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DFR PROPERTY OF FIRST-PASSAGE TIMES AND ITS PRESERVATION UNDER GEOMETRIC COMPOUNDING[1]

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It is shown that if a discrete-time Markov chain on the state space $\{0, 1, \dots\}$ has a transition probability matrix $\mathbf{P}$ and a transition survival probability matrix $\mathbf{Q}$ which is totally positive of order two (TP_2), where $Q(i, j) = \sum_{k \geq j} P(i, k)$, then the first-passage time from state 1 to state 0 has decreasing failure rate (DFR). This result is used to show that (i) the sum of a geometrically distributed number (i.e., geometric compound) of i.i.d. DFR random variables is DFR, and (ii) the number of customers served during a busy period in an M/G/1 queue with increasing failure-rate service times is DFR. Recent results of Szekli (1986) and the closure property of i.i.d. DFR random variables under geometric compounding are combined to show that the stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (increasing mean residual life) service times is DFR. We also provide sufficient conditions on the inter-renewal times under which the renewal function is concave. These results shed some light on a conjecture of Brown (1981).

1. Introduction. Brown (1980) proved that the renewal function for a renewal process with decreasing failure rate (DFR) inter-renewal times is concave. In a subsequent paper [Brown (1981)] he conjectured that DFR is also a necessary condition for concavity. A consequence of this conjecture, as pointed out by Brown, is that DFR distributions are closed under geometric compounding (i.e., the sum of a geometrically distributed number of i.i.d. DFR random variables is DFR). Thus a counterexample to this (possible) closure property of DFR random variables under geometric compounding would provide a counterexample to the concavity conjecture.

In an attempt to verify Brown's conjecture we obtained sufficient conditions for (i) the first-passage time from state 1 to state 0 of a Markov chain on the state space $\{0, 1, \dots\}$ to be DFR, and (ii) the renewal function to be concave when the i.i.d. inter-renewal times have the distribution of this passage time. Specifically, in Section 3 we show that if a discrete-time Markov chain on the state space $\{0, 1, \dots\}$ has a transition probability matrix $\mathbf{P}$ and a transition survival probability matrix $\mathbf{Q}$ which is totally positive of order two (TP_2), where $Q(i, j) = \sum_{k \geq j} P(i, k)$, then the first-passage time from state 1 to state 0 is DFR. This result is used in the same section to establish that:

(A) DFR distributions are closed under geometric compounding.

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This, therefore, closes one possible avenue to construct a counterexample to Brown's conjecture. Section 4 shows that the renewal function is concave if the inter-renewal times have the same distribution as the first-passage time from state 1 to state 0 of a stochastically monotone Markov chain on $\{0, 1, \dots\}$. Hence Brown's conjecture, if true, implies that:

(B) The first-passage time from state 1 to state 0 of a stochastically monotone Markov chain on $\{0, 1, \dots\}$ is DFR.

This DFR property is known to hold for birth and death processes [Keilson (1979)] and in general for any Markov process that can be uniformized such that the embedded Markov chain has a TP_2 transition probability matrix [Assaf, Shaked and Shanthikumar (1985)]. But it is not known whether it holds for this larger class. For this class of Markov chains (i.e., stochastically monotone) using ideas similar to that used in Marshall and Shaked (1983) or Shanthikumar (1984), it can be easily shown that the first-passage time from state 1 to state 0 has the new worse than used (NWU) property. The NWU property is weaker than the DFR property. Therefore, a counterexample to (B) will also be a counterexample to Brown's conjecture. In Section 4 we provide an example of a stochastically monotone discrete-time Markov chain on $\{0, 1, \dots\}$ for which this first-passage time is not DFR. Consequently,

(C) Brown's conjecture does not hold in the discrete-time case.

The discrete-time example does not generalize to the continuous-time case, and the truth of this conjecture is still unresolved.

The results presented in this paper, apart from providing some useful insights into Brown's conjecture, have other applications. The DFR property of the first-passage times of Markov chains is used in Section 5 to show that the number of customers served during a busy period in an M/G/1 queue with increasing failure-rate service times is DFR. Geometric compounding of i.i.d. random variables arises naturally in many applied probability models. A recent paper of Gertsbakh (1984) discusses a wide range of applications in reliability theory. In the queueing theory context it is well known that the stationary waiting time in a GI/G/1 queue can be represented as a geometric compound of i.i.d. random variables [cf. Feller (1971)].

Related results are (i) the class of completely monotone (CM) distributions is closed under geometric compounding and the stationary waiting time in an M/G/1 queue with CM service times is CM [Keilson (1978)]; and (ii) the distribution function of a geometric convolution of DFR distributions is concave and the stationary waiting time in a GI/G/1 (M/G/1) queue with DFR (increasing mean residual life) service times has a concave distribution function [Szekli (1986)]. In Section 5 we strengthen the above results of Szekli.

One aspect of our methodology which appears to be new is the consideration of a monotonicity property for a Markov chain which is stronger than stochastic monotonicity and weaker than TP_2. Stochastic monotonicity is based on the partial ordering of stochastic ordering, TP_2 on ordering by monotone-likelihood ratio. An intermediate ordering we use is the hazard-rate ordering [cf. Ross

(1983)]. This ordering is utilized by Brown (1980, 1983, 1984) to study properties of increasing mean residual life (IMRL) and DFR distributions.

2. Preliminaries. A continuous random variable X with (without) a possible mass at zero or its distribution F on $[0, \infty)$ is said to be DFR (IFR) if its failure rate $r_X(t) \equiv f(t)/\overline{F}(t)$ is decreasing (increasing) in $t \in [0, \infty)$: f and $\overline{F}$ are the density and survival functions of X ("increasing" and "decreasing" are not used in the strict sense). X is said to be IMRL if $E(X - t|X > t)$ is increasing in $t \in [0, \infty)$. Discrete cases are analogously defined.

Two nonnegative random variables X_1 and X_2 or their distributions F_1 and F_2 are ordered in the sense of usual stochastic (hazard rate) ordering if $\overline{F}_1(t) \geq \overline{F}_2(t)$, $t \geq 0$ $[\overline{F}_1(t)/\overline{F}_2(t)$ is increasing in $t \in [0, \infty)]$. Denote $X_1 \geq_{st} X_2$ $(X_1 \geq_h X_2)$. Note that [cf. Ross (1983)] $X_1 \geq_h X_2$ implies $\{X_1|X_1 > t\} \geq_{st} \{X_2|X_2 > t\}$, $t \geq 0$.

Let $\mathbf{t} = [t(i, j)]_{i, j \geq 1}$; $t(i, j) = 1$, $i \geq j$ and $t(i, j) = 0$ otherwise. Then its inverse is $\mathbf{t}^{-1} = [t^{-1}(i, j)]_{i, j \geq 1}$, where $t^{-1}(i, i) = 1$, $i \geq 1$; $t^{-1}(i, i-1) = -1$, $i \geq 2$ and $t^{-1}(i, j) = 0$ otherwise [cf. Keilson (1979)]. A discrete-time Markov chain on the state space $\{0, 1, \dots\}$ with transition probability matrix $\mathbf{P}$ is stochastically monotone if $\mathbf{t}^{-1}\mathbf{P}\mathbf{t} \geq \mathbf{0}$. A continuous-time Markov chain on the state space $\{0, 1, \dots\}$ with transition-rate matrix $\boldsymbol{\mu} = [\mu(i, j)]_{i, j \geq 0}$ is stochastically monotone if for any $k \geq 0$, $\Sigma_{j \geq k}\mu(i, j)$ is increasing in $i < k$ and $\Sigma_{j \leq k}\mu(i, j)$ is decreasing in $i > k$.

A function $\mathbf{a} = [a(i, j)]$ of two real variables ranging over linearly ordered sets X and Y, respectively, is TP_2 if for any $n_1 < n_2$ and $m_1 < m_2$ $(n_i \in X$, $m_i \in Y)$, $a(n_1, m_1)a(n_2, m_2) \geq a(n_1, m_2)a(n_2, m_1)$. Equivalently, using the convention $0/0 = 0$ one has $a(n_2, m_2)/a(n_2, m_1) - a(n_1, m_2)/a(n_1, m_1) \geq 0$ (when defined; otherwise set the difference equal to zero).

Suppose $\mathbf{a} = [a(i, j)]_{i, j \in N+}$ and $\mathbf{b} = [b(i, j)]_{i, j \in N+}$ are two nonnegative matrices (here $N+ = \{1, 2, \dots\}$).

LEMMA 2.1. *If* $\mathbf{at} \in TP_2$, $\mathbf{rt} \in TP_2$ *and* $\mathbf{t}^{-1}\mathbf{rt} \geq \mathbf{0}$, *then* $\mathbf{art} \in TP_2$.

PROOF. Let $\mathbf{A} = \mathbf{at}$, $\mathbf{R} = \mathbf{rt}$, $\mathbf{B} = \mathbf{art}$ and for $1 \leq n_1 < n_2$, $1 \leq m_1 < m_2$ and $B(n_1, m_1) > 0$, consider

$$\frac{B(n_1, m_2)}{B(n_1, m_1)} = \frac{\Sigma_{k=1}^{\infty} a(n_1, k)R(k, m_2)}{\Sigma_{l=1}^{\infty} a(n_1, l)R(l, m_1)}$$

$$(2.1) \qquad = \sum_{k=1}^{\infty} \frac{R(k, m_2)}{R(k, m_1)} \left[\frac{a(n_1, k)R(k, m_1)}{\Sigma_{l=1}^{\infty} a(n_1, l)R(l, m_1)}\right]$$

$$= \sum_{k=1}^{\infty} \left[\frac{R(k, m_2)}{R(k, m_1)} - \frac{R(k-1, m_2)}{R(k-1, m_1)}\right] \frac{\Sigma_{j=k}^{\infty} a(n_1, j)R(j, m_1)}{\Sigma_{l=1}^{\infty} a(n_1, l)R(l, m_1)},$$

where $R(0, m_2)/R(0, m_1) \equiv 0$. Since $\mathbf{R} \in TP_2$, the expression in the square brackets on the right-hand side of (2.1) is nonnegative. Consider

$$\frac{\sum_{j=k}^{\infty} a(n_1, j)R(j, m_1)}{\sum_{l=1}^{\infty} a(n_1, l)R(l, m_1)} = \frac{1}{1 + \dfrac{\sum_{l=1}^{k-1} a(n_1, l)R(l, m_1)}{\sum_{j=k}^{\infty} a(n_1, j)R(j, m_1)}}. \tag{2.2}$$

Now for $A(n_1, k) > 0$, set $R(0, m_1) = 0$ and consider

$$\begin{aligned}
&\frac{\sum_{l=1}^{k-1} a(n_1, l)R(l, m_1)}{\sum_{j=k}^{\infty} a(n_1, j)R(j, m_1)} \\
&= \frac{\sum_{l=1}^{k} A(n_1, l)[R(l, m_1) - R(l-1, m_1)] - A(n_1, k)R(k, m_1)}{\sum_{j=k}^{\infty} A(n_1, j)[R(j, m_1) - R(j-1, m_1)] + A(n_1, k)R(k-1, m_1)} \\
&= \frac{\sum_{l=1}^{k} (A(n_1, l)/A(n_1, k))[R(l, m_1) - R(l-1, m_1)] - R(k, m_1)}{\sum_{j=k}^{\infty} (A(n_1, j)/A(n_1, k))[R(j, m_1) - R(j-1, m_1)] + R(k-1, m_1)} \\
&\geq \frac{\sum_{l=1}^{k} (A(n_2, l)/A(n_2, k))[R(l, m_1) - R(l-1, m_1)] - R(k, m_1)}{\sum_{j=k}^{\infty} (A(n_2, j)/A(n_2, k))[R(j, m_1) - R(j-1, m_1)] + R(k-1, m_1)} \\
&= \frac{\sum_{l=1}^{k-1} a(n_2, l)R(l, m_1)}{\sum_{j=k}^{\infty} a(n_2, j)R(j, m)},
\end{aligned}$$

since $\mathbf{t}^{-1}\mathbf{rt} \geq \mathbf{0}$ implies $R(i, m_1) - R(i-1, m_1) \geq 0$, $i = 1, 2, \ldots$, and $\mathbf{A} = \mathbf{at} \in TP_2$ implies $A(n_1, l)/A(n_1, k) \geq (\leq) A(n_2, l)/A(n_2, k)$ for $l \leq (\geq) k$ and $n_1 < n_2$. Then from (2.2) one gets

$$\frac{\sum_{j=k}^{\infty} a(n_1, j)R(j, m_1)}{\sum_{l=1}^{\infty} a(n_1, l)R(l, m_1)} \leq \frac{\sum_{j=k}^{\infty} a(n_2, j)R(j, m_1)}{\sum_{l=1}^{\infty} a(n_2, l)R(l, m_1)}. \tag{2.3}$$

When $A(n_1, k) = 0$, the left-hand side of (2.3) is zero and the above inequality is trivially satisfied. Combining (2.1) and (2.3) one has

$$\frac{B(n_1, m_2)}{B(n_1, m_1)} \leq \frac{B(n_2, m_2)}{B(n_2, m_1)}, \qquad 1 \leq n_1 < n_2, 1 \leq m_1 < m_2. \tag{2.4}$$

Since $\mathbf{ar} \geq 0$, $B(i, j)$ is decreasing in j and therefore $B(n_1, m_1) = 0$ implies $B(n_1, m_2) = 0$. Hence (2.4) in this case is trivially satisfied. □

REMARK 2.2. Keilson and Kester (1978) show that if $\mathbf{at} \in TP_2$ and $\mathbf{t}^{-1}\mathbf{rt} \in TP_2$, then $\mathbf{art} \in TP_2$. This follows from the observations that $\mathbf{art} = \mathbf{att}^{-1}\mathbf{rt}$ and that the class of TP_2 matrices is closed under multiplication. Observe that $\mathbf{t}^{-1}\mathbf{rt} \in TP_2$ implies $\mathbf{rt} \in TP_2$ and $\mathbf{t}^{-1}\mathbf{rt} \geq \mathbf{0}$. Furthermore, counterexamples can be easily constructed to show that the reverse need not be true. Hence our condition $\mathbf{rt} \in TP_2$ and $\mathbf{t}^{-1}\mathbf{rt} \geq \mathbf{0}$ is weaker than the TP_2 condition of $\mathbf{t}^{-1}\mathbf{rt}$.

3. DFR first-passage times and closure of DFR under geometric compounding. Let $\underline{X} = \{X_n, n = 0, 1, \ldots\}$ be a temporally homogeneous discrete-time Markov chain on the state space $N = \{0, 1, \ldots\}$ with a transition probability matrix $\mathbf{P} = [P(i, j)]_{i, j \in N}(P(i, j) = P\{X_n = j | X_{n-1} = i\},\ i, j \in N)$. Define the first-passage time

$$(3.1) \qquad T = \{\min[n: X_n = 0,\ n = 1, 2, \dots] | X_0 = 1\}.$$

Since we are only interested in the first-passage time T of $\underline{X}$, without loss of generality we will assume that state 0 is absorbing [i.e., $P(0,0) = 1$].

Let $\mathbf{Q} = \mathbf{Pt}$ be the transition survival probability matrix of $\underline{X}$ (i.e., $Q(i, j) = \sum_{k=j}^{\infty} P(i, k),\ i, j \in N$). One has

Theorem 3.1. $\mathbf{Q} \in TP_2$ *implies* $T \in DFR$.

Proof. The failure rate r_T of T is given by

$$r_T(n) \equiv P\{T = n | T \geq n\} = P\{X_n = 0 | X_{n-1} \geq 1\}$$
$$= \sum_{i=1}^{\infty} P\{X_n = 0 | X_{n-1} = i,\ X_{n-1} \geq 1\} P\{X_{n-1} = i | X_{n-1} \geq 1\}.$$

That is,

$$(3.2) \qquad r_T(n) = 1 - E\left[Q\left(\tilde{X}_{n-1}, 1\right)\right],$$

where $\tilde{X}_{n-1} =_d \{X_{n-1} | X_{n-1} \geq 1\}$ ($=_d$ stands for equality in law). Let $\underline{\nu}_n$ be the probability vector of $\tilde{X}_n$ [i.e., $\nu_n(k) = P\{\tilde{X}_n = k\}$], $\mathbf{P}_l = [P(i, j)]_{i, j \in N+}$ be a submatrix of $\mathbf{P}$ and $\underline{\nu}_n^l = \underline{\nu}_0^l \mathbf{P}_l^n$, where $\underline{\nu}_0^l = (1, 0, 0, \dots) = \underline{\nu}_0$. Since state zero is absorbing, it is easily verified that

$$(3.3) \qquad \underline{\nu}_n = \underline{\nu}_0 \mathbf{P}_l^n / \underline{\nu}_0 \mathbf{P}_l^n \underline{e} = \underline{\nu}_n^l / \underline{\nu}_n^l \underline{e},$$

where $\underline{e} = (1, 1, \dots)'$. Note that $\mathbf{P}_l$ is the transition probability matrix of the lossy process $\underline{X}^l = \{X_n^l,\ n = 0, 1, \dots\}$ of $\underline{X}$ on the state space $N+$ [cf. Keilson (1979), page 44] and $\underline{\nu}_n^l$ is the probability vector of X_n^l [i.e., $\nu_n^l(k) = P\{X_n^l = k\}$]. Now consider $(\underline{\nu}_n^l,\ n = 0, 1, \dots)$. Clearly,

$$\begin{pmatrix} \underline{\nu}_0^l \\ \underline{\nu}_1^l \end{pmatrix} \in TP_2$$

and therefore from the closure property of TP_2 matrices under multiplication

$$\begin{pmatrix} \underline{\nu}_1^l \\ \underline{\nu}_2^l \end{pmatrix} \mathbf{t} = \begin{pmatrix} \underline{\nu}_0^l \\ \underline{\nu}_1^l \end{pmatrix} \mathbf{P}_l \mathbf{t} \equiv \begin{pmatrix} \underline{\nu}_0^l \\ \underline{\nu}_1^l \end{pmatrix} \mathbf{Q}_l$$

is TP_2 since $\mathbf{Q}_l = [Q(i, j)]_{i, j \in N+}$ is TP_2 ($\mathbf{Q} \in TP_2$ implies $\mathbf{Q}_l \in TP_2$). Now as an induction hypothesis assume that

$$\begin{pmatrix} \underline{\nu}_{n-1}^l \\ \underline{\nu}_n^l \end{pmatrix} \mathbf{t} \in TP_2.$$

We have shown that this is true for $n = 1, 2$. Since $\mathbf{Pt}$ is TP_2 and $\mathbf{P}$ is a transition probability matrix, one may verify that $\mathbf{t}^{-1}\mathbf{Pt} \geq \mathbf{0}$. Observe that

$\mathbf{t}^{-1}\mathbf{P}_l\mathbf{t}$ is a submatrix of $\mathbf{t}^{-1}\mathbf{P}\mathbf{t}$. Hence $\mathbf{t}^{-1}\mathbf{P}_l\mathbf{t} \geq 0$, and therefore from Lemma 2.1

$$\begin{pmatrix} \underline{\nu}_n^l \\ \underline{\nu}_{n+1}^l \end{pmatrix}\mathbf{t} = \begin{pmatrix} \underline{\nu}_{n-1}^l \\ \underline{\nu}_n^l \end{pmatrix}\mathbf{P}_l\mathbf{t}$$

is TP_2. Observe that this TP_2 property implies

$$\frac{P\{X_{n+1}^l \geq k+1\}}{P\{X_n^l \geq k+1\}} \geq \frac{P\{X_{n+1}^l \geq k\}}{P\{X_n^l \geq k\}}, \qquad k = 1,2,\dots. \tag{3.4}$$

Combining (3.3) and (3.4) one sees that

$$\tilde{X}_{n+1} \geq_h \tilde{X}_n, \qquad n = 0,1,\dots. \tag{3.5}$$

Since $\geq_h$ implies stochastic ordering and $Q(i,1)$ is increasing in i, one has from (3.2) and (3.5), $r_T(n)$ is decreasing in $n \in N+$. □

REMARK 3.2. It is known that if $\mathbf{P}$ is TP_2, then T has log-convex probability mass function which implies $T \in$ DFR [Assaf, Shaked and Shanthikumar (1985)]. However, our condition $\mathbf{Q}$ is TP_2 is weaker since $\mathbf{P} \in TP_2$ implies $\mathbf{Q}$ is TP_2 and not necessarily in the reverse direction.

REMARK 3.3. Let $T_{\bar{k}} = \{\min[n\colon X_n \leq k-1,\ n = 1,2,\dots]|X_0 = k\}$, $k \geq 1$. Considering a modification of $\underline{X}$ such that its states $\{0,1,\dots,k-1\}$ are lumped into one absorbing state, from Theorem 3.1 one sees that $\mathbf{Q} \in TP_2$ implies $T_{\bar{k}} \in$ DFR.

Next consider an absorbing, right-continuous continuous-time Markov chain $\underline{Y} = \{Y(t),\ t \geq 0\}$ with state space N, where 0 is the absorbing state. Let $\boldsymbol{\mu} = [\mu(i,j)]_{i,j \in N}$ be the transition-rate matrix of $\underline{Y}$ [$\mu(i,j)$ is the transition rate from state i to state j],

$$\mu_i = \sum_{j=0}^{\infty} \mu_{ij}, \qquad \mathbf{D}_\mu = \operatorname{diag}\{\mu_0, \mu_1, \dots\},$$

$$T^* = \{\inf[t\colon Y(t) = 0,\ t \geq 0]|Y(0) = 1\}. \tag{3.6}$$

Define $\mu^* = \sup\{\mu_i,\ i = 0,1,\dots\}$. Then using uniformization [cf. Keilson (1979), Chapter 2 or Assaf, Shaked and Shanthikumar (1985), Section 3] one has from Theorem 3.1

COROLLARY 3.4. *If there exists a* λ, $\mu^* \leq \lambda < \infty$, *such that*

$$[\mathbf{I} + (\boldsymbol{\mu} - \mathbf{D}_\mu)/\lambda]\mathbf{t}$$

is TP_2, *then* $T^* \in$ *DFR*.

We now present some applications of the above results. Let $\{Z_n,\ n = 1,2,\dots\}$ be a sequence of i.i.d. random variables with support $N+$ and K be a geometric random variable with $P\{K = k\} = (1-p)^{k-1}p$, $k \in N+$. $\{Z_n\}$ and K are mutually independent.

THEOREM 3.5. *$Z_n \in DFR$ implies $Z^* = \Sigma_{n=1}^{K} Z_n \in DFR$.*

PROOF. Let $\underline{X}$ be a temporally homogeneous Markov chain with state space N and transition probability matrix $\mathbf{P}$, where $P(0,0) = 1$, $P(i, i+1) = 1 - r_Z(i)$, $P(i,1) = (1-p)r_Z(i)$, $P(i,0) = pr_Z(i)$, $i \in N+$ and all other entries are zero. $r_Z(i) = P\{Z_n = i | Z_n \geq i\}$ is the failure rate of Z_n. Let T be as defined in (3.1) and define

$$T_1 = \{\min[n\colon X_n \leq 1,\ n = 1,2,\dots] | X_0 = 1\}. \tag{3.7}$$

Simple calculation shows that

$$\begin{aligned} P\{T_1 = k\} &= \prod_{l=1}^{k-1} (1 - r_Z(l)) r_Z(k) \\ &= P\{Z_n = k\}, \qquad k = 1,2,\dots. \end{aligned} \tag{3.8}$$

It can also be verified that $P\{X_{T_1} = 1\} = (1-p)$ and $P\{X_{T_1} = 0\} = p$. Since state 0 is absorbing, employing the Markov property of $\underline{X}$, (3.8) and the fact that T_1 is a stopping time of $\underline{X}$, it is not hard to see that

$$T =_d Z^*. \tag{3.9}$$

Computing $\mathbf{Q}$ one sees that $Q(i,0) = 1$, $i \in N$, $Q(i,1) = 1 - pr_Z(i)$, $Q(i,j) = 1 - r_Z(i)$, $2 \leq j \leq i$; $i \in N+$ and all other entries are zero. In this case $\mathbf{Q} \in TP_2$ as long as $r_Z(i)$ is decreasing in $i \in N+$. The required result now follows from (3.9) and Theorem 3.1. □

Suppose $\{W_n,\ n = 1,2,\dots\}$ is a sequence of i.i.d. continuous random variables with support $[0,\infty)$ and survival function $\overline{F}$, and K is a geometric random variable with $P\{K = k\} = (1-p)^{k-1}p$, $k = 1,2,\dots$. $\{W_n\}$ and K are mutually independent. Then approximating W_n by a sequence of discrete random variables that converges in distribution to W_n, one easily obtains from Theorem 3.5

COROLLARY 3.6. *$W_n \in DFR$ implies $W^* = \Sigma_{n=1}^{K} W_n \in DFR$.*

REMARK 3.7. Szekli (1986) shows that W^* has convex survival function (say $\overline{F}^*$). However, $W^* \in$ DFR is equivalent to that $\overline{F}^*$ is log-convex: a stronger property than convexity.

4. Concave renewal functions. Consider a discrete-time renewal process with the inter-renewal time having the first-passage time distribution of $\underline{X}$ from state 1 to state 0. Let $\gamma(n)$ be the probability that a renewal occurs at time $n = 1,2,\dots$ (γ is the renewal density). Then the expected number of renewals $M(n)$ during $\{1,2,\dots,n\}$ is equal to $\Sigma_{k=1}^{n} \gamma(k)$ (M is the renewal function.)

THEOREM 4.1. *$\mathbf{t}^{-1}\mathbf{Q} \geq \mathbf{0}$ implies $\gamma(n)$ is decreasing in $n \in N+$ (i.e., M is concave on $N+$).*

PROOF. Let $\underline{X}^* = \{X_n^*, n = 0, 1, \dots\}$ be a modification of $\underline{X}$ such that as soon as $\underline{X}$ reaches 0 it is placed back to state 1 (representing a renewal). Then $Q^*(i, j) = Q(i, j)$, $i \in N+$, $j = 2, 3, \dots$, and $Q^*(i, 1) = 1$, $i \in N+$. So $\mathbf{t}^{-1}\mathbf{Q} \geq \mathbf{0}$ implies $\mathbf{t}^{-1}\mathbf{Q}^* \geq \mathbf{0}$ and therefore $\underline{X}^*$ is stochastically monotone. Specifically, $\{X_n^* | X_0^* = 1\} \geq_{st} \{X_{n-1}^* | X_0^* = 1\}$, $n = 1, 2, \dots$ (note that the state space of $\underline{X}^*$ is $N+$). Then the observation that

$$\gamma(n) = 1 - E\{Q(X_{n-1}^*, 1)\}$$

and that $Q(i, 1)$ is increasing in i leads to the desired conclusion. □

We will next see that for inter-renewal times with a discrete distribution, this distribution need not be DFR for the renewal function to be concave. This will be achieved by showing that $\mathbf{t}^{-1}\mathbf{Q} \geq \mathbf{0}$ is not sufficient for the DFR property of T.

COUNTEREXAMPLE 4.2. Consider the first-passage time T [defined in (3.1)] of a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix}.$$

Then

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{4} \\ 1 & 1 & \frac{1}{4} \end{bmatrix}$$

and $\mathbf{t}^{-1}\mathbf{Q} \geq \mathbf{0}$. So the renewal density is decreasing (Theorem 4.1). Computing one gets $r_T(1) = \frac{3}{4}$; $r_T(2) = 0$; $r_T(3) = \frac{9}{16}$. So T is not DFR. Then one sees that in the discrete-time case the *DFR property of inter-renewal times is sufficient but not necessary* for a concave renewal function.

We next turn our attention to the continuous-time case. The inter-renewal times have the first-passage time distribution of $\underline{Y}$ from state 1 to state 0. γ and M are the renewal density and renewal function, respectively, of the renewal process. Similar to the discrete-time case one has

THEOREM 4.3. *$\underline{Y}$ is stochastically monotone implies $\gamma(t)$ is decreasing in $t \in [0, \infty)$ [i.e., M is concave on $[0, \infty)$].*

REMARK 4.4. An interesting consequence of Theorem 4.3 and the conjecture of Brown (1981), if true, is that the first-passage time T^* from state 1 to state 0 of a stochastically monotone Markov process $\underline{Y}$ with state space N is DFR.

5. Application in queueing. Consider a GI/G/1 queue at which customers arrive according to a renewal process with rate λ. The service times form a sequence of i.i.d. continuous random variables with a common distribution function F and mean $E[S]$. The arrival process and service times are mutually independent. We will first consider an M/G/1 queue.

THEOREM 5.1. *The number of customers served during a busy period of an* M/G/1 *queue with IFR service times is DFR.*

PROOF. Let $\tilde{X}_n$ be the number of customers in the M/G/1 queueing system just after the nth customer departure. Then $\underline{\tilde{X}} = \{\tilde{X}_n,\ n = 0, 1, \dots\}$ is a temporally homogeneous Markov chain with transition probabilities $\tilde{P}(i, j) = g(j + 1 - i)$, $i \in N+$; $\tilde{P}(0, j) = g(j)$, $j \in N$, where

$$g(k) = \begin{cases} 0, & k = -1, -2, \dots, \\ \displaystyle\int_0^\infty \frac{e^{-\lambda t}(\lambda t)^k}{k!}\, dF(t), & k = 0, 1, \dots \end{cases}$$

[cf. Ross (1983)]. Consider a modification $\underline{X}$ of $\underline{\tilde{X}}$ such that state 0 is absorbing in $\underline{X}$ [i.e., $P(i, j) = \tilde{P}(i, j)$, $i \in N+$, $j \in N$ and $P(0,0) = 1$]. Then if T is as defined in (3.1), it is the number of customers served during the first busy period. Consider $Q(i, j) = \overline{G}(j + 1 - i)$, $i, j \in N+$ $[Q(0,0) = 1,\ Q(0, j) = 0,\ j \in N+]$,

$$\overline{G}(k) = \begin{cases} 1, & k = -1, -2, \dots, \\ \displaystyle\sum_{l=k}^{\infty} g(l), & k = 0, 1, \dots. \end{cases}$$

It is known that if $\overline{F}(t + x)/\overline{F}(t)$ is decreasing in t (i.e., F is IFR), then $\overline{G}(n + k)/\overline{G}(k)$ is decreasing in k [Block and Savits (1980)]. This observation leads to a straightforward verification that $\mathbf{Q} \in TP_2$. The DFR property of T now follows from Theorem 3.1. □

Let H be the stationary distribution function of the waiting time in an M/G/1 queue with a server utilization $\rho \equiv \lambda E[S] < 1$. It is well known that [cf. Ross (1982)]

$$H(t) = (1 - \rho) + \rho(1 - \rho) \sum_{k=1}^{\infty} \rho^{k-1} F_R^{(k)}(t),$$

where $F_R^{(k)}$ is the k-fold convention of F_R with itself and F_R is the distribution function of the stationary residual life of the service times [i.e., $F_R(t) = (1/E[S]) \int_0^t (1 - F(x))\, dx$]. Observe that H is a mixture of a mass at the origin and a geometric compound of F_R. Since F is IMRL implies F_R is DFR from Corollary 3.6, it is not hard to see

THEOREM 5.2. *The stationary waiting time in an* M/G/1 *queue with IMRL service times is DFR.*

Next we will look at GI/G/1 queues. Let H be the stationary distribution function of the waiting time in a GI/G/1 queue with server utilization $\rho \equiv$

$\lambda E[S] < 1$. Then

$$H(t) = (1 - \alpha) + \alpha(1 - \alpha) \sum_{k=1}^{\infty} \alpha^{k-1} F_M^{(k)}(t),$$

where $\alpha = 1 - H(0)$ and F_M is obtained from F and the distribution function of the inter-arrival times [cf. Feller (1971)]. Szekli (1986) shows that if F is DFR, then F_M is also DFR [see Lemma 3.2 of Szekli (1986)]. It is then clear from Corollary 3.6 that one has

THEOREM 5.3. *The stationary waiting time in a* GI/G/1 *queue with DFR service times is DFR.*

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