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MAXIMAL INCREMENTS OF LOCAL TIME OF A RANDOM WALK[1]

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Let (S_j) be a lattice random walk, i.e., $S_j = X_1 + \cdots + X_j$, where $X_1, X_2, \dots$ are independent random variables with values in $\mathbb{Z}$ and common nondegenerate distribution F. Let $\{t_n\}$ be a nondecreasing sequence of positive integers, $t_n \le n$, and $L_n^* = \max_{0 \le j \le n - t_n}(L_{j+t_n} - L_j)$, where $L_n = \sum_{j=1}^n 1_{\{0\}}(S_j)$, the number of times zero is visited by the random walk by time n. Assuming that the random walk is recurrent and satisfies a more general condition than being in the domain of attraction of a stable law of index $\alpha > 1$, the following results are obtained:

(i) Constants β_n are defined such that $\limsup L_n^* \beta_n^{-1} = 1$ a.s.

(ii) If $\limsup n t_n^{-1} = \infty$, then constants γ_n are defined such that $\liminf L_n^* \gamma_n^{-1} = 1$ a.s. If $\limsup n t_n^{-1} < \infty$, then $\liminf(L_n^*/\gamma_n') = 0$ or ∞ for any choice of γ_n' and a simple test is given to determine which is the case.

(iii) If $\lim \log(n t_n^{-1})/\log_2 n = \infty$, then $\beta_n \sim \gamma_n$ and $\lim L_n^* \beta_n^{-1} = 1$ a.s. Also, the normalizers are found more explicitly in the domain of attraction case.

1. Introduction. Let $X_1, X_2, \dots$ be independent identically distributed random variables on a probability space (Ω, F, P) taking values in the one-dimensional integer lattice $\mathbb{Z}$ with the common distribution function F. The partial sums

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n, \qquad n \ge 1, \tag{1.1}$$

define a random walk. The local time L_n at 0 up to time n is the number of visits to 0 by the random walk between times 1 and n (inclusive), i.e.,

$$L_n = \sum_{j=1}^{n} 1_{\{0\}}(S_j), \qquad n \ge 1, \quad L_0 = 0, \tag{1.2}$$

where 1_A denotes the indicator function of A. If the random walk is recurrent, then $L_n \to \infty$ a.s. We will be concerned only with recurrent random walks here. Let $\{t_n\}$ be a *nondecreasing* positive integer sequence with $t_1 = 1$, $t_n \le n$, and let

$$L_{n,t_n}^* = \max_{0 \le j \le n - t_n} \left(L_{j+t_n} - L_j\right) \tag{1.3}$$

be the maximum number of visits to zero by the random walk over a time span t_n up to time n. Note that $L_{n,t_n}^* = L_n$ if $t_n = n$. To simplify notation if $\{t_n\}$ is fixed in a given context we will drop the subscript t_n and write L_n^* for L_{n,t_n}^*. Our

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aim here is to study the almost sure asymptotic behavior of L_n^* for a very general class of recurrent random walks. This is possible because of the very general large deviation probability estimates for the lower tail of partial sums of i.i.d. nonnegative random variables obtained in [6].

The class of random walks we consider here is the same one that we considered in [5] where some other aspects of the local time were explored. To describe this class, for $x > 0$ let

$$G(x) = P\{|X_1| > x\}, \qquad K(x) = x^{-2} \int_{|y| \le x} y^2 \, dF(y), \tag{1.4}$$

$$Q(x) = G(x) + K(x). \tag{1.5}$$

Our basic assumption is

$$\limsup_{x \to \infty} \frac{G(x)}{K(x)} < 1 \quad \text{and} \quad EX_1 = 0. \tag{A.1}$$

The first condition implies $E|X_1| < \infty$, so the second condition is to ensure recurrence. If X_1 is in the domain of attraction of a stable law of index α, then

$$\lim_{x \to \infty} \frac{G(x)}{K(x)} = \frac{2 - \alpha}{\alpha},$$

so that our assumption includes all distributions in the domain of attraction of a stable law of index $\alpha > 1$ which have zero mean.

In the next section we will give some basic notation and preliminary results. In Section 3 we define a real sequence $\{\beta_n\}$ such that $\limsup_n \beta_n^{-1} L_n^* = 1$ a.s. In Section 4 the lim inf behavior of L_n^* is analyzed. Here we obtain a dichotomy: If $\limsup_n n t_n^{-1} = \infty$, then there exists a real sequence $\{\gamma_n\}$ such that $\liminf_n \gamma_n^{-1} L_n^* = 1$ a.s., and in the contrary case for *any* real sequence $\{\gamma_n'\}$ either $\liminf_n (L_n^*/\gamma_n')$ is 0 a.s. or ∞ a.s. In Section 5 we show that if $\{t_n\}$ satisfies

$$\lim_{n \to \infty} \frac{\log(n/t_n)}{\log\log n} = \infty, \tag{1.6}$$

then in fact with $\{\beta_n\}$ as in Section 3

$$\lim_{n \to \infty} \beta_n^{-1} L_n^* = 1 \quad \text{a.s.} \tag{1.7}$$

An example in Section 6 shows that the lim sup and lim inf behaviors of L_n^* may be different if (1.6) is not satisfied. In this section we specialize the results to the domain of attraction case and give more explicit forms for β_n and γ_n, the normalizers for lim sup and lim inf, respectively.

Results of this type were obtained earlier by Csáki and Földes [1], motivating the present work. These authors consider only the case when X_1 has zero mean and finite variance, and $t_n \uparrow \infty$ and either satisfies (1.6) or for some $c > 0$, $t_n = c \log n$. For $\{t_n\}$ we assume only that it is nondecreasing. For other references and related work we refer to [1].

We will assume that the random walk satisfies

$$\{j\colon P\{S_n = j \text{ for some } n\} > 0\} = \mathbb{Z};$$

since the random walk is recurrent, this amounts to a rescaling of the state space. If the random walk is periodic (as a Markov chain), p will denote the period.

In the sequel it will be convenient to use the notation $a_n \approx b_n$ for two nonnegative sequences $\{a_n\}$ and $\{b_n\}$ to mean that there exist two positive constants c_1, c_2 such that

$$c_1 a_n \le b_n \le c_2 a_n, \qquad n \ge 1.$$

c, C with or without suffixes will be positive constants whose values will change from one context to another. The values of these constants are not important, but their dependence on certain parameters may be; when that is the case such dependence will be emphasized. As usual if x is real then $[x]$ will stand for the largest integer $\le x$. To simplify matters we will follow the convention that if a subscript should be an integer but is a real number x then it should be interpreted as $[x]$; e.g., L_t for $t \ge 0$ should be read as $L_{[t]}$. This should cause no confusion.

2. Preliminaries and probability estimates. In this section we derive some basic probability estimates which will be needed to prove the main results in the following sections. The first lemma gives some basic properties of the function Q defined in (1.5). For the proof see Lemma 2.4 of [8].

LEMMA 2.1. *The function Q defined in* (1.5) *is continuous and strictly decreasing for $x \ge x_0 = \sup\{y\colon G(y) = 1\}$ where G is defined in* (1.4). *The function $x^2 Q(x)$ is nondecreasing for $x \ge 0$ and under* (A.1) *there exist $x_1 \ge 0$ and $\lambda \in (1,2)$ such that $x^\lambda Q(x)$ is decreasing for $x \ge x_1$.*

We define a_y, $y \ge 0$, by $a_y = 1$ for $0 \le y \le y_0 = (Q(1))^{-1}$ and

$$Q(a_y) = y^{-1}, \quad \text{for } y > y_0. \tag{2.1}$$

The function a_y is nondecreasing, tends to ∞ as $y \to \infty$, and if (A.1) holds then by Lemma 2.1

$$a_y^\lambda y^{-1}\downarrow, \qquad a_y^2 y^{-1}\uparrow, \qquad y \ge \text{some } y_1. \tag{2.2}$$

We will frequently use (2.2) in the form: There exist positive constants c, C such that for $0 \le k \le n$

$$a_n \le c a_k \left[\frac{n+1}{k+1}\right]^{1/\lambda}, \qquad a_k \le C a_n \left[\frac{k+1}{n+1}\right]^{1/2}. \tag{2.3}$$

The sequence $\{a_n\}$ plays an important role because under (A.1) the sequence $\{a_n^{-1} S_n\}$ is stochastically compact, i.e., it is tight and contains no subsequence that converges to a degenerate law ([2] and [4]); furthermore, it satisfies an approximate local limit theorem [4].

We introduce the sequences

$$u_n = P\{S_n = 0\}, \qquad n \ge 1, \quad u_0 = 1; \tag{2.4}$$

$$r_n = P\{S_1 \ne 0, \dots, S_n \ne 0\}, \qquad n \ge 1, \quad r_0 = 1. \tag{2.5}$$

The next few results will give asymptotic relations among these sequences and some other probabilities of interest.

LEMMA 2.2. *Suppose* (A.1) *holds. Then*

$$u_n = O(a_n^{-1}). \tag{2.6}$$

Furthermore, if p is the period of the random walk, then there exist c and n_0 such that

$$u_{np} \ge c a_{np}^{-1}, \qquad n \ge n_0. \tag{2.7}$$

PROOF. This is an immediate consequence of Theorems 1 and 3 in [4]. □

LEMMA 2.3. *Suppose* (A.1) *holds. Then*

$$\sum_{k=0}^{n} r_k \approx a_n \approx (n+1) r_n \quad \textit{and} \quad \sum_{k=0}^{n} u_k \approx (n+1) a_n^{-1}. \tag{2.8}$$

PROOF. Decomposing according to the last visit to zero before time n and using (2.7) and the monotonicity of a_n and r_n gives (even if $p > 1$)

$$1 = \sum_{k=0}^{n} u_k r_{n-k} \ge c a_n^{-1} \sum_{k=0}^{n} r_k. \tag{2.9}$$

Similarly,

$$\sum_{k=0}^{n} u_k \ge c a_n^{-1}(n+1).$$

By (2.6) and (2.3),

$$\begin{aligned} \sum_{k=0}^{n} u_k &\le C \sum_{k=0}^{n} a_k^{-1} \\ &\le C_1 \sum_{k=0}^{n} a_n^{-1} \left\{ \frac{n+1}{k+1} \right\}^{1/\lambda} \\ &\approx (n+1) a_n^{-1}. \end{aligned} \tag{2.10}$$

Next, by (2.6), (2.9), and (2.3) for $\varepsilon < \frac{1}{2}$,

$$\begin{aligned} \sum_{k=(1-\varepsilon)n}^{n} u_k r_{n-k} &\le C a_n^{-1} \sum_{k=0}^{\varepsilon n} r_k \\ &\le C_1 a_n^{-1} a_{\varepsilon n} \le C_2 \varepsilon^{1/2}, \end{aligned}$$

where C_2 is independent of ε. Thus ε may be chosen to make this sum less than $\frac{1}{2}$. Then by (2.9) and (2.10)

$$\frac{1}{2} \le \sum_{k=0}^{(1-\varepsilon)n} u_k r_{n-k} \le r_{\varepsilon n} \sum_{k=0}^{(1-\varepsilon)n} u_k$$
$$\le C r_{\varepsilon n} n a_{(1-\varepsilon)n}^{-1} \le C\varepsilon^{-1} r_{\varepsilon n} \varepsilon n a_{\varepsilon n}^{-1}.$$

This gives the hardest inequality needed for the first assertion in (2.8). The final bound follows trivially from the monotonicity of r_n. □

Corollary 2.1. *Suppose* (A.1) *holds. For* $c > 0$ *we have*

$$a_{cn} \approx a_n, \qquad r_{cn} \approx r_n.$$

The constants in these relations depend on c.

Proof. The first is immediate from (2.3) and then the second follows using (2.8). □

Lemma 2.4. *Suppose* (A.1) *holds. Then there exist* $0 < c < C$ *such that if* $(m, n]$ *contains a multiple of* p, *the period of the random walk, we have*

$$c a_{n-m} a_n^{-1} \le P\{S_j = 0 \text{ for some } j \in (m, n]\} \le C a_{n-m} a_n^{-1}.$$

Proof. Note that by (2.7)

$$P\{S_j = 0 \text{ for some } j \in (m, n]\} = \sum_{j=m+1}^{n} u_j r_{n-j} \ge c a_n^{-1} \sum_{j=0}^{n-m-1} r_j$$

and then use (2.8) to estimate the sum. The same argument provides the upper bound if $m \ge n/2$; in the other case the trivial upper bound of 1 suffices. □

Corollary 2.2. *Suppose* (A.1) *holds. Given* $\eta \in (0,1)$ *there exists* n_0 *such that for* $n \ge n_0$

$$P\{S_j = 0 \text{ for some } j \in (\eta n, n]\} \approx 1.$$

Proof. This follows from Lemma 2.4 and Corollary 2.1. □

We will also need estimates for the probability that S_n has no zero in an interval. But for these we also need to include events giving information about the local time.

Lemma 2.5. *Assume* (A.1) *and let* $A \subset \mathbb{Z}$, $0 \le k \le m < n$. *Then there exists* C *such that*

$$P\{L_k^* \in A, S_j \ne 0 \text{ for } m < j \le n\} \le C r_n r_m^{-1} P\{L_k^* \in A\}.$$

Remark 2.1. The lemma also applies to L_k since one may take $t_k = k$.

PROOF. The bound is obvious if $m > n/2$ since then $r_m \approx r_n$. Thus assume $m \le n/2$ and let

$$U = \max\{j \le m: S_j = 0\}.$$

Then we have for $j \le k$

$$\begin{aligned}
P\{L_k^* \in A, U = j, S_i \neq 0 \text{ for } m < i \le n\} \\
&= P\{L_{j,t_k}^* \in A, S_j = 0\} r_{n-j} \\
&\le r_{n-m} r_m^{-1} P\{L_{j,t_k}^* \in A, S_j = 0\} r_{m-j} \\
&= r_{n-m} r_m^{-1} P\{L_k^* \in A, U = j\},
\end{aligned}$$

while for $k < j \le m$

$$\begin{aligned}
P\{L_k^* \in A, U = j, S_i \neq 0 \text{ for } m < i \le n\} \\
&= P\{L_k^* \in A, S_j = 0\} r_{n-j} \\
&\le r_{n-m} r_m^{-1} P\{L_k^* \in A, S_j = 0\} r_{m-j} \\
&\le r_{n-m} r_m^{-1} P\{L_k^* \in A, U = j\}.
\end{aligned}$$

Summing over all $j \le m$ then gives the result since $r_{n-m} \approx r_n$ in this case. □

The lower bound analogous to Lemma 2.5 is not valid in general. But we will prove it when $L_k^* \in A$ is replaced by $L_k \le x$. But first we need some information about the time intervals between successive zeroes of S_n. Let

$$\begin{aligned}
T_1 &= \min\{j > 0: S_j = 0\}, \\
T_k &= \min\{j > 0: S_{T_1 + \cdots + T_{k-1} + j} = 0\}, \qquad k > 1.
\end{aligned} \tag{2.11}$$

The random variables $T_1, T_2, \ldots$ are independent and identically distributed. Let F_1 denote their common distribution and G_1, K_1, and Q_1 the corresponding functions defined in (1.4) and (1.5). Then we have

LEMMA 2.6. *Assume* (A.1). *The distribution F_1 of T_1 satisfies the stochastic compactness assumption, i.e.,*

$$\limsup_{x \to \infty} \frac{G_1(x)}{K_1(x)} < \infty. \tag{2.12}$$

We also have the bounds

$$E(T_1 \wedge n) \approx n r_n \approx a_n \quad \textit{so that } ET_1 = \infty, \tag{2.13}$$

and there exist $0 < c < C$ such that for all $x \ge 0$, $n \ge 0$,

$$\begin{aligned}
c[(x+1) r_n \wedge 1] \le P\{L_n \le x\} &= P\{T_1 + \cdots + T_{x+1} > n\} \\
&\le C[(x+1) r_n \wedge 1].
\end{aligned} \tag{2.14}$$

PROOF. For $0 < \varepsilon < 1$,

$$n^2 K_1(n) \geq \int_{(\varepsilon n, n]} y^2 \, dF_1(y) \geq \varepsilon^2 n^2 P\{\varepsilon n < T_1 \leq n\}$$
$$= \varepsilon^2 n^2 (r_{\varepsilon n} - r_n)$$

and by (2.8) and (2.3)

$$r_{\varepsilon n} \geq c_1(\varepsilon n)^{-1} a_{\varepsilon n} \geq c_2(\varepsilon n)^{-1} a_n \varepsilon^{1/\lambda} \tag{2.15}$$
$$\geq c_3 r_n \varepsilon^{-1+1/\lambda} \geq 2r_n,$$

for appropriate ε and large n. Thus

$$n^2 K_1(n) \geq \varepsilon_1^2 n^2 r_n = \varepsilon_1^2 n^2 G_1(n),$$

which proves (2.12). Next,

$$E(T_1 \wedge n) = \sum_{k=1}^{n} P\{T_1 \geq k\} = \sum_{k=1}^{n} r_{k-1} \approx a_n \approx n r_n$$

by (2.8). For the upper bound in (2.14), we have

$$P\{T_1 + \cdots + T_{x+1} > n\} \leq \frac{(x+1)E\{T_1 \wedge (n+1)\}}{n+1} \approx (x+1) r_n.$$

For the lower bound we use

$$\{T_1 + \cdots + T_{x+1} > n\} \supset \bigcup_{k=1}^{j} \{T_k > n\},$$

with $j = (x+1) \wedge r_n^{-1}$. By inclusion–exclusion,

$$P\left(\bigcup_{k=1}^{j} \{T_k > n\} \right) \geq j r_n - \tfrac{1}{2} j(j-1) r_n^2$$
$$= j r_n \left(1 - \tfrac{1}{2}(j-1) r_n\right) \geq \tfrac{1}{2} j r_n,$$

which completes the proof. □

Now we can prove the lower bound somewhat analogous to Lemma 2.5.

LEMMA 2.7. *Assume* (A.1). *For* $x \geq 0$, $0 \leq k \leq m < n$, *there exists* $c > 0$ *such that*

$$P\{L_k \leq x,\, S_j \neq 0 \text{ for } m < j \leq n\} \geq c r_n r_m^{-1} P\{L_k \leq x\}.$$

PROOF. We let

$$F(i, j; k, m, n) = \{T_1 + \cdots + T_i \leq k,\, T_1 + \cdots + T_{i+1} > k,$$
$$T_1 + \cdots + T_j \leq m,\, T_1 + \cdots + T_{j+1} > n\}.$$

Then we have

$$P\{L_k \leq x,\, S_j \neq 0 \text{ for } m < j \leq n\} = \sum_{i=0}^{x} \sum_{j=i}^{\infty} P(F(i, j; k, m, n)).$$

By (2.15) we can find C so that

$$P\{k < T_1 \le Ck\} = r_k - r_{Ck} \ge r_k/2. \tag{2.16}$$

Now if $m < 2Ck$ we simply use

$$\begin{aligned} P(F(i, i; k, m, n)) &= P\{T_1 + \cdots + T_i \le k,\, T_1 + \cdots + T_{i+1} > n\} \\ &\ge P\{T_1 + \cdots + T_i \le k,\, T_{i+1} > n\} \\ &= r_n P\{T_1 + \cdots + T_i \le k\}. \end{aligned}$$

By (2.14) each such term will be at least $c_1 r_n$ if $i \le c_2 r_k^{-1}$, so summing on i leads to

$$\begin{aligned} P\{L_k \le x,\, S_j \ne 0 \text{ for } m < j \le n\} &\ge c_3 r_n\big((x+1) \wedge r_k^{-1}\big) \\ &\approx r_n r_m^{-1} P\{L_k \le x\} \end{aligned}$$

by (2.14) again since $r_m \approx r_k$ in this case. If $m \ge 2Ck$, then by (2.16)

$$\begin{aligned} &P(F(i, j; k, m, n)) \\ &\quad \ge P\{T_1 + \cdots + T_i \le k,\, k < T_{i+1} \le Ck, \\ &\qquad\qquad T_1 + \cdots + T_i + T_{i+2} + \cdots + T_j \le m/2\} r_n \\ &\quad \ge c_4 r_k r_n P\{T_1 + \cdots + T_i \le k,\, T_1 + \cdots + T_{j-1} \le m/2\}. \end{aligned}$$

Now using (2.14) this exceeds $c_5 r_k r_n$ provided that $i \le c_6 r_k^{-1}$, $j \le c_6 r_m^{-1}$. Since $r_m \le r_{2Ck} \le r_{Ck} \le r_k/2$ in this case we will obtain

$$\begin{aligned} &P\{L_k \le x,\, S_j \ne 0 \text{ for } m < j \le n\} \\ &\quad \ge c_5 r_k r_n\big((x+1) \wedge c_6 r_k^{-1}\big) c_6\big(r_m^{-1} - r_k^{-1}\big) \\ &\quad \ge c_7 r_n r_m^{-1}\big((x+1) r_k \wedge 1\big) \approx r_n r_m^{-1} P\{L_k \le x\}, \end{aligned}$$

which completes the proof. □

COROLLARY 2.3. *Assume* (A.1). *For* $x \ge 0$, $0 \le k \le m < n$, *we have*

$$P\{L_k \le x,\, S_j \ne 0 \textit{ for } m < j \le n\} \approx r_n r_m^{-1} P\{L_k \le x\} \approx r_n r_m^{-1}\big((x+1) r_k \wedge 1\big)$$

and

$$P\{S_j \ne 0 \textit{ for } m < j \le n\} \approx r_n r_m^{-1},$$

where the constants are independent of x, k, m, *and* n.

PROOF. Use Lemmas 2.5–2.7. For the last statement, let $x \to \infty$. □

Next we obtain a bound analogous to (2.13) for the delayed return times to zero for S_n. Let $\{t_n\}$ be a nondecreasing sequence of positive integers with $t_n \le n$ and define $\tau_0 = 0$,

$$\tau_k = \min\{j \ge \tau_{k-1} + t_n \colon S_j = 0\}, \qquad \xi_k = \tau_k - \tau_{k-1}, \qquad k \ge 1, \tag{2.17}$$

and let

$$\theta_n = a_{t_n} a_n^{-1} n t_n^{-1}.$$

LEMMA 2.8. *Assume* (A.1). *For* $k \geq t_n$ *we have*

$$P\{\xi_1 > k\} \approx r_k r_{t_n}^{-1}, \qquad P\{\xi_1 > n\} \approx r_n r_{t_n}^{-1} \approx \theta_n^{-1},$$

$$E(\xi_1 \wedge n) \approx n\theta_n^{-1} \approx n r_n r_{t_n}^{-1}.$$

Furthermore, there exist positive constants c, C *and* $\lambda \in (1,2)$ *such that*

$$c(n/t_n)^{1-1/\lambda} \leq \theta_n \leq C(n/t_n)^{1/2}.$$

If $n \leq 2m$, *then there exists* $C_1 > 0$ *such that*

$$\theta_k \leq C_1 \theta_m \quad \textit{uniformly in } k \in [m, n]. \tag{2.18}$$

If, in addition, $t_n \leq 2t_m$, *then*

$$\theta_k \approx \theta_m \quad \textit{uniformly for } k \in [m, n]. \tag{2.18a}$$

PROOF. The first two statements follow immediately from Corollary 2.3 and Lemma 2.3. Then using Lemma 2.3 again

$$E(\xi_1 \wedge n) = \sum_{k=1}^{n} P\{\xi_1 \geq k\} \leq t_n + \sum_{k=t_n}^{n-1} P\{\xi_1 > k\}$$

$$\leq t_n + C \sum_{k=t_n}^{n-1} r_k r_{t_n}^{-1} \leq C \sum_{k=0}^{n-1} r_k r_{t_n}^{-1} \approx n r_n r_{t_n}^{-1}.$$

For the lower bound, if $t_n \geq n/2$, then

$$E(\xi_1 \wedge n) \geq t_n \geq n/2 \approx n\theta_n^{-1},$$

since $\theta_n \approx 1$ in this case. If $t_n < n/2$ then

$$E(\xi_1 \wedge n) \geq \sum_{k=t_n}^{n-1} P\{\xi_1 > k\} \geq c \sum_{k=t_n}^{n-1} r_k r_{t_n}^{-1}$$

$$\geq c r_n r_{t_n}^{-1}(n - t_n) \approx n r_n r_{t_n}^{-1}.$$

The bounds for θ_n follow immediately from (2.3). For (2.18), we have

$$\theta_m \approx \frac{r_{t_m}}{r_m} \approx \frac{r_{t_m}}{r_n} \geq \frac{r_{t_k}}{r_k} \geq \frac{r_{t_n}}{r_m}$$

and $r_{t_k} r_k^{-1} \approx \theta_k$ while $r_{t_n} r_m^{-1} \approx \theta_m$ under the additional hypothesis. This proves the lemma. □

The main results of this section are estimates for the upper tail of the distribution of L_n and both tails of the distribution of L_n^*. Note that an estimate for the lower tail of the distribution of L_n has already been obtained in (2.14). We let

$$a = \min\{j\colon P\{T_1 = j\} > 0\}, \qquad q = P\{T_1 = a\}. \tag{2.19}$$

Next define for $s > 0$

$$\varphi(s) = Ee^{-sT_1}, \qquad g(s) = -\varphi'(s)/\varphi(s), \tag{2.20}$$
$$h(s) = -(g(s))^{-1}\log\varphi(s) - s.$$

One can easily check that $g\downarrow$, $h\uparrow$, and

$$g(0+) = \infty, \qquad g(\infty) = a, \qquad h(0+) = 0, \qquad h(\infty) = -a^{-1}\log q.$$

The general form of the upper tail of the distribution of L_n is really in Theorem 2.1 of [6], which will now be converted to a form appropriate for our application. Since this theorem was proved in complete generality, we do not need assumption (A.1) here but only recurrence of $\{S_n\}$.

THEOREM 2.1. *Let $t_n \uparrow \infty$ be an integer sequence and ρ_n a positive sequence such that $\rho_n t_n \to \infty$. Let*

$$\beta_n = \left[(t_n - 1)a^{-1}\right] + 1, \quad \textit{if } \rho_n \geq -a^{-1}\log q, \tag{2.21}$$

$$\beta_n = t_n/g(\lambda_n), \quad \textit{if } \rho_n < -a^{-1}\log q, \tag{2.22}$$

where λ_n is the unique solution of $h(\lambda_n) = \rho_n$. Then $\beta_n \to \infty$ and given $c > 0$, $0 < \varepsilon, \eta < 1$, there exists n_0 such that for $n \geq n_0$

$$\log P\{L_{ct_n} > c\beta_n\} \leq -ct_n\rho_n(1-\eta), \tag{2.23}$$

$$\log P\{L_{ct_n} > c(1-\varepsilon)\beta_n\} \geq -ct_n\rho_n(1-\varepsilon)(1+\eta). \tag{2.24}$$

PROOF. To show that $\beta_n \to \infty$, we only need to consider the case when $\beta_n = t_n/g(\lambda_n)$ and then only for $\lambda_n \to 0$ along a subsequence. But then

$$\beta_n = \frac{t_n}{g(\lambda_n)} \geq -t_n\frac{h(\lambda_n)}{\log\varphi(\lambda_n)} = -\frac{t_n\rho_n}{\log\varphi(\lambda_n)} \to \infty,$$

along the subsequence since $t_n\rho_n \to \infty$, $\varphi(\lambda_n) \to 1$.

For the bounds, it suffices, by looking at subsequences, to consider separately the two forms for β_n. To prove (2.23), let $\alpha_n = [c\beta_n] + 1$. Then

$$P\{L_{ct_n} > c\beta_n\} = P\left\{T_1 + \cdots + T_{\alpha_n} \leq \alpha_n\frac{ct_n}{\alpha_n}\right\}.$$

If β_n is given by (2.21), then $ct_n\alpha_n^{-1} < a$ so the probability is zero. In the other case when β_n is given by (2.22), we have $ct_n\alpha_n^{-1} < t_n\beta_n^{-1} = g(\lambda_n)$ and so by Theorem 2.1 of [6]

$$\log P\{L_{ct_n} > c\beta_n\} \leq \log P\{T_1 + \cdots + T_{\alpha_n} \leq \alpha_n g(\lambda_n)\} \sim -\alpha_n h(\lambda_n)g(\lambda_n)$$
$$\leq -c\beta_n\rho_n t_n\beta_n^{-1} = -c\rho_n t_n.$$

The factor of $1-\eta$ is to allow for the $\sim$.

To prove (2.24), we abuse the notation slightly by letting

$$\alpha_n = [(1-\varepsilon)c\beta_n] + 1.$$

Then

$$P\{L_{ct_n} > c(1-\varepsilon)\beta_n\} = P\left\{T_1 + \cdots + T_{\alpha_n} \le \alpha_n \frac{ct_n}{\alpha_n}\right\}.$$

If β_n is given by (2.21), then

$$ct_n\alpha_n^{-1} \sim (1-\varepsilon)^{-1} t_n \beta_n^{-1} \sim (1-\varepsilon)^{-1} a,$$

so that $ct_n\alpha_n^{-1} \ge (1+\varepsilon)a$ for large n. Thus for $\eta_1 \in (0, \varepsilon] \cap (0, \eta)$

$$\log P\{L_{ct_n} > c(1-\varepsilon)\beta_n\} \ge \log P\{T_1 + \cdots + T_{\alpha_n} \le \alpha_n(1+\eta_1)a\}$$
$$\sim -\alpha_n h(\lambda) g(\lambda),$$

where $g(\lambda) = a(1+\eta_1)$ by Theorem 2.1 of [6]. Then

$$\begin{aligned}\alpha_n h(\lambda) g(\lambda) &\sim (1-\varepsilon)c\beta_n h(\lambda) a(1+\eta_1)\\ &\sim (1-\varepsilon)ct_n h(\lambda)(1+\eta_1)\\ &\le (1-\varepsilon)ct_n(-a^{-1}\log q)(1+\eta_1)\\ &\le (1-\varepsilon)ct_n\rho_n(1+\eta_1),\end{aligned}$$

which is sufficient for this case. Finally, if β_n is given by (2.22), then

$$\begin{aligned}ct_n\alpha_n^{-1} &\sim (1-\varepsilon)^{-1} t_n \beta_n^{-1}\\ &= (1-\varepsilon)^{-1} g(\lambda_n),\end{aligned}$$

so that

$$ct_n\alpha_n^{-1} \ge g(\lambda_n),$$

for large n. Thus, again by Theorem 2.1 of [6],

$$\log P\{L_{ct_n} > c(1-\varepsilon)\beta_n\} \ge \log P\{T_1 + \cdots + T_{\alpha_n} \le \alpha_n g(\lambda_n)\}$$
$$\sim -\alpha_n h(\lambda_n) g(\lambda_n) \sim -(1-\varepsilon)c\rho_n t_n.$$

This completes the proof. □

We close this section with the bounds for the distribution of L_n^*. We start with an easy intermediate result.

LEMMA 2.9. *Define*

$$L_n^{\sim} = \max_{0 \le j \le n} (L_{j+n} - L_j) = L_{2n,n}^*.$$

Then for every $\varepsilon > 0$

(2.25) $$P\{L_n > x\} \le P\{L_n^{\sim} > x\} \le ([\varepsilon^{-1}] + 1) P\{L_{(1+\varepsilon)n} > x - 1\}.$$

Also, if (A.1) *holds, there is a positive c such that*

(2.26) $$cP\{L_n \le x\} \le P\{L_n^{\sim} \le x\} \le P\{L_n \le x\}.$$

PROOF. For the upper bound in (2.25), use

$$\{L_n^{\sim} > x\} \subset \bigcup_{k=0}^{[\varepsilon^{-1}]} \{L_{n+(k+1)\varepsilon n} - L_{k\varepsilon n} > x\}.$$

Then to estimate the probability, start over at the first zero after $k\varepsilon n$. This reduces the number of zeros by 1 which is the reason for $x - 1$ in the bound. For the lower bound in (2.26), use

$$P\{L_n \le x,\, S_j \ne 0 \text{ for } n < j \le 2n\} \le P\{L_n^{\sim} \le x\}$$

and apply Lemma 2.7. The other bounds are trivial. □

The probability estimates for L_n^* can now be given in terms of θ_n and $P\{L_{t_n}^{\sim} > x\}$.

THEOREM 2.2. *Suppose* (A.1) *holds and that* $t_n \le n/4$. *Then there exist positive constants* c_1, c_2, C_1, *and* C_2 *such that for* $x > 0$, $n \ge 1$, *we have*

$$c_1(\theta_n p_n(x) \wedge 1) \le P\{L_n^* > x\} \le C_1(\theta_n p_n(x) \wedge 1), \tag{2.27}$$

and

$$c_2\left\{\frac{1 - p_n(x)}{\theta_n p_n(x)} \wedge 1\right\} \le P\{L_n^* \le x\} \le C_2\left\{\frac{1 - p_n(x)}{\theta_n p_n(x)} \wedge 1\right\}, \tag{2.28}$$

where $p_n(x) = P\{L_{t_n}^{\sim} > x\}$ *and* $\theta_n = a_{t_n} a_n^{-1} n t_n^{-1}$.

PROOF. Define τ_k and ξ_k as in (2.17) and let

$$J_k = \max_{0 \le j \le t_n} \left(L_{\tau_k + j + t_n} - L_{\tau_k + j}\right), \qquad k \ge 0,$$

$$N_n = \max\{k\colon \tau_k + 2t_n \le n\}, \qquad M_n = \max\{k\colon \tau_k \le n\}.$$

Note that the J_k are identically distributed with the distribution of $L_{t_n}^{\sim}$ and that for any k, $\{J_i\colon i < k\}$ is independent of $\{J_i\colon i > k\}$.

(i) *Upper bound in* (2.27). Since

$$\{L_n^* > x\} \subset \bigcup_{k=0}^{\infty} \{J_k > x,\, M_n \ge k\},$$

we have

$$\begin{aligned} P\{L_n^* > x\} &\le \sum_{k=0}^{\infty} P\{J_k > x,\, M_n \ge k\} \\ &= \sum_{k=0}^{\infty} P\{J_k > x,\, \tau_k \le n\} \\ &= \sum_{k=0}^{\infty} P\{J_k > x\} P\{\tau_k \le n\} \\ &= p_n(x) \sum_{k=0}^{\infty} P\{M_n \ge k\} = p_n(x)(EM_n + 1). \end{aligned}$$

Now $M_n + 1$ is a stopping time and

$$\sum_{k=1}^{M_n+1} (\xi_k \wedge n) \le \tau_{M_n} + n \le 2n,$$

so by Wald's identity

$$E(M_n + 1)E(\xi_1 \wedge n) \le 2n.$$

Applying Lemma 2.8 gives $E(M_n + 1) = O(\theta_n)$ which completes the proof of the upper bound in (2.27).

(ii) *Lower bound in* (2.27). For this bound we will use inclusion–exclusion. For $0 < \delta < 1$, let

$$B_n = \left\{k\colon k \text{ is even and } 0 \le k \le \delta\left(\theta_n \wedge \{p_n(x)\}^{-1}\right)\right\}.$$

Then

$$\begin{aligned} P\{L_n^* > x\} &\ge P\left(\bigcup_{k\in B_n} \{J_k > x,\ N_n \ge k\}\right) \\ &\ge \sum_{k\in B_n} P\{J_k > x,\ \tau_k \le n - 2t_n\} - \sum_{\substack{j,\,k\in B_n\\ j<k}} P\{J_j > x,\ J_k > x\} \\ &\ge p_n(x) \sum_{k\in B_n} P\{\tau_k \le n/2\} - \tfrac{1}{2}p_n^2(x)\mathrm{Card}(B_n)[\mathrm{Card}(B_n) - 1]. \end{aligned}$$

Now by Lemma 2.8

$$P\{\tau_k > n/2\} = P\left\{\sum_{j=1}^{k} (\xi_j \wedge n) > n/2\right\} \le 2kn^{-1}E(\xi_1 \wedge n) \le ck\theta_n^{-1}.$$

Thus if $\delta \le (2c)^{-1}$, we have $P\{\tau_k > n/2\} \le \frac{1}{2}$ for all $k \in B_n$, so

$$\begin{aligned} P\{L_n^* > x\} &\ge \tfrac{1}{2}p_n(x)\mathrm{Card}(B_n)\left(1 - p_n(x)[\mathrm{Card}(B_n) - 1]\right) \\ &\ge \frac{\delta}{4}p_n(x)\left(\theta_n \wedge \{p_n(x)\}^{-1}\right)(1 - \delta). \end{aligned}$$

This is adequate for the lower bound in (2.27).

(iii) *Upper bound in* (2.28). For n fixed, let $z_k = [k\{p_n(x)\}^{-1}]$. Then $z_{k-1} < z_k$, $k \ge 1$, and

$$P\{L_n^* \le x\} = \sum_{k=1}^{\infty} P\{L_n^* \le x,\ z_{k-1} \le N_n < z_k\}. \tag{2.29}$$

Letting $B = \{J_i \le x \text{ for } i \le z_{k-1}\}$, and

$$B_j = \{J_i \le x \text{ for } i \ne j - 2,\ j - 1,\ i \le z_{k-1}\},$$

we have

$$
\begin{aligned}
P\{L_n^* \le x,\ z_{k-1} \le N_n < z_k\} &\le P\{\tau_{z_k} > n - 2t_n,\ J_i \le x \text{ for } i \le z_{k-1}\} \\
&\le P\left\{\sum_{j=1}^{z_k} (\xi_j \wedge n) > n/2,\ J_i \le x \text{ for } i \le z_{k-1}\right\} \\
&\le \sum_{j=1}^{z_k} \frac{2}{n} \int_B (\xi_j \wedge n)\, dP \\
&\le \frac{2}{n} \sum_{j=1}^{z_k} \int_{B_j} (\xi_j \wedge n)\, dP \\
&\le \frac{2}{n} z_k E(\xi_1 \wedge n) P(B_j) \\
&\le Ck \frac{1 - p_n(x)}{\theta_n p_n(x)} e^{2p_n(x) - (k-1)/2} \\
&\le Ce^3 \frac{1 - p_n(x)}{\theta_n p_n(x)} ke^{-k/2},
\end{aligned}
$$

since

$$
\begin{aligned}
(1 - p_n(x))^{-1} P(B_j) &\le (1 - p_n(x))^{(z_{k-1} - 3)/2} \\
&\le \exp\Big(-p_n(x)\big((k-1)\{p_n(x)\}^{-1} - 4\big)/2\Big)
\end{aligned}
$$

by discarding the J_i's with odd indices to achieve independence. Summing over k and recalling (2.29) completes the proof.

(iv) *Lower bound in* (2.28). Define

$$F_j = \{J_k \le x,\ 0 \le k < j,\ \xi_j > n\}.$$

Then we use

$$P(F_j) \ge P\{\xi_j > n\} - \sum_{k=0}^{j-1} P\{J_k > x,\ \xi_j > n\}.$$

For $k < j - 2$, J_k and ξ_j are independent but for the last two terms we need to use Lemma 2.5. By starting over at τ_{j-2}, we have

$$
\begin{aligned}
P\{J_{j-2} > x,\ \xi_j > n\} &= P\{L_{t_n}^{\sim} > x,\ \xi_2 > n\} \\
&= P\{L_{t_n}^{\sim} > x,\ \xi_1 \le 2t_n,\ \xi_2 > n\} \\
&\quad + P\{L_{t_n}^{\sim} > x,\ \xi_1 > 2t_n,\ \xi_2 > n\} \\
&\le P\{L_{t_n}^{\sim} > x,\ S_j \ne 0 \text{ for } 3t_n < j \le n\} \\
&\quad + P\{L_{t_n} > x,\ \xi_2 > n\} \\
&\le C\theta_n^{-1} P\{L_{t_n}^{\sim} > x\} + P\{L_{t_n} > x\} P\{\xi_2 > n\} \\
&\le C_3 \theta_n^{-1} p_n(x).
\end{aligned}
$$

Similarly

$$\begin{aligned}P\{J_{j-1} > x, \xi_j > n\} &= P\{L^{\sim}_{t_n} > x, \xi_1 > n\}\\ &= P\{L_{t_n} > x, S_j \neq 0 \text{ for } t_n < j \leq n\}\\ &\leq C\theta_n^{-1}P\{L_{t_n} > x\} \leq C\theta_n^{-1}p_n(x).\end{aligned}$$

Thus we have by Corollary 2.3

$$P(F_j) \geq 2c\theta_n^{-1} - jC\theta_n^{-1}p_n(x) \geq c\theta_n^{-1},$$

for $j \leq cC^{-1}\{p_n(x)\}^{-1}$. Since

$$P(F_jF_k) \leq P\{\xi_j > n, \xi_k > n\} \leq C_4\theta_n^{-2},$$

we have by inclusion–exclusion that if $j_0 \leq cC^{-1}\{p_n(x)\}^{-1}$

$$\begin{aligned}P\{L_n^* \leq x\} \geq P\Big(\bigcup_{j \leq j_0} F_j\Big) &\geq j_0 c\theta_n^{-1} - j_0^2 C_4\theta_n^{-2}\\ &= j_0 c\theta_n^{-1}(1 - j_0 c^{-1}C_4\theta_n^{-1}).\end{aligned}$$

Thus by taking $j_0 = c_3(\theta_n \wedge \{p_n(x)\}^{-1})$ for appropriate c_3 we obtain

$$P\{L_n^* \leq x\} \geq c_4\big(1 \wedge \{\theta_n p_n(x)\}^{-1}\big).$$

In making this estimate we have tacitly assumed that

$$\theta_n \wedge \{p_n(x)\}^{-1} > c_3^{-1}.$$

For the remaining case we use Lemma 2.7 to obtain

$$\begin{aligned}P\{L_n^* \leq x\} &\geq P\{L_{t_n} \leq x, \xi_1 > n\}\\ &\geq c\theta_n^{-1}P\{L_{t_n} \leq x\} \geq c\theta_n^{-1}(1 - p_n(x)).\end{aligned} \tag{2.30}$$

Then if $p_n(x) \geq c_3$ we have $p_n(x) \approx 1$ so this bound suffices while if $\theta_n \leq c_3^{-1}$, $p_n(x) < c_3$, then $\theta_n^{-1}(1 - p_n(x)) > c_3(1 - c_3)$ so the bound in (2.30) is adequate in this case also. □

COROLLARY 2.4. *Assume* (A.1) *holds*, $0 < \varepsilon < 1$, *and* $t_n \leq \varepsilon n/4$. *Then there exists a positive constant* C_3 *such that*

$$P\{L_n^* \leq x\} \leq P\{L^*_{\varepsilon n, t_n} \leq x\} \leq C_3 P\{L_n^* \leq x\}.$$

PROOF. Since $p_n(x)$ depends only on x and t_n but not on n itself, this follows from Theorem 2.2 and Corollary 2.1. □

Now we may state the probability estimates for L_n^* in the form in which they will be used.

THEOREM 2.3. *Assume* (A.1). *Then for every* $\varepsilon > 0$, *there exist positive constants* c_4, c_5, C_4, C_5 *such that for* $x > 0$, $n \geq 1$,

$$c_4(\theta_n q_n(x) \wedge 1) \leq P\{L_n^* > x\} \leq C_4(\theta_n q_n^{\sim}(x) \wedge 1), \tag{2.31}$$

$$c_5\left[\frac{1 - q_n(x)}{\theta_n q_n^{\sim}(x)} \wedge 1\right] \leq P\{L_n^* \leq x\} \leq C_5\left[\frac{1 - q_n(x)}{\theta_n q_n(x)} \wedge 1\right], \tag{2.32}$$

where $q_n(x) = P\{L_{t_n} > x\}$, $q_n^{\sim}(x) = P\{L_{(1+\varepsilon)t_n} > x - 1\}$.

REMARK 2.2. These bounds are valid even if $t_n > n/4$.

PROOF. This follows immediately from the bounds in Theorem 2.2, Lemma 2.9, and (2.30) if $t_n \leq n/4$. If $t_n > n/4$, then $\theta_n \approx 1$ and (2.31) is essentially the same as (2.25). Also, in this case, $P\{L_n^* \leq x\} \approx 1 - q_n(x)$ as in (2.26) which gives the upper bound in (2.32). It also is sufficient for the lower bound since if $q_n(x) \geq \frac{1}{2}$, then $q_n^{\sim}(x) \geq \frac{1}{2}$ and so the lower bound in (2.32) is $\approx 1 - q_n(x)$. On the other hand, if $q_n(x) < \frac{1}{2}$, then $1 - q_n(x) \approx 1$ which also exceeds the lower bound in (2.32). □

3. Lim sup behavior of L_n^*. The main result of this section is Theorem 3.1 which shows that under condition (A.1) there exist constants β_n such that $\limsup_n(L_n^*/\beta_n) = 1$ a.s. First we take care of the case when $\{t_n\}$ is a bounded sequence.

PROPOSITION 3.1. *If* $\{t_n\}$ *is a bounded nondecreasing sequence, i.e.,* $t_n = k$ *for all n sufficiently large, then*

$$P\{L_n^* \neq \beta \text{ i.o.}\} = 0,$$

where $\beta = \beta_n = [(k-1)a^{-1}] + 1$ *and a is defined in* (2.19). *In particular,* $\lim_n(L_n^*/\beta_n) = 1$ *a.s.*

PROOF. Recall that $\{T_k\}$, the time intervals between successive zeroes of S_n, are independent, identically distributed random variables. Thus, with probability one, there will be $\beta - 1$ consecutive T_k's equal to a. Once this occurs, $L_n^* = \beta$ from that time on. □

To state the theorem, we first define β_n. Let $\{a_n\}$ be the sequence defined by (2.1) and let

$$\theta_n = a_{t_n} a_n^{-1} n t_n^{-1}. \tag{3.1}$$

By (2.3) we have $\theta_n \geq c > 0$ for all n. Let

$$\rho_n = t_n^{-1}(\log \theta_n + \log_2 n) \tag{3.2}$$

and if $0 < \rho_n < -a^{-1}\log q$, let $0 < \lambda_n < \infty$ be the unique solution of

$$h(\lambda_n) = \rho_n, \tag{3.3}$$

where h is defined in (2.20). [Here a and q are defined in (2.19).] We then define

$$(3.4)\qquad \begin{aligned}\beta_n &= \left[(t_n - 1)a^{-1}\right] + 1, \quad \text{if } \rho_n \geq -a^{-1}\log q,\\ &= t_n/g(\lambda_n), \quad \text{if } \rho_n < -a^{-1}\log q,\end{aligned}$$

where g is defined in (2.20).

THEOREM 3.1. *Suppose* (A.1) *holds and* $\{t_n\}$ *is a nondecreasing sequence of positive integers. If* $\{\beta_n\}$ *is defined by* (3.4), *then* $\limsup_n(L_n^*/\beta_n) = 1$ *a.s.*

PROOF. Since $\theta_n \geq c > 0$, if $\{t_n\}$ is bounded then $\rho_n \to \infty$ so β_n is given by the first expression in (3.4) for n sufficiently large and this agrees with the definition of β_n in Proposition 3.1. Thus the theorem is an immediate consequence of Proposition 3.1 when $\{t_n\}$ is bounded. For the rest we assume $t_n \uparrow \infty$.

First we prove the upper bound. Let $0 < \varepsilon < 1$ be given and let $n(0) = 1$,

$$n(k) = \min\{j > n(k-1)\colon t_j > (1+\varepsilon)t_{n(k-1)}\} \wedge 2n(k-1), \qquad k \geq 1.$$

For each k we either have $n(k) = 2n(k-1)$ or $t_{n(k)} > (1+\varepsilon)t_{n(k-1)}$. If we consider $k = 1, 2, \dots, 2r$, one or the other of these must occur at least half the time so either $n(2r) \geq 2^r$ or $n(2r) \geq t_{n(2r)} \geq (1+\varepsilon)^r t_1$. In either case, there exists $c > 0$ such that for all $k \geq 1$

$$(3.5)\qquad ck \leq \log n(k) \leq \log 2^k.$$

Let

$$\Sigma_k = \bigcup_{n=n(k-1)}^{n(k)-1} \left\{L_n^* > (1+\varepsilon)^3\beta_n\right\}.$$

Then

$$\begin{aligned}\Sigma_k &\subset \left\{L_{n(k)-1}^* > (1+\varepsilon)^3\beta_{m(k)}\right\}\\ &\subset \left\{L_{n(k)-1,(1+\varepsilon)t_{m(k)}}^* > (1+\varepsilon)^3\beta_{m(k)}\right\} = \Sigma_k',\end{aligned}$$

where $m(k) \in [n(k-1), n(k))$ is such that

$$\beta_{m(k)} = \min\{\beta_n\colon n(k-1) \leq n < n(k)\}.$$

By Theorem 2.3, for all k sufficiently large

$$P(\Sigma_k') \leq C\theta_{n(k)-1}P\left\{L_{(1+\varepsilon)^2 t_{m(k)}} > (1+\varepsilon)^2\beta_{m(k)}\right\}$$

and then by (2.23) with $\eta = \varepsilon(1+\varepsilon)^{-1}$

$$\begin{aligned}P(\Sigma_k') &\leq C\theta_{n(k)-1}\exp\{-(1+\varepsilon)t_{m(k)}\rho_{m(k)}\}\\ &= C\theta_{n(k)-1}\theta_{m(k)}^{-1-\varepsilon}(\log m(k))^{-1-\varepsilon}.\end{aligned}$$

Since $t_{n(k)-1} \leq (1+\varepsilon)t_{m(k)}$ we have $\theta_{n(k)-1} \approx \theta_{m(k)}$ by (2.18a) and $\log m(k) \approx k$ by (3.5). Thus $P(\Sigma_k') = O(k^{-1-\varepsilon})$ and so $\sum P(\Sigma_k) < \infty$. By the Borel–Cantelli lemma, this implies $\limsup_n(L_n^*/\beta_n) \leq (1+\varepsilon)^3$ a.s. for any $\varepsilon > 0$.

We now consider the lower bound. Let $0 < \varepsilon < \frac{1}{3}$ and for any integer n let $n' = [(1-3\varepsilon)n]$. In particular, $t'_n = [(1-3\varepsilon)t_n]$. For a positive integer $t \le (1-\varepsilon)n$ we write

$$L^*_{n,t}(\varepsilon) = \max_{\varepsilon n \le j \le n-t} \left(L_{j+t} - L_j\right).$$

We will continue to write L^*_n for L^*_{n,t_n}. Let $n(j) = 2^j$, $0 < \eta < 1$, $\varepsilon < \eta/3$, and

$$\Gamma_j = \left\{L^*_{n(j),\, t'_{n(j)}}(\varepsilon) > (1-\eta)\beta_{n(j)}\right\}.$$

For the lower bound it suffices to show that for any $0 < \eta < 1$ there is an ε such that $P(\Gamma_j \text{ i.o.}) = 1$. Since this probability is 0 or 1 by the Hewitt–Savage zero-one law, it will be enough to prove (see Proposition 3, page 317 of [9]) that there exists $C > 0$ such that for $N \ge 1$

$$\sum P(\Gamma_j) = \infty \tag{3.6}$$

and

$$\sum_{j,k=1}^{N} P(\Gamma_j \Gamma_k) \le C \sum_{j,k=1}^{N} P(\Gamma_j)P(\Gamma_k).$$

By Lemma 3.1 below and then Theorem 2.3 and Theorem 2.1 with $c = 1 - 3\varepsilon$

$$\begin{aligned} P(\Gamma_j) &\ge cP\left\{L^*_{n(j),\, t'_{n(j)}} > (1-\eta)\beta_{n(j)}\right\} \\ &\ge c_1\theta_{n(j)}P\left\{L_{t'_{n(j)}} > (1-\eta)\beta_{n(j)}\right\} \\ &\ge c_1\theta_{n(j)}\exp\left\{-(1-\eta^2)\left(\log\theta_{n(j)} + \log_2 n(j)\right)\right\} \\ &= c_1\theta^{\eta^2}_{n(j)}(\log n(j))^{-1+\eta^2} \ge c_2 j^{-1}, \end{aligned}$$

for large j where (3.5) has been used at the last step. Thus the first condition in (3.6) is satisfied. To check the second condition, we have for (j,k) such that $n(j) < \varepsilon n(k)$

$$\begin{aligned} P(\Gamma_j\Gamma_k) &\le P\{\Gamma_j,\, S_i \ne 0 \text{ for } i \in (n(j), \varepsilon n(k))\} \\ &\quad + P\{\Gamma_j,\, S_i = 0 \text{ for some } i \in (n(j), \varepsilon n(k)),\, \Gamma_k\} \\ &\le P\left\{L^*_{n(j),\, t'_{n(j)}} > (1-\eta)\beta_{n(j)},\, S_i \ne 0 \text{ for } i \in (n(j), \varepsilon n(k))\right\} \\ &\quad + P(\Gamma_j)P\left\{L^*_{n(k),\, t'_{n(k)}} > (1-\eta)\beta_{n(k)}\right\} \\ &\le CP(\Gamma_j)r_{\varepsilon n(k)}r^{-1}_{n(j)} + CP(\Gamma_j)P(\Gamma_k), \end{aligned}$$

where we used the Markov property of $\{S_n\}$ at the second step and Lemmas 2.5 and 3.1 at the last step. This bound is adequate for the second condition in (3.6) since

$$r_{\varepsilon n(k)}r^{-1}_{n(j)} \approx a_{n(k)}a^{-1}_{n(j)}2^{j-k} = O\left(2^{(j-k)(1-\lambda^{-1})}\right)$$

by Lemma 2.3 and (2.3). □

It remains to prove

LEMMA 3.1. *Assume* (A.1). *For* $0 < \varepsilon < \frac{1}{3}$ *there exist* n_0 *and* C *such that for all* x, $t \le (1 - 3\varepsilon)n$, *and* $n \ge n_0$

$$P\{L^*_{n,t}(\varepsilon) > x\} \le P\{L^*_{n,t} > x\} \le CP\{L^*_{n,t}(\varepsilon) > x\}.$$

PROOF. The first inequality is clear. For the second,

$$P\{L^*_{n,t} > x\} \le P\{L^*_{n,t}(\varepsilon) > x\} + P\{L^*_{\varepsilon n+t,t} > x\}$$

and then by Corollary 2.2 we have for $n \ge n_0$

$$\begin{aligned} P\{L^*_{\varepsilon n+t,t} > x\} &\le CP\{S_j = 0 \text{ for some } j \in (\varepsilon n, 2\varepsilon n]\}P\{L^*_{\varepsilon n+t,t} > x\} \\ &\le CP\{L^*_{n,t}(\varepsilon) > x\} \end{aligned}$$

by starting over at the first zero in $(\varepsilon n, 2\varepsilon n]$. □

4. Lim inf behavior of L^*_n. Unlike the lim sup case, we cannot always find normalizers γ_n such that

$$\liminf \gamma_n^{-1} L^*_n = 1 \quad \text{a.s.} \tag{4.1}$$

Indeed, in Theorem 4.2 we will show that if $\limsup nt_n^{-1} = \infty$, then a sequence $\{\gamma_n\}$ can be found which satisfies (4.1). On the other hand, if $\limsup nt_n^{-1} < \infty$, then for any choice of γ_n the right side in (4.1) is either 0 a.s. or ∞ a.s. and we give a simple test to determine which is the case. This gives a complete answer to the problem except that the definition of γ_n, assuming only $\limsup nt_n^{-1} = \infty$, is not very satisfactory. In Theorem 4.1 we will show that if a stronger condition holds, namely that

$$\sum_{n=1}^{\infty} (n\theta_n)^{-1} < \infty, \tag{4.2}$$

then the γ_n can be defined essentially in the same manner as in the lim sup case. This will allow a comparison between β_n and γ_n (Section 5) to show that in fact $\lim(\gamma_n/\beta_n) = 1$ under an additional condition [see (5.1)]; it will also lead to more explicit expressions for these normalizers in the domain of attraction situation (Section 6).

We assume (4.2) first, and define a nonnegative sequence $\{s_n\}$ which satisfies $s_n \to \infty$ and for every $\varepsilon \in (0,1)$

$$\sum_n (n\theta_n)^{-1} s_n^{1-\varepsilon} < \infty, \qquad \sum_n (n\theta_n)^{-1} s_n^{1+\varepsilon} = \infty, \tag{4.3}$$

and given $c_0 > 1$, there exist $\alpha \in [0,1]$, C_1, and C_2 such that

$$C_1 \frac{s_n}{\theta_n^\alpha} \le \frac{s_{cn}}{\theta_{cn}^\alpha} \le C_2 \frac{s_n}{\theta_n^\alpha}, \quad \text{for } 1 \le c \le c_0,\ n \ge 1. \tag{4.4}$$

It is always possible to find such an $\{s_n\}$ sequence. If we set

$$s_n = \left\{ \sum_{k=n}^{\infty} (k\theta_k)^{-1} \right\}^{-1},$$

then (4.3) is satisfied and (4.4) is also with $\alpha = 0$, $C_1 = 1$, since

$$\frac{s_{cn}}{s_n} \le 1 + \left\{ \sum_{k=n}^{[c_0 n]} (k\theta_k)^{-1} \Big/ \sum_{k=[c_0 n]}^{2[c_0 n]} (k\theta_k)^{-1} \right\} \le C_2$$

by (2.18).

REMARK 4.1. We would like to note that other choices of s_n are possible and we will use this flexibility in the next two sections. In particular, if $\theta_n \ge \log n \log_2 n \cdots (\log_k n)^{1+\delta}$ for some k and $\delta > 0$ (where $\log_k$ stands for the k times iterated log), then we can take $s_n = \theta_n / \log n \log_2 n \cdots \log_k n$. It is easy to check that s_n satisfies (4.3) and (4.4) with $\alpha = 1$ in this case.

To state Theorem 4.1 we assume (4.2) and choose $\{s_n\}$ satisfying (4.3) and (4.4). Then let

$$\hat{\rho}_n = t_n^{-1} \log s_n \tag{4.5}$$

and

$$\begin{aligned} \gamma_n &= \left[(t_n - 1)a^{-1}\right] + 1, \quad \text{if } \hat{\rho}_n \ge -a^{-1}\log q, \\ &= t_n / g(\hat{\lambda}_n), \quad \text{if } \hat{\rho}_n < -a^{-1}\log q, \end{aligned} \tag{4.6}$$

where g, h are defined in (2.20) and if $\hat{\rho}_n < -a^{-1}\log q$ then $\hat{\lambda}_n$ is the unique solution of $h(\hat{\lambda}_n) = \hat{\rho}_n$.

THEOREM 4.1. *Suppose* (A.1) *holds and* $\Sigma(n\theta_n)^{-1} < \infty$. *Then* $nt_n^{-1} \to \infty$ *and if the sequence* $\{\gamma_n\}$ *is defined by* (4.6)

$$\liminf_n (L_n^* / \gamma_n) = 1 \quad a.s.$$

PROOF. First observe that

$$\sum_{k=n}^{2n} (k\theta_k)^{-1} \ge c\theta_n^{-1}$$

by (2.18), so $\theta_n \to \infty$ and then so does nt_n^{-1} by Lemma 2.8. Fix $0 < \eta < 1$ and choose $\delta > 0$ small enough that

$$(1 + 2\delta)^{1/2}(1 + \eta)^{-(1-1/\lambda)} = \rho < 1,$$

with λ as in (2.2). Then define $n(k) = [(1+\delta)^k]$ and $m(k,0) = n(k)$,

$$m(k, j) = \min\{i > m(k, j-1): t_i > (1+\eta)t_{m(k,j-1)}\}, \qquad j \ge 1,$$

and let $j_k = \max\{j: m(k, j) < n(k+1)\}$. Now for k large and $0 \le j < j_k$, we have by (2.2)

$$\begin{aligned} \frac{\theta_{m(k,j+1)}}{\theta_{m(k,j)}} &= \frac{(m(k,j+1))^{1/2}}{a_{m(k,j+1)}} \frac{a_{m(k,j)}}{(m(k,j))^{1/2}} \left\{ \frac{m(k,j+1)}{m(k,j)} \right\}^{1/2} \\ &\quad \times \frac{a_{t_{m(k,j+1)}}}{(t_{m(k,j+1)})^{1/\lambda}} \frac{(t_{m(k,j)})^{1/\lambda}}{a_{t_{m(k,j)}}} \left\{ \frac{t_{m(k,j)}}{t_{m(k,j+1)}} \right\}^{1-1/\lambda} \\ &\le (1+2\delta)^{1/2}(1+\eta)^{-(1-1/\lambda)} = \rho < 1. \end{aligned} \tag{4.7}$$

Now we define

$$\Lambda_k = \left\{ \min_{n(k) \le n < n(k+1)} (L_n^*/\gamma_n) \le 1 - \eta \right\} \subset \bigcup_{j=0}^{j_k} \Lambda_{k,j},$$

where

$$\Lambda_{k,j} = \left\{ L_{m(k,j)}^* \le (1-\eta)\gamma_{\nu(k,j)} \right\}$$

and $\nu(k, j) \in [m(k, j), m(k, j+1))$ is such that

$$\gamma_{\nu(k,j)} = \max\{\gamma_n: m(k,j) \le n < m(k, j+1)\}.$$

[Here we let $m(k, j_k + 1) = n(k+1)$ so as to keep $\nu(k, j) \in [n(k), n(k+1))$ for all j.] Now by Theorem 2.3

$$P(\Lambda_{k,j}) \le C\theta_{m(k,j)}^{-1} q_{k,j}^{-1},$$

where by Theorem 2.1 with $c = (1+\eta)^{-1}$ we have

$$\begin{aligned} q_{k,j} &= P\left\{ L_{t_{m(k,j)}} > (1-\eta)\gamma_{\nu(k,j)} \right\} \\ &\ge P\left\{ L_{(1+\eta)^{-1} t_{\nu(k,j)}} > (1-\eta)\gamma_{\nu(k,j)} \right\} \\ &\ge \exp\left\{ -(1-\eta^2)\log s_{\nu(k,j)} \right\} = \left(s_{\nu(k,j)} \right)^{-(1-\eta^2)}. \end{aligned}$$

Thus by (2.18), (4.4), and (4.7)

$$\begin{aligned} P(\Lambda_k) &\le \sum_{j=0}^{j_k} P(\Lambda_{k,j}) \\ &\le C \sum_{j=0}^{j_k} s_{\nu(k,j)}^{1-\eta^2} \theta_{\nu(k,j)}^{-\alpha(1-\eta^2)} \theta_{m(k,j)}^{-1+\alpha(1-\eta^2)} \\ &\le C_1 s_{n(k+1)}^{1-\eta^2} \theta_{n(k+1)}^{-\alpha(1-\eta^2)} \sum_{j=0}^{j_k} \theta_{m(k,j)}^{-1+\alpha(1-\eta^2)} \\ &\le C_2 s_{n(k+1)}^{1-\eta^2} \theta_{n(k+1)}^{-1}, \end{aligned}$$

since the series behaves like a geometric series. Finally, by (4.4) and (2.18) we have

$$\begin{aligned} &\sum_{n=n(k+1)}^{n(k+2)-1} (n\theta_n)^{-1} s_n^{1-\eta^2} \\ &\qquad \ge c_1 s_{n(k+1)}^{1-\eta^2} \theta_{n(k+1)}^{-\alpha(1-\eta^2)} \sum_{n=n(k+1)}^{n(k+2)-1} \left(n\theta_n^{1-\alpha(1-\eta^2)} \right)^{-1} \\ &\qquad \ge c_2 s_{n(k+1)}^{1-\eta^2} \theta_{n(k+1)}^{-1} \sum_{n=n(k+1)}^{n(k+2)-1} n^{-1} \\ &\qquad \ge c_3 s_{n(k+1)}^{1-\eta^2} \theta_{n(k+1)}^{-1}. \end{aligned}$$

Then using (4.3) we see that $\Sigma P(\Lambda_k) < \infty$ and so

$$\liminf(L_n^*/\gamma_n) \geq 1 - \eta \quad \text{a.s.}$$

by the Borel–Cantelli lemma.

To establish the upper bound, let $n(k) = 2^k$ and for $\varepsilon > 0$

$$\Gamma_k = \left\{L_{n(k)}^* \leq (1 + \varepsilon)^2 \gamma_{n(k)}\right\}.$$

We will show $P\{\Gamma_k \text{ i.o.}\} = 1$ for all $\varepsilon > 0$ by checking (3.6) for the events Γ_k in the present context. By Proposition 3.1 we may assume $t_n \uparrow \infty$ and then $\gamma_n \to \infty$ by Theorem 2.1. By Theorem 2.3, for all k sufficiently large

$$P(\Gamma_k) \geq c\left\{\frac{1 - p_k}{\theta_{n(k)} p_k'} \wedge 1\right\},$$

where

$$p_k = P\left\{L_{t_{n(k)}} > (1 + \varepsilon)^2 \gamma_{n(k)}\right\},$$

$$p_k' = P\left\{L_{(1+\varepsilon)t_{n(k)}} > (1 + \varepsilon)\gamma_{n(k)}\right\}.$$

By Theorem 2.1 with $\eta = \varepsilon(1 - \varepsilon)(1 + \varepsilon)^{-1}$, for all k sufficiently large

$$p_k \leq p_k' \leq \exp\left\{-(1 + \varepsilon^2)\log s_{n(k)}\right\} = s_{n(k)}^{-(1+\varepsilon^2)}.$$

Thus $p_k \to 0$ and so for all sufficiently large k

$$P(\Gamma_k) \geq c\left(s_{n(k)}^{1+\varepsilon^2}\theta_{n(k)}^{-1} \wedge 1\right).$$

To show $\Sigma P(\Gamma_k) = \infty$, it suffices to check that $\Sigma s_{n(k)}^{1+\varepsilon^2}\theta_{n(k)}^{-1} = \infty$. If $\alpha(1 + \varepsilon^2) \leq 1$, then by (4.4) and (2.18)

$$\begin{aligned}
&\sum_{n=n(k-1)}^{n(k)-1} s_n^{1+\varepsilon^2}(n\theta_n)^{-1} \\
&\quad \leq C s_{n(k)}^{1+\varepsilon^2}\theta_{n(k)}^{-\alpha(1+\varepsilon^2)} \sum_{n=n(k-1)}^{n(k)-1} n^{-1}\theta_n^{-1+\alpha(1+\varepsilon^2)} \\
&\quad \leq C_1 s_{n(k)}^{1+\varepsilon^2}\theta_{n(k)}^{-1} \sum_{n=n(k-1)}^{n(k)-1} n^{-1} \leq C_2 s_{n(k)}^{1+\varepsilon^2}\theta_{n(k)}^{-1}.
\end{aligned}$$

If, on the other hand, $\alpha(1 + \varepsilon^2) > 1$ then a similar argument shows that

$$\sum_{n=n(k-1)}^{n(k)-1} s_n^{1+\varepsilon^2}(n\theta_n)^{-1} \leq C s_{n(k-1)}^{1+\varepsilon^2}\theta_{n(k-1)}^{-1}.$$

Thus in either case the series diverges by (4.3). We will now check the second condition in (3.6). For $j + 1 < k$, we have

$$\begin{aligned}
P(\Gamma_j\Gamma_k) \leq\; & P\left\{\Gamma_j;\, S_n = 0 \text{ for some } n \in [n(j), n(k-1)], \Gamma_k\right\} \\
& + P\left\{\Gamma_j;\, S_n \neq 0 \text{ for } n \in [n(j), n(k-1)]\right\}.
\end{aligned}$$

We use the Markov property of $\{S_n\}$ and Corollary 2.4 on the first term and Lemma 2.5 on the second to obtain

$$P(\Gamma_j\Gamma_k) \le P(\Gamma_j)P\left\{L^*_{n(k-1),\,t_{n(k)}} \le (1+\varepsilon)^2\gamma_{n(k)}\right\} + CP(\Gamma_j)r_{n(k-1)}r^{-1}_{n(j)}$$
$$\le C_1P(\Gamma_j)P(\Gamma_k) + CP(\Gamma_j)r_{n(k-1)}r^{-1}_{n(j)}.$$

This bound is enough for the second condition in (3.6) as in the proof of Theorem 3.1. □

We now prove the general theorem.

THEOREM 4.2. *If* $\limsup nt_n^{-1} = \infty$, *then there exists a sequence* $\{\gamma_n\}$ *such that* (4.1) *holds. If* $\limsup nt_n^{-1} < \infty$, *and* $\{\gamma_n\}$ *is any sequence of positive numbers, then*

$$\liminf(L^*_n/\gamma_n) = 0 \quad a.s. \quad or \quad \infty \quad a.s.$$

More precisely, let $2^k \le m(k) < 2^{k+1}$ *be such that*

$$\gamma_{m(k)} = \max\left\{\gamma_n : 2^k \le n < 2^{k+1}\right\};$$

then

$$\Sigma_k\gamma_{m(k)}r_{m(k)} < \infty\ (=\infty)\ \textit{implies}\ \liminf_n L^*_n/\gamma_n = \infty\ (=0) \quad a.s.$$

PROOF. We first observe that by Lemma 2.8 the condition $\limsup(n/t_n) = \infty$ is equivalent to the condition $\limsup\theta_n = \infty$. Let $\{n(k)\}$ be an increasing positive integer sequence such that $\theta_{n(k)} > 2^k$, $k \ge 1$. Let

$$\hat{\rho}_{n(k)} = t^{-1}_{n(k)}\log\theta_{n(k)}.$$

We then define $\gamma_{n(k)}$ by (4.6) with $n = n(k)$ and $\hat{\rho}_{n(k)}$ as defined above, and $\gamma_n = \gamma_{n(k)}$ for $n(k) \le n < n(k+1)$. By Proposition 3.1 we may assume $t_n \uparrow \infty$ and then $\gamma_n \to \infty$ by Theorem 2.1. For $\varepsilon > 0$, by Theorem 2.3 we have

$$P\left\{L^*_{n(k)} \le (1+\varepsilon)^2\gamma_{n(k)}\right\} \ge c\left\{\frac{1-q_k}{\theta_{n(k)}q'_k} \wedge 1\right\},$$

for all sufficiently large k, where by Theorem 2.1

$$q'_k = P\left\{L_{(1+\varepsilon)t_{n(k)}} > (1+\varepsilon)\gamma_{n(k)}\right\}$$
$$\le \exp\left\{-(1+\varepsilon^2)\log\theta_{n(k)}\right\} = \theta_{n(k)}^{-(1+\varepsilon^2)},$$

so $(1-q_k)/\theta_{n(k)}q'_k \ge \theta^{\varepsilon^2}_{n(k)}/2 \to \infty$. Thus for all k sufficiently large we have

$$P\left\{L^*_{n(k)} \le (1+\varepsilon)^2\gamma_{n(k)}\right\} \ge c.$$

It follows that

$$P\left\{L^*_{n(k)} \le (1+\varepsilon)^2\gamma_{n(k)}\ \text{i.o.}\right\} \ge c > 0.$$

By the Hewitt–Savage 0–1 law

$$\liminf(L_n^*/\gamma_n) \le (1 + \varepsilon^2) \quad \text{a.s.},$$

for any $\varepsilon > 0$. For the lower bound, for any $\varepsilon > 0$, by Theorem 2.3, for all large k

$$P\{L_{n(k)}^* \le (1-\varepsilon)\gamma_{n(k)}\} \le \frac{C}{\theta_{n(k)} q_k},$$

where by Theorem 2.1

$$q_k = P\{L_{t_{n(k)}} > (1-\varepsilon)\gamma_{n(k)}\} \ge \exp\{-(1-\varepsilon^2)\log\theta_{n(k)}\} = \theta_{n(k)}^{\varepsilon^2 - 1}.$$

Therefore, for all k sufficiently large

$$P\{L_{n(k)}^* \le (1-\varepsilon)\gamma_{n(k)}\} \le C\theta_{n(k)}^{-\varepsilon^2} \le C 2^{-k\varepsilon^2},$$

by the choice of $\theta_{n(k)}$. Since for $\varepsilon > 0$ we have $\Sigma_k 2^{-k\varepsilon^2} < \infty$, by the Borel–Cantelli lemma

$$P\{L_{n(k)}^* \le (1-\varepsilon)\gamma_{n(k)} \text{ i.o.}\} = 0,$$

so $\liminf(L_{n(k)}^*/\gamma_{n(k)}) \ge 1 - \varepsilon$ a.s. Since $L_n^* \uparrow$, for $n(k) \le n < n(k+1)$ we have $L_n^*/\gamma_n = L_n^*/\gamma_{n(k)} \ge L_{n(k)}^*/\gamma_{n(k)}$, and so $\liminf(L_n^*/\gamma_n) \ge 1 - \varepsilon$ a.s. for every $\varepsilon > 0$. This proves the first part of the theorem.

We now assume $\limsup \theta_n < \infty$, so $\theta_n \approx 1$. As observed earlier there exists $C > 0$ such that

$$t_n \le n \le C t_n. \tag{4.8}$$

Suppose $\Sigma_k \gamma_{m(k)} r_{m(k)} < \infty$. Then $\gamma_{m(k)} r_{m(k)} \to 0$ and by (2.14), for any $M > 0$ we have with $n(k) = 2^k$

$$\begin{aligned} P\{L_{n(k)}^* \le M\gamma_{m(k)}\} &\le P\{L_{t_{n(k)}} \le M\gamma_{m(k)}\} \le c_2 M \gamma_{m(k)} r_{t_{n(k)}} \\ &\le c_3 M \gamma_{m(k)} r_{m(k)}, \end{aligned}$$

where (4.8) and Corollary 2.1 are used at the last step. By the Borel–Cantelli lemma we have a.s. $L_{n(k)}^* > M\gamma_{m(k)}$ for all k sufficiently large. Since $L_n^* \uparrow$, this implies that $\liminf(L_n^*/\gamma_n) \ge M$ a.s. for every M, so $\liminf(L_n^*/\gamma_n) = \infty$ a.s. in this case.

Now assume $\Sigma_k \gamma_{m(k)} r_{m(k)} = \infty$. Then by (2.14) we have

$$P\{L_{m(k)}^* \le \varepsilon\gamma_{m(k)}\} \ge P\{L_{m(k)} \le \varepsilon\gamma_{m(k)}\} \ge c_1 \varepsilon \gamma_{m(k)} r_{m(k)},$$

so the probabilities sum to ∞. This shows the first condition in (3.6) holds with $\Gamma_k = \{L_{m(k)}^* \le \varepsilon\gamma_{m(k)}\}$. The second condition in (3.6) is checked very much along the same lines as before, so we delete this argument. Thus $\liminf(L_n^*/\gamma_n) \le \varepsilon$ a.s. for every $\varepsilon > 0$, and the theorem is proved. □

5. Limit behavior of L_n^*. The main result here, Theorem 5.1, says that if t_n does not get too close to n, then $\beta_n \sim \gamma_n$, where β_n are the normalizers for the lim sup behavior in Section 3 and γ_n the normalizers for the lim inf behavior in Section 4. Thus in this case we have $\lim \beta_n^{-1} L_n^* = 1$ a.s.

THEOREM 5.1. *If the sequence* $\{t_n\}$ *satisfies the condition*

(5.1) $$\lim_n \log(nt_n^{-1})/\log_2 n = \infty,$$

then we have

$$\lim_n \beta_n^{-1} L_n^* = 1 \quad a.s.,$$

where the sequence $\{\beta_n\}$ *is defined by* (3.4).

REMARK 5.1. If condition (5.1) is not satisfied, it will be shown in the next section that the lim sup and lim inf behaviors may be different.

PROOF. By Proposition 3.1 we only need to consider the case $t_n \uparrow \infty$. By Lemma 2.8 the condition in (5.1) is equivalent to

(5.2) $$\lim_n (\log \theta_n / \log_2 n) = \infty.$$

In view of this, by Remark 4.1 we take $s_n = \theta_n/\log n$. Then $\hat{\rho}_n$ in (4.5) is defined by

$$\hat{\rho}_n = t_n^{-1}(\log \theta_n - \log_2 n).$$

We use this definition of $\hat{\rho}_n$ to define γ_n by (4.6). Also, by (3.2)

$$\rho_n = t_n^{-1}(\log \theta_n + \log_2 n),$$

which is used to define β_n via (3.4). By (5.2) we have $\hat{\rho}_n \rho_n^{-1} \to 1$ as $n \to \infty$. By Lemma 5.1 below, we have $\beta_n \sim \gamma_n$; the theorem then follows from Theorems 3.1 and 4.1. □

LEMMA 5.1. *Suppose that* β_n, γ_n *are defined by* (3.4) *and* (4.6), *respectively, and* $\rho_n \sim \hat{\rho}_n$. *Then* $\beta_n \sim \gamma_n$.

PROOF. If $\{t_n\}$ is bounded, then both $\rho_n, \hat{\rho}_n \to \infty$ so $\beta_n = \gamma_n$ for large n. Thus we may assume $t_n \uparrow \infty$. If, along a subsequence $\{n_k\}$, we have $\rho_{n_k} \geq -a^{-1}\log q$, then $\beta_{n_k} \sim a^{-1}t_{n_k}$ and also $\liminf \hat{\rho}_{n_k} \geq -a^{-1}\log q$. Then even if $\hat{\rho}_{n_k} < -a^{-1}\log q$ we have $\hat{\rho}_{n_k} \to -a^{-1}\log q$ so that $\hat{\lambda}_{n_k} \to \infty$ and $g(\hat{\lambda}_{n_k}) \to a$; thus in any case $\gamma_{n_k} \sim a^{-1}t_{n_k}$ also. Since the same argument applies if $\hat{\rho}_{n_k} \geq -a^{-1}\log q$ for a subsequence $\{n_k\}$ we may assume that $0 < \rho_n, \hat{\rho}_n < -a^{-1}\log q$ for the rest of the argument. It suffices to consider those n for which $\hat{\rho}_n < \rho_n$. By the generalized mean value theorem there exists ξ_n, $\hat{\lambda}_n < \xi_n < \lambda_n$ such that

$$\frac{h(\lambda_n) - h(\hat{\lambda}_n)}{g(\lambda_n) - g(\hat{\lambda}_n)} = \frac{h'(\xi_n)}{g'(\xi_n)} = \frac{\log \varphi(\xi_n)}{g^2(\xi_n)},$$

where φ is given by (2.20). Therefore

(5.3) $$\begin{aligned} \frac{[h(\lambda_n) - h(\hat{\lambda}_n)]g(\xi_n)}{[g(\lambda_n) - g(\hat{\lambda}_n)]h(\xi_n)} &= \frac{\log \varphi(\xi_n)}{h(\xi_n)g(\xi_n)} \\ &= \frac{-1}{1 + \xi_n g(\xi_n)[\log \varphi(\xi_n)]^{-1}} < -1, \end{aligned}$$

since $g \downarrow$ so that

$$\log \varphi(x) = \int_0^x - g(s)\,ds < -xg(x)$$

and so

$$0 < 1 + \xi_n g(\xi_n)[\log \varphi(\xi_n)]^{-1} < 1.$$

Therefore

$$0 < \frac{g(\hat{\lambda}_n) - g(\lambda_n)}{g(\xi_n)} < \frac{h(\lambda_n) - h(\hat{\lambda}_n)}{h(\xi_n)} \to 0,$$

since $\rho_n \sim \hat{\rho}_n$. Thus $g(\lambda_n) \sim g(\hat{\lambda}_n)$ and so $\beta_n \sim \gamma_n$. □

REMARK 5.2. Note that in Theorem 5.1 we can take ρ_n in the definition of β_n to be $t_n^{-1}\log \theta_n$. This is clear from Lemma 5.1.

6. Domains of attraction context. In this section we consider the special case when X_1 is in the domain of attraction of a stable law of index α. Our basic condition (A.1) implies $1 < \alpha \le 2$ and that $EX_1 = 0$. There are recurrent random walks in the domain of attraction of a Cauchy law ($\alpha = 1$), but that case has to be excluded in the present context.

We now assume $EX_1 = 0$ and that X_1 is in the domain of attraction of a stable law of index α, $1 < \alpha \le 2$. The following facts about G and Q [see (1.4) and (1.5)] can be found in [3]. There exists a slowly varying (near infinity) function ℓ such that as $x \to \infty$

$$G(x) \sim \frac{2-\alpha}{\alpha} x^{-\alpha}\ell(x), \tag{6.1}$$

and

$$Q(x) \sim \frac{2}{\alpha} x^{-\alpha}\ell(x). \tag{6.2}$$

Furthermore, if X_1 is in the domain of *normal* attraction, then ℓ may be chosen to be a positive constant c_0, and

$$c_0 = EX_1^2 = \sigma^2, \quad \text{when } \alpha = 2. \tag{6.3}$$

Recalling that a_n is defined by the equation $nQ(a_n) = 1$, we conclude that there exists a slowly varying function ℓ_1 such that

$$a_n = n^{1/\alpha}\ell_1(n) \tag{6.4}$$

and in the domain of *normal* attraction case

$$a_n = \left(2c_0\alpha^{-1}\right)^{1/\alpha} n^{1/\alpha}, \qquad \alpha \le 2. \tag{6.5}$$

In particular,

$$a_n = \sigma n^{1/2}, \quad \text{when } \alpha = 2. \tag{6.6}$$

Let G_1, K_1, Q_1 be the functions associated with the distribution of T_1, the time of first return to 0.

PROPOSITION 6.1. *If X_1 is in the domain of attraction of a stable law H of index α, $1 < \alpha \le 2$, $EX_1 = 0$, then T_1 is in the domain of attraction of a stable law of index $\beta = 1 - \alpha^{-1}$. More specifically, as $x \to \infty$*

$$G_1(x) \sim \frac{x^{-\beta}\ell_1(x)}{\psi(0)\Gamma(\beta)\Gamma(1-\beta)}, \tag{6.7}$$

where ℓ_1 is the slowly varying function appearing in (6.4), *and ψ is the density of the stable law H.*

PROOF. We use a Tauberian argument. For any $n \ge 1$

$$1 = \sum_{k=0}^{n} r_{n-k}u_k;$$

therefore

$$\frac{1}{1-s} = R_1(s)U(s), \qquad 0 < s < 1,$$

where $R_1(s) = \sum_{k=0}^{\infty} r_k s^k$, $U(s) = \sum_{k=0}^{\infty} u_k s^k$. By the local limit theorem (it is not necessary that the random walk be strongly aperiodic, i.e., $p = 1$)

$$u_0 + \cdots + u_n \sim \psi(0)n^{\beta}/\beta\ell_1(n);$$

hence by Theorem 5 ([3], page 447) we have

$$U(s) \sim \psi(0)\Gamma(\beta)/(1-s)^{\beta}\ell_1(1/(1-s)) \quad \text{as } s \uparrow 1.$$

It follows that

$$R_1(s) \sim \frac{\ell_1(1/(1-s))}{\psi(0)\Gamma(\beta)}\frac{1}{(1-s)^{1-\beta}}.$$

Since $r_n \downarrow$, by Theorem 5 ([3], page 447) again

$$r_n \sim \frac{1}{\psi(0)\Gamma(\beta)\Gamma(1-\beta)}n^{-\beta}\ell_1(n).$$

Since $G_1(n) = P\{T_1 > n\} = r_n$, and G_1 is monotone, it follows that G_1 satisfies (6.7) as $x \to \infty$. □

THEOREM 6.1. *Let $\beta = 1 - 1/\alpha$ and*

$$\rho_n = t_n^{-1}(\beta \log(n/t_n) + \log_2 n).$$

Let

$$\beta_n = \begin{cases} [(t_n - 1)a^{-1}] + 1, & \text{if } \rho_n \ge -a^{-1}\log q, \\ t_n/g(\lambda_n), & \text{if } \rho_n < -a^{-1}\log q, \end{cases}$$

where λ_n is the unique solution of $h(\lambda_n) = \rho_n$. Then

$$\limsup_n (L_n^*/\beta_n) = 1 \quad a.s.$$

If, in addition, $\lim_n[\log(n/t_n)/\log_2 n] = \infty$, *then*

$$\lim_n (L_n^*/\beta_n) = 1 \quad a.s.$$

In particular, if $t_n = c \log n$ *for some* $c > 0$, *then there exists* $c_1 > 0$ *such that* $\beta_n \sim c_1 t_n$, *where* $c_1 = a^{-1}$ *if* $c^{-1}\beta \geq -a^{-1}\log q$, *otherwise* $c_1 = g(\lambda_0)^{-1}$, *where* λ_0 *is determined by* $h(\lambda_0) = c^{-1}\beta$. *Furthermore,* $\beta_n/t_n \to 0$ *iff* $t_n^{-1}\log n \to 0$, *in which case*

$$(6.8)\qquad \beta_n \sim \frac{\psi(0)\Gamma(\beta)t_n^\beta[\beta\log(n/t_n) + \log_2 n]^{1-\beta}}{\beta^\beta(1-\beta)^{1-\beta}\ell_1\big(\beta^{-1}(1-\beta)(\beta\log(n/t_n) + \log_2 n)^{-1}t_n\big)}.$$

REMARK 6.1. As a very special case, if $\alpha = 2$, $\ell_1 \equiv \sigma$, i.e., the case of normal attraction to the Gaussian, and $t_n = n$, then $L_n^* = L_n$ and the theorem says that

$$\limsup_n L_n/(2n\log_2 n)^{1/2} = \sigma^{-1},$$

a result of Kesten [7]. The theorem also contains the results of [1] for various choices of t_n when $\alpha = 2$ and $\ell_1 \equiv \sigma$ (the only case considered there).

PROOF. To show that $\limsup(L_n^*/\beta_n) = 1$ or $\lim(L_n^*/\beta_n) = 1$, it suffices to check that the ρ_n of the theorem and the ρ_n of (3.2) are asymptotically equivalent, and then the result follows from Lemma 5.1 and Theorem 3.1 or Theorem 5.1. Recalling that $a_n = n^{1/\alpha}\ell_1(n)$, we have

$$\theta_n = \left\{\frac{n}{t_n}\right\}^\beta \frac{\ell_1(t_n)}{\ell_1(n)}$$

and using the representation theorem for slowly varying functions (corollary, page 282, [3]) we easily see that

$$\log\theta_n \sim \beta\log(n/t_n) \quad \text{as } n \to \infty,$$

which gives the asymptotic equivalence of the two ρ_n's.

The assertion when $t_n = c\log n$ follows immediately from the definition of β_n.

Finally, $\beta_n/t_n \to 0$ iff $g(\lambda_n) \to \infty$ iff $\lambda_n \to 0$ iff $h(\lambda_n) = \rho_n \to 0$. Now

$$\rho_n = \{\beta(\log n - \log t_n) + \log_2 n\}t_n^{-1} \to 0,$$

iff $t_n \uparrow \infty$ and $\{\beta\log n + \log_2 n\}t_n^{-1} \to 0$ iff $t_n^{-1}\log n \to 0$. It only remains to establish (6.8).

In the notation of Proposition 2.1 [6]

$$h(\lambda) = \frac{R(\lambda)}{g(\lambda)} \sim \frac{1-\beta}{\beta}\lambda \quad \text{as } \lambda \to 0.$$

Therefore

$$(6.9)\qquad \lambda_n \sim \frac{\beta}{1-\beta}h(\lambda_n) = \frac{\beta}{1-\beta}\,\frac{\beta\log(n/t_n) + \log_2 n}{t_n}.$$

Again by Proposition 2.1 [6],

$$g(\lambda_n) \sim \beta\Gamma(1-\beta)G_1(1/\lambda_n)\lambda_n^{-1}$$

and using (6.7) we get

$$(6.10)\qquad \begin{aligned} g(\lambda_n) &\sim \frac{\beta\Gamma(1-\beta)\lambda_n^{\beta-1}\ell_1(1/\lambda_n)}{\psi(0)\Gamma(\beta)\Gamma(1-\beta)} \\ &= \frac{\beta}{\psi(0)\Gamma(\beta)}\lambda_n^{\beta-1}\ell_1(1/\lambda_n). \end{aligned}$$

Since $\beta_n = t_n/g(\lambda_n)$ when $\lambda_n \to 0$, (6.8) follows from (6.9) and (6.10). This proves the theorem. □

In the next example we show that the lim sup and lim inf behaviors of L_n^* may be different if (5.1) does not hold.

EXAMPLE 6.1. Suppose $EX_1 = 0$, $EX_1^2 = 1$. Let $C > 2$ and $t_n = n/(\log n)^C$. Then

$$\frac{\log(n/t_n)}{\log_2 n} = C,$$

so condition (5.1) does not hold. However ρ_n in Theorem 6.1 equals $(C+2) \times (\log n)^C \log_2 n/2n$, which tends to 0 as $n \to \infty$. Therefore by Theorem 6.1 ($\beta = \frac{1}{2}$, $\psi(0) = (2n)^{-1/2}$, $\ell_1 \equiv 1$)

$$\beta_n \sim n^{1/2}(\log n)^{-C/2}\{(C+2)\log_2 n\}^{1/2}.$$

On the other hand, since (4.2) is satisfied we can use Remark 4.1 and Theorem 4.1 and take

$$\hat{\rho}_n = \frac{\frac{1}{2}\log(n/t_n) - \log_2 n}{t_n} = \frac{(C-2)(\log n)^C \log_2 n}{2n},$$

which again goes to 0 as $n \to \infty$, so (as in Theorem 6.1)

$$\hat{\gamma}_n \sim n^{1/2}(\log n)^{-C/2}\{(C-2)\log_2 n\}^{1/2},$$

and we have $\limsup(L_n^*/\beta_n) = 1$, $\liminf(L_n^*/\hat{\gamma}_n) = 1$ by Theorems 6.1 and 4.1, respectively. Thus the two behaviors are different. By letting C depend on n and approach 2 it is clear that one may make $\beta_n/\hat{\gamma}_n \to \infty$. But we may examine the rate for some well behaved $\{t_n\}$ sequences of this type. For example, let

$$t_n = n/\left\{\log n(\log_2 n)^{1+\varepsilon}\right\}^2,$$

for some $\varepsilon > 0$. Then

$$\theta_n = (n/t_n)^{1/2} = \log n(\log_2 n)^{1+\varepsilon}$$

so that (4.2) is satisfied. By Theorem 6.1,

$$\beta_n \sim \frac{2(n\log_2 n)^{1/2}}{\log n(\log_2 n)^{1+\varepsilon}}.$$

On the other hand, by Remark 4.1 we may take

$$\hat{\rho}_n = t_n^{-1}\varepsilon \log_3 n$$
$$= \varepsilon n^{-1}\left\{\log n(\log_2 n)^{1+\varepsilon}\right\}^2 \log_3 n$$

and so (as in Theorem 6.1)

$$\hat{\gamma}_n \sim \frac{(2\varepsilon n \log_3 n)^{1/2}}{\log n(\log_2 n)^{1+\varepsilon}}.$$

Thus $\beta_n/\hat{\gamma}_n \sim (2\log_2 n/\varepsilon \log_3 n)^{1/2}$.

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