{for each } u > 0, \quad \sum \left\{ 2^{\operatorname{card} I_n};\ \eta_n \geq u \right\} \leq \exp\left(K a^2 / u^2 \right).$$

To understand this result better, one should note that if C is the closed convex hull of $\bigcup_n C(I_n, \eta_n)$, a simple computation (that is done in the course of the proof of Theorem 4) shows that condition (2) implies that $E\ \mathrm{Sup}_C X_t \leq Ka$. So, Theorem 3 means that given any sign-invariant GB-set A, one can find a sign-invariant set C containing A, and such that $E\ \mathrm{Sup}_C X_t \leq KE\ \mathrm{Sup}_A X_t$, where C is the convex hull of a sequence of finite-dimensional spheres, whose radius and dimensions are related by (2), and hence is obviously a sign-invariant GB-set.

Theorem 3 can be translated into a statement about certain Gaussian measures on some special Banach spaces. Let us recall that a (separable) Banach space Y has an unconditional basis if there is a sequence $(f_n)_{n \geq 1}$ of norm one elements of Y such that each x in Y is the sum of a series $\Sigma_{n \geq 1} x_n f_n$, and that the norm of the sum $\Sigma_{n \geq 1} x_n f_n$ is the same as the norm of $\Sigma_{n \geq 1} \varepsilon_n x_n f_n$ for any sequence $\varepsilon = (\varepsilon_n)$ in L. We denote by f_n^* the element of Y^* given by $f_n^*(\Sigma_{k \geq 1} x_k f_k) = x_n$. A centered Gaussian measure μ on Y is a Borel probability measure on Y such that the law of each continuous linear functional on Y is (centered) Gaussian.

THEOREM 4. *Let μ be a Gaussian measure on a Banach space Y with unconditional basis $(f_n)_{n\geq 1}$. Assume that the functionals (f_n^*) are independent. Let $\sigma_n = (\int f_n^{*2}\,d\mu)^{1/2}$. Assume that $\sigma_n > 0$ for each n. Then there is a sequence I_n of finite subsets of $\mathbb{N}$, and a sequence $\eta_n > 0$, such that if we consider the pseudo-norm of Y (valued in $\mathbb{R} \cup \{\infty\}$) given by*

$$N(x) = \operatorname*{Sup}_n \eta_n \left(\sum_{i\in I_n} \left(f_i^*(x)/\sigma_i \right)^2 \right)^{1/2}, \tag{3}$$

then the following properties hold:

$$\forall x \in Y, \qquad \|x\| \leq N(x),$$

$$\int_Y N(x)\,d\mu(x) \leq K\int_Y \|x\|\,d\mu(x). \tag{4}$$

This means that the norm of Y is a weakening of a norm N of the very explicit type (3), so we have completely understood what the object (μ, Y) is.

2. Proofs. Before we start the proofs, we explain what the issue is. For $t > 0$, let $h(t) = (\log 1/t)^{1/2}$. In [2], we deduce Theorem 1 from the following result.

THEOREM 5. *For any subset A of H, there is a probability measure m on A such that if D is the diameter of A, we have*

$$\forall x \in A, \qquad \int_0^D h(\operatorname{Sup}\{m(\{u\});\ \|u - x\| \leq \eta\})\,d\eta \leq KE\operatorname*{Sup}_A X_t. \tag{5}$$

The essential step to prove Theorem 2 is to show that when A is sign-invariant, the measure m of Theorem 5 can be taken sign-invariant (in the obvious sense that it is invariant under each M_ε). In [2] the existence of m is obtained through an involved analysis. It does not seem possible in this analysis to take into account the fact that A is sign-invariant. We first note that (5) implies that

$$\forall x \in A, \qquad \int_0^D h(m(B(x,\eta)))\,d\eta \leq KE\operatorname*{Sup}_A X_t, \tag{6}$$

where $B(x,\eta)$ is the ball of center x and radius η. We note now that h is convex for $t \leq e^{-1/4}$. Averaging the measures $M_\varepsilon m$ over ε in L, L being provided with the canonical measure, we get the following.

PROPOSITION 6. *For every sign-invariant subset A of H, there is a sign-invariant probability measure m on A such that*

$$\forall x \in A, \qquad \int_0^D h(m(B(x,\eta)))\,d\eta \leq KE\operatorname*{Sup}_A X_t.$$

The line of attack is to use the above probability measure to produce a new one that satisfies (5) and is sign-invariant.

LEMMA 7. *There exists a universal constant K_1 with the following property. For $a = \sum_{i \geq 1} a_i e_i$ in H, $a_i \geq 0$ and $\eta > 0$, denote by $N(a, \eta)$ the largest possible number of closed disjoint balls of radius η that are centered at points of the type $M_\varepsilon a$ (ε in L). Then if $k = [K_1 \log N(a, \eta)]$, we can find c in H which has only at most k nonzero components and satisfies $\|a - c\| \leq K_1 \eta$.*

PROOF. There is no loss of generality to assume that the sequence (a_i) is nonincreasing. Consider a sequence (δ_i) of independent random variables, with $P(\delta_i = 0) = 1/2$ and $P(\delta_i = 1) = 1/2$. We have, for $i \geq 1$ and $\lambda > 0$,

$$E \exp(-\lambda a_i^2 \delta_i) = \tfrac{1}{2}(1 + \exp(-\lambda a_i^2)),$$

so also

$$E \exp\left(-\sum_{i=1}^{\infty} \lambda a_i^2 \delta_i\right) = \prod_{i=1}^{\infty} \tfrac{1}{2}(1 + \exp(-\lambda a_i^2)).$$

It follows that

$$P\left(\sum_{i=1}^{\infty} a_i^2 \delta_i \leq \eta^2\right) \leq (\exp(\lambda \eta^2)) \prod_{i=1}^{\infty} \tfrac{1}{2}(1 + \exp(-\lambda a_i^2)). \tag{7}$$

Consider now $0 < \alpha_1 \leq 1$ such that $1 + \exp(-x) \leq 2\exp(-\alpha_1 x)$ for $0 \leq x \leq 1$. Let $\alpha_2 = \log(\frac{1}{2}(1 + 1/e)) > 0$. Since we can assume $K_1^2 \geq 1/\alpha_1$, there is nothing to prove if $\sum_{i \geq 1} a_i^2 \leq \eta^2/\alpha_1$, since we can then take $c = 0$. If $\sum_{i \geq 1} a_i^2 > \eta^2/\alpha_1$, we consider the largest integer k such that $\sum_{i \geq k} a_i^2 \geq \eta^2/\alpha_1$. In (7), we take $\lambda = 1/a_k^2$. For $i < k$, we have $a_i^2/a_k^2 \geq 1$, so

$$\frac{1}{2}(1 + \exp - \lambda a_i^2) \leq \frac{1}{2}\left(1 + \frac{1}{e}\right) = \exp(-\alpha_2).$$

For $i \geq k$, we have $a_i^2/a_k^2 \leq 1$, so

$$\tfrac{1}{2}(1 + \exp(-\lambda a_i^2)) \leq \exp(-\alpha_1 a_i^2/a_k^2).$$

It follows that

$$\begin{aligned} P\left(\sum a_i^2 \delta_i \leq \eta^2\right) &\leq \exp\left[-(k-1)\alpha_1 + \left(\eta^2 - \alpha_1 \sum_{i \geq k} a_i^2\right)\Big/ a_k^2\right] \\ &\leq \exp(-(k-1)\alpha_1). \end{aligned} \tag{8}$$

Fix now $\varepsilon' = (\varepsilon_i')$ in L. For ε in L, we have

$$\|M_\varepsilon a - M_{\varepsilon'} a\|^2 = \sum_{i \geq 1} a_i^2 (\varepsilon_i - \varepsilon_i')^2.$$

For the canonical probability P on L, the sequence $((\varepsilon_i - \varepsilon_i')^2)$ is distributed like the sequence $(4\delta_i)$. It follows from (8) that we can find a subset X of L, with card $X \geq \exp(-(k-1)\alpha_1)$, such that $\|M_\varepsilon a - M_{\varepsilon'} a\| > 2\eta$ for $\varepsilon, \varepsilon'$ in X, $\varepsilon \neq \varepsilon'$. This shows that $N(a, \eta) \geq \exp(k-1)\alpha_1$, so $k - 1 \leq (1/\alpha_1)\log N(a, \varepsilon)$. We note that if $k = 1$, $\sum_{i \geq 1} a_i^2 \geq \eta^2/\alpha_1 \geq \eta^2$, so $N(a, \eta) \geq 2$. If $k \geq 2$, we have $k \leq 2(k-1)$; so for some universal constant K_1, we have $k \leq [K_1 \log N(a, \eta)]$. This completes the proof, by taking $c = \sum_{i \leq k} a_i e_i$, so that $\|a - c\| \leq \eta \alpha_1^{-1/2} \leq K_1 \eta$. □

We now perform the main construction.

PROPOSITION 8. *There exist universal constants K_2, K_3 with the following property. For each sign-invariant totally bounded subset A of H, each sign-invariant probability measure m on A and each $\eta > 0$, there is a finite sign-invariant subset B of H and a sign-invariant probability measure μ on B such that*

$$(9)\quad \forall x \in A, \qquad h\big(\sup\{\mu(\{t\});\ t \in B(x, K_2\eta)\}\big) \le K_3 h\big(m(B(x,\eta))/2\big).$$

PROOF. *Step* 1. We first describe the basic construction. Let A' be a sign-invariant subset of A. Let a in A' be such that

$$2m(B(a,\eta)) \ge \sup\{m(B(b,\eta));\ b \in A'\}.$$

By $B(a,\eta)$, we denote here and in the sequel the ball in H, *not* its restriction to A'. Using Lemma 7, we find k, such that there is a point b in H that has exactly k nonzero components, with $\|a-b\| \le K_1\eta$, and that at least $\exp(k/K_1)$ of the balls $B(M_\varepsilon a, \eta)$ are disjoint. Let B be the set of points of the type $M_\varepsilon b$, $\varepsilon \in L$, so card $B = 2^k$. For t in B, we set $\mu(\{t\}) = 2^{-k} m(B(a,\eta))$.

We note that all the balls $B(M_\varepsilon a, \eta)$ have the same measure for m, since m is sign-invariant; since at least $\exp(k/K_1)$ of these balls are disjoint, we have

$$\log(1/m(B(a,\eta))) \ge k/K_1.$$

So, for each point t of the type $M_\varepsilon b$, we have

$$\begin{aligned}\log(1/\mu(\{t\})) &= k\log 2 + \log(1/m(B(a,\eta)))\\ &\le (K_1\log 2 + 1)\log(1/m(B(a,\eta))).\end{aligned}$$

Let $(K_1\log 2 + 1)^{1/2} = K_3$. Consider the set D, union of all the balls $M_\varepsilon(B(a, 2\eta))$. Let $K_2 = K_1 + 2$. For each x in D, there is t in B with $\|x - t\| \le K_2\eta$. We note also that $\mu(B) \le m(B(a,\eta))$, and that for x in D, we have $m(B(x,\eta)) \le 2m(B(a,\eta))$.

Step 2. By induction over p, we construct sign-invariant subsets A_p of A, sign-invariant sets D_p, finite sign-invariant sets B_p, points a_p of A_p and a sign-invariant measure μ on $\bigcup_{i \le p} B_i$, such that the following conditions hold:

(10) $a_p \in A_p$;

(11) $\mu(B_p) \le m(B(a_p,\eta))$;

(12) for each t in B_p, $\big(h(\mu(\{t\})) \le K_3 h\big(m(B(a_p,\eta))\big)\big)$;

(13) D_p is the union of the balls $B(M_\varepsilon a_p, 2\eta)$ for ε in L;

(14) $\forall x \in D_p, \exists\, t \in B_p,\ \|x - t\| \le K_2\eta$;

(15) for each x in A_p, $m(B(x,\eta)) \le 2m\big(B(a_p,\eta)\big)$;

(16) $A_{p+1} = A_p \setminus D_p$.

The construction starts with $A_0 = A$, and is immediate by induction, using step 1. The construction continues until $A_p = \varnothing$, which occurs in a finite number of steps since A is totally bounded. We set $B = \bigcup B_p$. From (13) and (16), we see that any two balls $B(a_p, \eta)$, $B(a_q, \eta)$, $p \neq q$, are disjoint; it follows from (11) that $\|\mu\| \leq 1$. We now check (9). Let x in A. Let p be the largest integer such that $x \in A_p$, so $x \in D_p$. From (12), (14) we see that there is t in $B(x, K_2\eta)$ such that

$$h(\mu(\{t\})) \leq K_3 h\big(m\big(B(a_p, \eta)\big)\big).$$

From (15), we have

$$h\big(m\big(B(a_p, \eta)\big)\big) \leq h(m(B(x, \eta))/2).$$

This proves (9). Proposition 8 is proved, except for the fact that μ is not a probability but $\|\mu\| \leq 1$. However, (9) will still hold if we replace μ by a larger sign-invariant probability. The proof is complete. □

We can now prove the version of Theorem 5 that we need.

PROPOSITION 9. *For any sign-invariant subset A of H, there is a sign-invariant probability measure μ on A such that if D is the diameter of A, we have*

$$\text{(17)} \qquad \operatorname*{Sup}_{x \in A} \int_0^D h(\sup\{\mu(\{u\});\ \|x - u\| \leq \eta\})\, d\eta \leq KE \operatorname*{Sup}_A X_t.$$

PROOF. Let m be the sign-invariant measure on A given by Proposition 6. According to Proposition 8, for each $k \geq 0$, there exists a sign-invariant probability μ_k on B such that for all x in A,

$$h\big(\operatorname{Sup}\{\mu_k(\{t\});\ t \in B\big(x, K_2 D 2^{-k}\big)\}\big) \leq K_3 h\big(m\big(B(x, 2^{-k}D)\big)/2\big).$$

Let $\mu = \sum_{k=0}^{\infty} 2^{-k-1}\mu_k$, so μ is a probability. Moreover,

$$\begin{aligned} &h\big(\operatorname{Sup}\{\mu(\{t\});\ t \in B\big(x, K_2 D 2^{-k}\big)\}\big) \\ &\quad \leq h\big(\operatorname{Sup}\{2^{-k-1}\mu_k(\{t\});\ t \in B\big(x, K_2 D 2^{-k}\big)\}\big) \\ &\quad \leq h(2^{-k-1}) + h\big(\operatorname{Sup}\{\mu_k(\{t\});\ t \in B\big(x, K_2 D 2^{-k}\big)\}\big). \end{aligned}$$

We note now that for a decreasing function f, we have

$$\sum_{k \geq 0} 2^{-k-1} D f(2^{-k}D) \leq \int_0^D f(\eta)\, d\eta \leq \sum_{k \geq 0} 2^{-k} D f(2^{-k}D).$$

It then follows that for x in A

$$\int_0^D h(\operatorname{Sup}\{\mu(\{t\});\ t \in B(x, K_2\eta)\})\, d\eta \leq K_4\Big(D + \int_0^D h(m(B(x, \eta)))\, d\eta\Big).$$

Making a change of variables and noting that $D \leq K_5 E \operatorname{Sup}_A X_t$, we obtain the result. □

PROOF OF THEOREM 2. Since the proof is very similar to the proof of the Theorem 2 of [1], we only indicate the necessary modifications. Consider the probability μ given by Proposition 8. Then, for each $k \geq 1$, consider the set

$$B_k = \Big\{(t_1 - t_2)\big(2^{-k}Dh\big(2^{-k}\mu(\{t_1\})\mu(\{t_2\})\big)\big)^{-1};\ \|t_1 - t_2\| \leq 2^{-k}D,$$
$$\mu(\{t_1\}), \mu(\{t_2\}) > 0\Big\}.$$

Since μ is sign-invariant, so is B_k. Hence $B' = \bigcup B_k$ is sign-invariant. As in [2], one checks that B' satisfies

$$\forall\, u > 0, \qquad \operatorname{card}\{b \in B';\ \|b\| \geq u\} \leq \exp\big(Ka^2/u^2\big)$$

and that each t in A can be written as a sum $\sum_{b \in B'} \alpha_b(t) b$, where $\alpha_b(t) \geq 0$, $\sum_{b \in B'} \alpha_b(t) \leq 1$. This completes the proof, by setting $B = a^{-1}K^{-1/2}B'$. □

PROOF OF THEOREM 3. Let B be as in Theorem 2. B is sign-invariant; we pick one element in each class of the action of M_ε on B; in other words, we consider a sequence (b^n) in B such that b^p is not of the type $M_\varepsilon b^n$ if $p \neq n$, but that each b in B is of the type $M_\varepsilon b^n$ for some n, some ε in L. Let I_n be the support of b^n, and let $\eta_n = \|b^n\|$. Then $C(I_n, \eta_n)$ contains all the points $M_\varepsilon b^n$, and there are $2^{\operatorname{card} I_n}$ of them. It follows that $B \subset \bigcup_n C(I_n, \eta_n)$. Moreover, for $u > 0$,

$$\sum \big\{2^{\operatorname{card} I_n};\ \eta_n \geq u\big\} \leq \operatorname{card}\{b \in B;\ \|b\| \geq u\} \leq \exp\big(1/u^2\big).$$

This completes the proof. □

PROOF OF THEOREM 4. Denote by Y_1^* the unit ball of Y^*. Denote by A the image of Y_1^* in $L^2(\mu)$ by the canonical injection from Y^* into $L^2(\mu)$. Denote by H the closed linear span of A in $L^2(\mu)$. For each $n \geq 1$, let $e_n = f_n^*/\sigma_n$. Then $(e_n)_{n \geq 1}$ is an orthonormal basis of H. Each x^* in Y^* is the sum of the series $\sum_{n=1}^\infty x^*(f_n) f_n^*$, where the series is weak* convergent. For $\varepsilon = (\varepsilon_n)$ in L, we have $\|\sum_{n=1}^\infty \varepsilon_n x^*(f_n) f_n^*\| = \|x^*\|$. This shows that, when H is provided with the basis $(e_n)_{n \geq 1}$, the set A is sign-invariant. Moreover, it is clear that $E \operatorname{Sup}_A X_t = \int_Y \|x\|\, d\mu(x)$. Consider a sequence of finite sets I_n and a sequence $\eta_n > 0$ that satisfy the conclusions of Theorem 3. Fix x in Y. For each x^* in Y_1^*, we have

$$x^*(x) = \sum_{i \geq 1} x^*(f_i) f_i^*(x) = \sum_{i \geq 1} \big(f_i^*(x)/\sigma_i\big)\big(\sigma_i x^*(f_i)\big)$$
$$\leq \operatorname{Sup}\Big\{\sum_{i \geq 1} a_i\big(f_i^*(x)/\sigma_i\big);\ a = (a_i)_{i \geq 1} \in A\Big\}.$$

If $a \in C(I_p, \eta_p)$, we have

$$\sum_{i \geq 1} a_i f_i^*(x)/\sigma_i \leq \eta_p \bigg(\sum_{i \in I_p} \big(f_i^*(x)/\sigma_i\big)^2\bigg)^{1/2} \leq N(x),$$

so this inequality still holds whenever $a \in A$, so $x^*(x) \leq N(x)$. Since this is true whenever $x^* \in Y_1^*$ we have $\|x\| \leq N(x)$.

It remains to show that $\int_Y N(x)\, d\mu(x) \leq K \int_Y \|x\|\, d\mu(x) = Ka$. If C denotes the union of the sets $C_n = C(I_n \eta_n)$, we have $\int_Y N(x)\, d\mu(x) = E \operatorname{Sup}_C X_t$. Now, for some universal constant β and for each n, we can find a subset B_n of H, consisting of vectors of length $2\eta_n$, such that card $B_n \leq 2^{\beta \operatorname{card} I_n}$, and such that the convex hull of B_n contains C_n. Let B be the union of the sets B_n. Then the convex hull of B contains C, so $E\operatorname{Sup}_C X_t \leq E\operatorname{Sup}_B X_t$. On the other hand, for $u > 0$, we have, from condition (2)

$$\begin{aligned}\operatorname{card}\{b \in B:\, b \geq u\} &\leq \sum\left\{2^{\beta \operatorname{card} I_n};\ \eta_n \geq u/2\right\}\\ &\leq \left(\sum\left\{2^{\operatorname{card} I_n};\ \eta_n \geq u/2\right\}\right)^{\beta}\\ &\leq \exp\left(4K\beta a^2/u^2\right).\end{aligned}$$

As mentioned in the Introduction, this implies (by homogeneity) that $E\operatorname{Sup}_B X_t \leq Ka$, and finishes the proof. □

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