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BOOK REVIEW

ULRICH KRENGEL, *Ergodic Theorems*, de Gruyter Studies in Mathematics, volume 6, de Gruyter, Berlin, 1985, viii + 357 pages, $49.95.

REVIEW BY ROBERT SINE

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Ergodic theory was born in a flurry of papers in the *Proceedings of the National Academy of Sciences* in 1931–1932. The mathematical setup was a normalized measure space (Ω, F, μ) and a one parameter group $\{\phi_t\}$ of measure perserving maps of Ω. For a subset A in F we can consider the average amount of time that a point spends in A when transported by the dynamics at hand $\{\phi_t\}$. This mean sojourn time takes the form

$$\lim(1/T)\int_0^T 1_A(\phi_t\omega)\,dt.$$

Since the time of Boltzmann, physicists hoped to show the equality of the average time spent in the set A and the size of the set $\mu(A)$. B. O. Koopman (who received his degree from Birkhoff in 1926 and would go on to write joint papers in ergodic theory with both Birkhoff and von Neumann) published a note in 1931 where he observed that the group of measure isometries induces a group of unitary isometries of L_2 and he suggested relating the spectral properties of the generator of that unitary group with properties of the Hamiltonian system flowing in the background. von Neumann learned of this from Koopman in the spring of 1930 and, based on Koopman's observation, proved that the limit in question existed in the L_2 sense and that the limiting value did indeed have the hoped for constant value provided the system was "ergodic"—that is, the only invariant measurable sets were of measure 0 or 1. von Neumann's paper was not communicated until December 10, 1931, but he informed Birkhoff of his result in a letter in October of that year in which he suggested that a.e. convergence may in fact hold. This Birkhoff was able to show by devising a maximal lemma. The Birkhoff proof, described by Halmos [10] as maximally confusing, has been reworked by Hopf, Khintchine, Riesz and Garsia.

Two elements played the major roles in this initial work—the existence of limits of averaged quantities and the identification of these limits. It is these two elements of ergodic theory that are the focal points of the book under review.

We will give a chapter-by chapter outline and then some general comments.

Chapter 1. Measure preserving and null preserving point mappings (70 pages). Garsia's elementary proof of the maximal lemma is used to obtain the almost everywhere theorem. Induced transformations, recurrence and the Hopf decomposition of the transformation into conservative and dissipative parts are developed. One of the first applications of ergodic theory was to limit theorems of stationary processes made by Doob, Hopf and Khintchine in 1934.

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If ϕ is again our measure preserving transformation and f is in L_1, then set $F_k = f + f \circ \phi + \cdots + f \circ \phi^k$. The sequence $\{F_k\}$ then satisfies

$$F_{n+k} = F_n + F_k \circ \phi^n$$

and the Birkhoff theorem asserts the existence of the limit of F_n/n. Kingman's subadditive ergodic theorem hypothesizes a sequence of measurable functions which satisfies

$$F_{n+k} \leq F_n + F_k \circ \phi^n.$$

If F_1^+ is integrable, then F_n/n converges a.e. to an invariant function with integrable positive part. This extension (given here in simplified form) can be applied in many situations where the Birkhoff theorem itself is inadequate. After proving Kingman's theorem, Krengel uses it to prove the extension of Furstenberg and Kesten's work on random matrix products due to Oseledec. This result is currently of very great interest in dynamics, control and stochastic equations. Recent references not included in the book are [2]–[4]. [3] gives an extensive survey of current trends in Lyapounov exponents together with 22 papers from a Bremen workshop in late 1984.

Chapter 2. Mean ergodic theory (42 pages). One direction for the extension of the von Neumann theorem is to other B-spaces. It has been known for some time that all contractions are mean ergodic on reflexive spaces. In the direction of a converse, Zaharopol [20] has recently shown that a countably order complete B-lattice which is mean ergodic for all contractions must be reflexive. (A different paper, cited on p. 85 for this result, apparently contains an error.) The general case is still wide open.

Another direction for extension is to special classes of operators. Here we find quasicompact operators which are related to the Doeblin condition (which itself is not mentioned in the text). A reference too recent to be included is [15]; this entire volume should be of interest.

The elegant and highly useful decomposition for almost periodic transformations is treated. Krengel starts from Ellis' deep theorem of algebraic analysis to develop the splitting in the case of weakly almost periodic operators. The decomposition theory of Jacobs and deLeeuw–Glicksberg has not received much in the way of a utility grade exposition before the presentation here. This situation should be further remedied with a forthcoming book by Bergland, Junghenn and Milnes. A great deal of the behavior of constricted systems can be explained easily from the splitting theory [4].

Ergodic theory has borrowed from combinatorics on occasions. The accumulated debts have been repaid in full with Furstenberg's proof of the Szemeredi theorem. Krengel has included a brief introduction to the work on recurrence but defers to Furstenberg's gem of exposition for details [8].

A necessary and sufficient condition on $\sigma(T)$ for $\|T^n(I - T)\| \to 0$ where T is a contraction on a B-space has recently been given by Katznelson and Tzafriri [12]. The recent notes [17] may be of interest.

Chapter 3. Positive contractions in L_1 (46 pages). If P is a Markov kernel on a measurable space, we call μ a reference measure for P if $P\mu$ is absolutely continuous with respect to μ. In this case, P induces on $L_1(\mu)$ a positive L_1 contraction by way of the Radon–Nikodym theorem. Krengel developes the Hopf decomposition and the Chacon–Ornstein ratio theorem which plays the role of the Birkhoff theorem in this setting. The Akcoglu–Sucheston extension of the subadditive ergodic theorem is given and the chapter closes with a discussion of filling schemes.

Chapter 4. Extensions of the L_1 theory (18 pages). This chapter is devoted to nonpositive L_1 contractions using the modulus operator and to vector valued ergodic theorems.

Chapter 5. Operators in $C(K)$ and L_p $(1 < p < \infty)$ (18 pages). The Breiman–Jamison strong law is stated for mean ergodic operators on $C(K)$, but the argument given apparently assumes unique ergodicity. Jamison's category argument that for an irreducible $C(K)$ Markov operator, weak almost periodicity implies strong almost periodicity is also given. The major L_p result is the a.e. theorem for positive L_p contractions due to Akcoglu.

Chapter 6. Pointwise ergodic theorems for multiparameter and amenable semigroups (34 pages). Here we will only mention two references too recent to be included in the text. The paper of Frangos and Sucheston [7] continues the reduction of multiparameter theorems to one parameter arguments and gives a multiparameter Chacon–Ornstein theorem. The lecture notes of Moulin Ollagnier [16] deal extensively with amenable actions.

Chapter 7. Local ergodic theorems and differentiation (22 pages). This topic is developed for positive one parameter semigroups and then extended in the direction of nonpositive, multiparameter and vector valued functions.

Chapter 8. Subsequences and generalized means (16 pages). Here various weighted and sampled averages are taken up.

Chapter 9. Special topics (34 pages). The first topic surveyed is ergodic theory in von Neumann algebras. A recent reference here is [1], which consists of 37 papers of a 1983 conference in Romania where ergodic theorems of entropy and information and their multiparameter generalizations are taken up. Krengel has informed the reviewer that the problem of extending the pointwise ergodic theorem for information to countable partitions has recently been solved by Ornstein and Weiss.

Krengel next gives Baillon's nonlinear ergodic theorem for a nonexpansive map on Hilbert space. A multiparameter version is given but the discussion is restricted to Hilbert space except for remarks in the notes. The work of Akcoglu, Krengel and Lin on nonlinear ergodic theory in L_1 was too recent to be included in the text. This work is surveyed in [13]. This chapter of miscellaneous results closes with a section of miscellanea. There is a guide to the literature of about a paragraph each on random sets, empirical distribution functions, martingales and ergodic theorems, nonhomogeneous Markov chains, ergodic theorems in demography and nonlinear averages. A recent survey of limit theorems for random sets is [9]; see also the survey in [21].

The book has a 19 page supplement by A. Brunel giving a quick tour of Harris processes. Both the transition kernel and the abstract L_1 operator formulation are presented. A discussion of special functions and a proof of the Ornstein–Metivier–Brunel theorem are given. The zero-two law of Ornstein and Sucheston is proven. A development too recent to be included in the text is the L_p zero-two law of Zaharopol [12] and [21].

There are of course some minor misprints. Here are a few that have been brought to the reviewer's attention:

Page 34. A line of the text was omitted. The correct statement at line 17 should be "For a Gaussian automorphism θ ergodicity, weak mixing and r-fold mixing are equivalent. Mixing and r-fold mixing are equivalent to $r(n) \to 0$."

Page 146. On line 7, the quoted paper of Maharam gives conditions for existence of a σ-finite measure.

Page 210. For Theorem 2.14 one must assume condition 2.12 to hold even for nonconvex A.

The text is quite lucid and readable and the typography and layout are well done. Every chapter of the book leads off with a paragraph or more letting the reader know what is in store for him. Each section closes with a very informative set of notes giving background, stating related results and extensions and pointing to related literature. The bibliography of 1040 entries covering 26 pages is very complete and up to date even without taking into consideration the rapid development and broad application of the subject. Some items not in the bibliography are the papers of Feldman, Feller, and Furstenberg on boundaries and the Vancouver [5] and North Carolina lecture notes of Foguel [6].

The goal of the book has been a comprehensive treatment of the convergence theorems of ergodic theory. It is very successful in achieving that goal. Of previous books, it is perhaps closest to Peterson [18] and there is some overlap on the material of maximal theorems and recurrence. The emphasis on subadditivity and multiparameter extensions makes this text unique. A large part of ergodic theory is the classification of transformations. This is not covered at all nor was it any part of the goal of this book which is titled *Ergodic Theorems* and not *Ergodic Theory*. For the reader who is interested in exploring that part of the ergodic landscape, Birkhäuser has just published a two-volume collection of papers of Kakutani [11]. The largest chapter is that on ergodic theory. A feature of this collection which should be of great interest is extensive current commentary of each paper included.

There has been an enormous amount of interest and activity in ergodic theory recently. For anyone working in the field and its numerous applications, this is a book deserving a handy place on the bookshelf.

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