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CONDITIONAL BROWNIAN MOTION IN RAPIDLY EXHAUSTIBLE DOMAINS

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Let D be a domain in $\mathbb{R}^d$ and let Δ_1 be the set of minimal points of the Martin boundary of D. For $x \in D$ and $z \in \Delta_1$, let (X_t) under the law $P^{x;z}$ be Brownian motion in D, starting at x and conditioned to converge to z. Let τ be the lifetime of (X_t), so $X_{\tau-} = z$ $P^{x;z}$ a.s. Let $q \in L^p(D)$ for some $p > d/2$. Under the assumption that D is what we call *rapidly exhaustible*, which is essentially a very weak boundary smoothness condition, we show that if the quantity

$$E^{x;z}\left\{\exp\left[\int_0^\tau q(X_s)\,ds\right]\right\}$$

is finite for one $x \in D$ and one $z \in \Delta_1$, then this quantity is bounded on $D \times \Delta_1$. This result may be viewed as saying, in a fairly strong sense, that the amount of time (X_t) spends in each part of D does not depend very much on the minimal Martin boundary point z to which (X_t) is conditioned to converge.

Let D be a domain, i.e., an open connected set in $\mathbb{R}^d$, where $d \geq 2$. Let ∂ be a point not belonging to D (we think of ∂ as a "cemetery point") and let Ω be the set of functions ω from $[0, \infty]$ into $D \cup \{\partial\}$, such that $\{t\colon \omega(t) = \partial\}$ is of the form $[\tau(\omega), \infty]$ for some $\tau(\omega) \in (0, \infty]$ and such that ω is continuous on $[0, \tau(\omega))$. For $0 \leq t \leq \infty$, define X_t on Ω by $X_t(\omega) = \omega(t)$. Let $p_t(x, y)$ be the transition density for Brownian motion killed on exit from D. Given a positive harmonic function h in D, let ${}_hp_t(x, y) = h(x)^{-1}p_t(x, y)h(y)$. For each x in D, there is a probability measure ${}_hP^x$ on Ω under which (X_t) is a Markov process starting from x, with transition density ${}_hp_t$. This so-called h-path Brownian motion was defined and studied by Doob [7], who showed that ${}_hP^x$ a.s., the limit

$$X_{\tau-} = \lim_{t \uparrow \tau} X_t$$

exists in the Martin compactification of D and belongs to the set Δ_1 of minimal Martin boundary points of D. Moreover, he showed that if h is the minimal positive harmonic function in D corresponding to a point $z \in \Delta_1$, then $X_{\tau-} = z$ ${}_hP^x$ a.s., so that ${}_hP^x$ may be interpreted as P^x conditioned on the event that $X_{\tau-} = z$. In this case, ${}_hP^x$ is also denoted by $P^{x;z}$. [Of course, $P^x = {}_1P^x$ is the probability measure on Ω under which (X_t) is ordinary Brownian motion starting at x and killed on exit from D.] For the sake of concreteness, let us recall (see [11]) that if D is bounded and Lipschitz, then Δ_1 may be identified with ∂D, the ordinary boundary of D.

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Now let q be a real-valued Borel function in D, let

$$e_q(t) = \exp\left[\int_0^t q(X_s)\,ds\right]$$

for $0 \le t \le \tau$, and define the *gauge* u: $D \to [0, \infty]$ and the *conditional gauge* v: $D \times \Delta_1 \to [0, \infty]$ by

$$u(x) = E^x\{e_q(\tau)\}, \qquad v(x, z) = E^{x; z}\{e_q(\tau)\}.$$

It is understood that suitable hypotheses must be imposed on D and q to ensure that $e_q(\tau)$ is defined $P^{x; z}$ a.s. The main object of this paper is to prove a version of the following result:

CONDITIONAL GAUGE THEOREM. *Under suitable hypotheses on D and q, if v is finite at some point* $(x_0, z_0) \in D \times \Delta_1$, *then v is bounded on* $D \times \Delta_1$.

In [9] we proved this result for D bounded, with a C^2 boundary, and q bounded. Here we shall prove it for D *rapidly exhaustible* (see Definition 2) and $q \in L^p(D)$ where $p > d/2$. The condition of rapid exhaustibility is primarily a very weak boundary smoothness condition on D. We prove that every bounded domain that satisfies an *interior cone condition* (see Definition 4) is rapidly exhaustible. However, even this small amount of boundary smoothness is by no means necessary for a domain to be rapidly exhaustible, as we show by an example. Let us also mention that if the dimension $d = 2$ and q is bounded, then the conditional gauge theorem holds with no boundary smoothness condition on D; it suffices that D have finite area. In [14] Zhao showed that if D is bounded with C^2 boundary and q belongs to the Kato class (which contains $L^p(D)$ for $p > d/2$—see [1]) and if u is finite at some point $x_0 \in D$ then v is bounded on $D \times \Delta_1$. Subsequently, he has proved the conditional gauge theorem for D bounded with $C^{1,1}$ boundary and q in the Kato class. We remark that although Zhao considers a wider class of functions q than we do, the boundary smoothness conditions he imposes on D are far more stringent then rapid exhaustibility. One of the keys to Zhao's proof of the conditional gauge theorem was the following result:

BASIC ESTIMATE. *Under suitable hypotheses on D and q,* $E^{x; z}[\int_0^\tau |q(X_s)|\,ds]$ *is bounded for* (x, z) *varying over* $D \times \Delta_1$ *and moreover, if C is a (Borel) subset of D whose Lebesgue measure* $\lambda(C)$ *is small then* $E^{x; z}[\int_0^{\tau_c} |q(X_s)|\,ds]$ *is uniformly small in* $(x, z) \in D \times \Delta_1$. (*Here and elsewhere in this paper,* $\tau_c = \inf\{t > 0\colon X_t \notin C\}$.)

Chung [4], in an admirable simplification of the argument Zhao used, has shown that in fact the conditional gauge theorem holds whenever the basic estimate holds. He did not, however, improve on Zhao's condition for the basic estimate to hold: namely, D bounded with $C^{1,1}$ boundary and q in the Kato class. We are indebted to Chung for sending us a preprint of [4]. In Theorem 1, we prove the basic estimate for D rapidly exhaustible and $q \in L^p(D)$ for some

$p > d/2$. The proofs of Lemma 2 and Theorem 2 (the conditional gauge theorem) below are taken from [4] with a number of small changes that were needed to dispense with some unnecessary assumptions.

It is a pleasure to thank Mike Cranston for giving us a preprint of his paper [5] on the expected lifetime of conditional Brownian motion in Lipschitz domains. We learned some key ideas from this paper. Corollary 1 is a generalization of the main result of [5]. In this connection, let us point out that a rapidly exhaustible domain need not be Lipschitz. Even if it satisfies an interior cone condition, there is no reason why it should satisfy an exterior cone condition. A more striking example is given below after the proof of Theorem 2.

We should explain why one cannot hope for the conditional gauge theorem to hold without some boundary smoothness hypothesis on D (at least for $d \geq 3$). In [6], Cranston and McConnell have given an example of a bounded domain D in $\mathbb{R}^3$ having a minimal Martin boundary point z^* such that for all $x \in D$, $P^{x;\,z^*}\{\tau = \infty\} = 1$. Since D is bounded, $E^x\{e^{\varepsilon\tau}\}$ is bounded for some $\varepsilon > 0$. Then $E^{x;\,z}\{e^{\varepsilon\tau}\} < \infty$ for almost all (in the sense of harmonic measure) minimal Martin boundary points z. But of course $E^{x;\,z^*}\{e^{\varepsilon\tau}\} = \infty$, so the conditional gauge theorem does not hold for this domain. For those not familiar with the example of Cranston and McConnell, let us just mention that the bad boundary point z^* is at the tip of a cusp that points outward from D. This cusp is actually infinitely long, but is rolled up to fit into a bounded region.

Now let us explain why the conditional gauge theorem may be viewed as saying, in a sense, that the amount of time (X_t) spends in each part of D does not depend very much on which minimal Martin boundary point we condition (X_t) to converge to. Suppose D is bounded and Lipschitz, so $\Delta_1 = \partial D$ and D is rapidly exhaustible. Let z_1, z_2 be distinct points of ∂D. Consider

$$q(x) = \frac{C}{\|x - z_2\|^{2-\varepsilon}},$$

where $C, \varepsilon > 0$. Then $q \in L^p(D)$ for some $p > d/2$. Hence if

$$\exp\left[\int_0^\tau q(X_t)\,dt\right]$$

has finite expectation when (X_t) is conditioned to converge to z_1, it will still have finite expectation when (X_t) is conditioned to converge to z_2. But in the former case, one might have thought that (X_t) would not spend too much time near the pole of q, whereas in the latter case it slams head on into this singularity.

Next, let us recall some standard notation. For $0 \leq t \leq \infty$, θ_t: $\Omega \to \Omega$ is the usual translation operator defined by $(\theta_t\omega)(s) = \omega(s + t)$. G: $D \times D \to [0, \infty]$ is the Green function of D. It is related to the transition density for Brownian motion killed on exit from D by the formula $G(x, y) = \int_0^\infty p_t(x, y)\,dt$. If f is a real-valued function on D then $Gf(x) = \int_D G(x, y)f(y)\,dy$ for all $x \in D$ for which the integral makes sense. We now turn to the detailed statements and proofs of our results.

DEFINITION 1. An open ball $B(p, r)$ with center p and radius r will be called *dilatable* iff $B(p, 2r) \subseteq D$. By a *dilatable chain of length n* from x to y, we shall mean a sequence $B_1, \ldots, B_n$ of n *dilatable* open balls, where successive balls have nonempty intersection, such that $x \in B_1$ and $y \in B_n$.

REMARK. It follows from Harnack's inequality that there is a finite positive constant c, depending only on the dimension d, such that if h is a positive harmonic function in D and if x and y are points of D that can be joined by a dilatable chain of length n, then

$$h(x) \leq c^n h(y). \tag{1}$$

NOTATION. Let us fix a *reference point* $x_* \in D$. For each $x \in D$, let $k(x)$ denote the minimum number of balls in a dilatable chain from x_* to x. For $n \in \mathbb{N}$, let $E_n = \{x \in D: k(x) \geq n\}$. Let λ denote Lebesgue measure on $\mathbb{R}^d$.

DEFINITION 2. We shall say that D is *rapidly exhaustible* iff for all $\alpha > 0$, we have

$$\sum_{n=0}^{\infty} \lambda(E_n)^\alpha < \infty.$$

REMARK. It often happens that $\lambda(E_n)$ goes to 0 geometrically fast. For instance, this is so if D is an open ball $B(x_*, r)$. In this case, $E_0 = E_1 = D$, while for $n \geq 2$,

$$E_n = \{x: (1 - 2^{1-n})r \leq \|x - x_*\| < r\}.$$

On the other hand, although $\lambda(E_n)$ must go to 0 if $\lambda(D)$ is finite, it is easy to construct examples in which $\lambda(E_n)$ goes to 0 as slowly as desired and thus to construct examples of domains D that are not rapidly exhaustible. For instance, suppose D consists of a sequence of open balls, together with very thin tubes joining successive balls. Then a large number of balls will be required in a dilatable chain passing through the tube from one ball to the next. Hence it is clear that by choosing the dimensions of the tubes correctly, one can make D fail to be rapidly exhaustible.

REMARK. It is clear that whether D is rapidly exhaustible or not is independent of the choice of the reference point x_*, although the sets E_n depend on x_*. It is also clear that whether D is rapidly exhaustible or not is independent of the particular dilation factor (namely 2) that we choose to use in Definition 1. If we define a ball $B(x, r)$ to be a-dilatable iff $B(x, ar) \subseteq D$, and if we define an a-dilatable chain accordingly, then given a number a satisfying $1 < a < \infty$, there is a positive integer m such that if x and y lie in a dilatable ball B then they can be joined by an a-dilatable chain of length m and vice versa. Hence there is no real gain in generality to be obtained by considering dilation factors other than 2.

DEFINITION 3. By a *cone* we shall mean the convex hull of a closed ball and a point, called the *vertex* of the cone, which lies outside the ball. The line segment from the vertex of the cone to the center of the ball we shall call the *axis* of the cone, while any line through the vertex of the cone and tangent to the ball will be called a *generator* of the cone. All generators of a cone make the same acute angle with its axis; this angle we shall call the *vertex semiangle* of the cone.

DEFINITION 4. We shall say that D satisfies an *interior cone condition* iff there is a cone C such that for all $x \in D$, there is a cone $C(x)$ congruent to C such that $C(x)$ is contained in D, and x is the vertex of $C(x)$.

REMARK. In an earlier draft of this paper, we used a (strictly) stronger interior cone condition. We are grateful to Jean Brossard for suggesting the more general condition stated in Definition 4. Let us also note that this condition implies that each point $z \in \partial D$ is the vertex of a cone $C(z)$ congruent to C, such that the interior of $C(z)$ is contained in D. (To see this, let $x \to z$.) The latter is a more usual definition of the interior cone condition. We have not tried to determine whether the condition stated in Definition 4 is strictly stronger. For our purposes, the important thing is that it is reasonably general and that it is strong enough for the following result.

PROPOSITION 1. *Suppose D is bounded and satisfies an interior cone condition. Then D is rapidly exhaustible.*

PROOF. In fact, we shall show that $\lambda(E_n)$ goes to 0 geometrically fast. Let C and r be as in Definition 4. Observe that we can find a number $\rho > 0$ and a positive integer N, such that if $x \in D$ with $\operatorname{dist}(x, \partial D) < \rho$, then there is a dilatable chain of length N from x to a point $y \in D$ satisfying

$$\operatorname{dist}(y, \partial D) \geq 2 \operatorname{dist}(x, \partial D).$$

This follows, by a similarity argument, from the fact that if $\operatorname{dist}(x, \partial D)$ is small enough, then there is a point y_0 on the axis of $C(x)$, such that $\operatorname{dist}(y_0, x) = \operatorname{dist}(x, \partial D)/4$.

In view of the above observation, we will be done if we can show that there is a constant $A < \infty$ such that

$$\lambda(\partial_\varepsilon D) \leq A\varepsilon, \tag{2}$$

for all sufficiently small $\varepsilon > 0$, where

$$\partial_\varepsilon D = \{x \in D \colon \operatorname{dist}(x, \partial D) < \varepsilon\}.$$

[We remark in passing that if ∂D were smooth, then $\lambda(\partial_\varepsilon D)$ would be approximately ε times the $d - 1$ dimensional measure of ∂D.] Let θ be the vertex semiangle of the cone C. It is clear that there are constants $\alpha > 0$ and $\beta < \infty$ such that if $\varepsilon > 0$ and L is any line through the vertex of C, which makes an angle of no more than $\theta/2$ with the axis of C, then the linear (i.e., one dimensional) measure of the line segment $L \cap C$ is at least α, while the linear measure of

$$\{y \in L \cap C \colon \operatorname{dist}(y, \partial C) < \varepsilon\}$$

is no more than $\beta\varepsilon$.

Let us digress for a moment. Consider a Borel set $S \subseteq \mathbb{R}$ such that for each $x \in S$, there is an interval I of length at least α, such that $x \in I$ and the linear measure of $I \cap S$ satisfies $|I \cap S| \leq \beta\varepsilon$. Suppose n is a positive integer such that diam $S \leq n\alpha$. Then $|S| \leq 2n\beta\varepsilon$. To see this, it suffices to consider the case $n = 1$, since if $n > 1$ then S can be divided into n pieces each of diameter $\leq \alpha$. Let $a = \inf S$, $b = \sup S$. Then $0 < b - a \leq \alpha$ (unless $|S| = 0$). If I is an interval of length at least α and I contains a point $x \in S$, then either a or b must belong to I or be an endpoint of I. Thus for each $x \in S$, either $|[a, x] \cap S| \leq \beta\varepsilon$ or $|[x, b] \cap S| \leq \beta\varepsilon$. Let

$$c = \sup\{x \in S\colon |[a, x] \cap S| \leq \beta\varepsilon\}.$$

Then $a < c \leq b$, $|[a, x] \cap S| \leq \beta\varepsilon$, and for all $x \in (c, b] \cap S$, $|[x, b] \cap S| \leq \beta\varepsilon$, whence $|[c, b] \cap S| \leq \beta\varepsilon$. Therefore $|S| \leq 2\beta\varepsilon$. This concludes the digression.

Now by the compactness of the unit sphere in $\mathbb{R}^d$, there is some finite set $\{L_1, \dots, L_m\}$ of lines through the origin such that any line through the origin makes an angle no greater than $\theta/2$ with some L_i. Let $L_i(x)$ denote the line through x parallel to L_i. Given ε satisfying $0 < \varepsilon < r$, for $i = 1, \dots, m$ let B_i be the set of points $x \in \partial_\varepsilon D$ such that the line $L_i(x)$ intersects the axis of the cone $C(x)$ at an angle no greater than $\theta/2$. Then $\partial_\varepsilon D = B_1 \cup \cdots \cup B_m$, so

$$\lambda(\partial_\varepsilon D) \leq \sum_{i=1}^{m} \lambda(B_i). \tag{3}$$

By the choice of α and β, if $x \in B_i$, then on the line $L_i(x)$ there is an interval containing x, of length at least α, whose intersection with B_i has linear measure at most $\beta\varepsilon$. Let $\delta = \operatorname{diam} D$. By the digression, it follows that the linear measure of $B_i \cap L_i(x)$ is at most $2n\beta\varepsilon$, where n is a positive integer chosen so that $\delta \leq n\alpha$. Then by Fubini's theorem,

$$\lambda(B_i) \leq 2n\beta\delta^{d-1}\varepsilon. \tag{4}$$

From (3) and (4), (2) follows with $A = 2mn\beta\delta^{d-1}$. This completes the proof of the proposition. □

REMARK. In a sense (2) says that the dimension of ∂D is no more than $d - 1$, at least as viewed from inside D. More generally, if $0 < \gamma < d$, an estimate of the form $\lambda(\partial_\varepsilon D) \leq A\varepsilon^\gamma$ would say that in a sense the dimension of ∂D is no more than $d - \gamma$. Note that such an estimate could have taken the place of (2) in the proof of Proposition 1. As we shall see, there are rapidly exhaustible domains whose boundaries are of fractional dimension strictly larger than $d - 1$.

NOTATION. We shall denote by H the set of positive harmonic functions h in D satisfying $h(x_*) = 1$.

THEOREM 1. *Suppose D is rapidly exhaustible and q is a nonnegative function belonging to $L^p(D)$ for some $p > d/2$. Then*

$$\sup_{x \in D,\, h \in H} {}_hE^x\left\{\int_0^\tau q(X_t)\,dt\right\} < \infty. \tag{5}$$

Moreover, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for any Borel set $C \subseteq D$ satisfying $\lambda(C) \le \delta$, we have

$$\sup_{x \in D,\, h \in H} {}_hE^x\left\{\int_0^{\tau_c} q(X_t)\,dt\right\} < \varepsilon\|q\|_p, \tag{6}$$

where $\tau_c = \inf\{t > 0\colon X_t \notin C\}$.

PROOF. Since D is rapidly exhaustible, $\lambda(D) < \infty$. Hence q also belongs to $L^r(D)$ for $1 \le r \le p$. Let r be chosen to satisfy $d/2 < r < p$ and let $\beta > 0$ be defined by $p = (1+\beta)r$. Let $\alpha = \beta/p$. We shall show that there is a constant $c_1 < \infty$, such that

$$ {}_hE^x\left\{\int_0^{\tau_c} q(X_t)\,dt\right\} \le c_1\|q\|_p \sum_{n=0}^{\infty} \lambda(E_n \cap C)^{\alpha}. \tag{7}$$

Since D is rapidly exhaustible, (5) and (6) follow from (7). [To deduce (6), apply a dominated convergence argument to the series on the right in (7).]

Let c be as in (1). For each integer n, positive or negative, let

$$ {}_hA_n = \{x \in D\colon c^{n-1} < h(x) < c^{n+1}\},$$
$$ {}_hB_n = \{x \in D\colon h(x) = c^n\}.$$

Since $h(x_*) = 1$, it follows from (1) that ${}_hA_n \subseteq E_{|n|}$ for all integers n. Let

$$R = \inf\{t > 0\colon X_t \notin {}_hA_n \text{ but } X_0 \in {}_hA_n, \text{ for some } n\}.$$

Note that

$$h(X_t)/h(X_0) < c \quad \text{for } 0 \le t < R. \tag{8}$$

Let $S_1 = R$,

$$S_{k+1} = S_k + R \circ \theta_{S_k}, \qquad k = 1, 2, 3, \dots.$$

Then

$$S_k \wedge \tau \uparrow \tau. \tag{9}$$

Indeed, if $t < \tau(\omega)$, then since any open cover of the compact set $[0, t]$ has a Lebesgue number, there is a positive integer m such that for each $l \in \{1, \dots, m\}$, there is an integer n for which $X_s(\omega) \in {}_hA_n$ for all s in the interval $I_l = [(l-1)t/m, lt/m]$. But then each interval I_l can contain at most one $S_k(\omega)$, so $S_{m+1}(\omega) > t$.

Now let $T = \tau_c$ and let $S_0 = 0$. Note that for $0 \le t < \infty$, we have

$$T = t + T \circ \theta_t \quad \text{identically on } \{T > t\}. \tag{10}$$

By (9) and (10),

$$\int_0^T q(X_t)\,dt = \sum_{k=0}^{\infty} 1_{\{S_k < T\}} \int_{S_k}^{S_{k+1} \wedge T} q(X_t)\,dt$$
$$= \sum_{k=0}^{\infty} 1_{\{S_k < T\}} \left\{\left[\int_0^{R \wedge T} q(X_t)\,dt\right] \circ \theta_{S_k}\right\}.$$

Therefore, by the strong Markov property,

$$
\begin{aligned}
& {}_hE^x\left\{\int_0^T q(X_t)\,dt\right\} \\
(11)\qquad & = {}_hE^x\left\{\int_0^{R\wedge T} q(X_t)\,dt\right\} \\
& \quad + \sum_{k=1}^{\infty}\sum_n {}_hE^x\left\{{}_hE^{X(S_k)}\left[\int_0^{R\wedge T} q(X_t)\,dt\right];\ S_k < T,\ X(S_k)\in{}_hB_n\right\}.
\end{aligned}
$$

Now

$$
\begin{aligned}
{}_hE^y\left\{\int_0^{R\wedge T} q(X_t)\,dt\right\} &= \int_0^\infty {}_hE^y\{q(X_t);\ t < R\wedge T,\ t<\tau\}\,dt \\
&= \int_0^\infty E^y\{q(X_t)h(X_t)/h(X_0);\ t < R\wedge T,\ t<\tau\}\,dt \\
&\le cE^y\left\{\int_0^{R\wedge T} q(X_t)\,dt\right\} \quad \text{by (8)}.
\end{aligned}
$$

Thus if we let

$$
{}_hM_n = \sup_{y\in{}_hA_n} E^y\left\{\int_0^{R\wedge T} q(X_t)\,dt\right\},
$$

then the first term on the right in (11) is no greater than $c\Sigma_n\,{}_hM_n$, while the general term in the double sum there is less than or equal to

$$
c\,{}_hM_n\,{}_hP^x\{X(S_k)\in{}_hB_n\}.
$$

If $k\ge 2$ and $X(S_k)\in{}_hB_n$, then during $[S_{k-1}, S_k]$, the positive ${}_hP^x$-supermartingale $(1/h(X_t))$ has either performed an upcrossing of $[c^{-n-1}, c^{-n}]$ or a downcrossing of $[c^{-n}, c^{-n+1}]$. Hence by the upcrossing and downcrossing inequalities, we have

$$
\sum_{k=1}^{\infty} {}_hP^x\{X(S_k)\in{}_hB_n\} \le 1 + \frac{1}{c-1} + \frac{c}{c-1} = \frac{2c}{c-1}.
$$

Inserting these estimates into (11), we obtain

$$
(12)\qquad {}_hE^x\left\{\int_0^T q(X_t)\,dt\right\} \le \frac{3c^2-c}{c-1}\sum_n {}_hM_n.
$$

It remains only to estimate ${}_hM_n$. We have

$$
\begin{aligned}
E^y\left\{\int_0^{R\wedge T} q(X_t)\,dt\right\} &\le E^y\left\{\int_0^{\tau}\left(q1_{C\cap{}_hA_n}\right)(X_t)\,dt\right\} \\
(13)\qquad &= G\left(q1_{C\cap{}_hA_n}\right)(y) \\
&\le G\left(q1_{C\cap E_{|n|}}\right)(y) \\
&\le \gamma\|q1_{C\cap E_{|n|}}\|_r,
\end{aligned}
$$

for a suitable constant $\gamma<\infty$. In the last step we have used Hölder's inequality:

since $r > d/2$, $\sup_{y \in D} \|G(y, \cdot)\|_s = \gamma < \infty$ (where $r^{-1} + s^{-1} = 1$). If $d \geq 3$, this follows from the fact that G is bounded by the explicitly known Green function of $\mathbb{R}^d$; if $d = 2$, it follows from Lemma 1 below. Next, for any Borel set $F \subseteq D$,

$$
\begin{aligned}
\|q 1_F\|_r &= \left\{ \int q^r 1_F \right\}^{1/r} \\
&\leq \left\{ \|q^r\|_{(1+\beta)} \|1_F\|_{(1+\beta)'} \right\}^{1/r} \\
&= \left\{ \left[\int q^p \right]^{1/(1+\beta)} \lambda(F)^{\beta/(1+\beta)} \right\}^{1/r} \\
&= \|q\|_p \lambda(F)^{\alpha}.
\end{aligned}
\tag{14}
$$

On taking $F = C \cap E_{|n|}$ in (14), and combining this with (12) and (13), we obtain (7) with

$$c_1 = 2\gamma \frac{3c^2 - c}{c - 1}.$$

This completes the proof of the theorem. □

REMARK. In [6] Cranston and McConnell showed that if the dimension $d = 2$, then ${}_hE^x\{\tau_C\} \leq c_2 \lambda(C)$ for a suitable constant $c_2 < \infty$. This holds for an arbitrary domain $D \subseteq \mathbb{R}^2$ and c_2 is even independent of D. (See [3] for a simpler proof.) Thus if q is bounded and $p = \infty$, then the conclusions of Theorem 1 hold for an arbitrary domain $D \subseteq \mathbb{R}^2$ whose area is finite.

COROLLARY 1. *If D is rapidly exhaustible then* $\sup_{x \in D,\, h \in H} {}_hE^x\{\tau\} < \infty$.

PROOF. This follows immediately from Theorem 1 on taking $q \equiv 1$, but let us point out that in fact,

$${}_hE^x\{\tau\} \leq c_1 \sum_{n=0}^{\infty} \lambda(E_n)^{2/d},$$

for a suitable constant $c_1 < \infty$ which depends only on d. This follows from the proof of Theorem 1, together with the estimate $E^x\{\tau_C\} \leq c_2 \lambda(C)^{2/d}$, where $c_2 < \infty$ is another constant depending only on d. The latter estimate is proved in [6] and also in [2], page 148ff. □

LEMMA 1. *Suppose the dimension $d = 2$ and the area of D is finite. Then there are constants $c_1, c_2 < \infty$ such that the Green function G of D satisfies*

$$G(x, y) \leq \max\left\{ -\frac{1}{\pi} \ln\|x - y\| + c_1, c_2 \right\}$$

for all $x, y \in D$.

PROOF. Recall that $G(x, y) = \int_0^\infty p_t(x, y)\,dt$. Let $t_0 = \lambda(D)/\pi$. Then

$$P^x(\tau > t_0) \le \frac{\lambda(D)}{2\pi t_0} = \frac{1}{2}$$

for all $x \in D$. By the Markov property,

$$P^x(\tau > nt_0) \le 2^{-n}.$$

Thus for $t \ge nt_0$,

$$\int_D p_t(x, y)\,dy \le P^x(\tau > t) \le P^x(\tau > nt_0) \le 2^{-n}.$$

But then for $t \ge (n+1)t_0$,

$$\begin{aligned} p_t(x, y) &= \int_D p_{t-t_0}(x, x')p_{t_0}(x', y)\,dx' \\ &\le \frac{1}{2\pi t_0}\int_D p_{t-t_0}(x, x')\,dx' \\ &\le \frac{2^{-n}}{2\pi t_0}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{2t_0}^\infty p_t(x, y)\,dt &= \sum_{n=1}^\infty \int_{(n+1)t_0}^{(n+2)t_0} p_t(x, y)\,dt \\ &\le \sum_{n=1}^\infty \frac{2^{-n}}{2\pi t_0} t_0 = \frac{1}{2\pi}. \end{aligned} \tag{15}$$

Next, by letting $b = \|x - y\|$, we have

$$\begin{aligned} \int_0^{2t_0} p_t(x, y)\,dt &\le \int_0^{2t_0} \frac{1}{2\pi t} e^{-b^2/2t}\,dt \\ &= \frac{1}{2\pi}\int_{b^2/4t_0}^\infty \frac{e^{-s}}{s}\,ds, \end{aligned}$$

where we have made the substitution $s = b^2/2t$. If $b^2/4t_0 \le 1$, we have

$$\begin{aligned} \int_0^{2t_0} p_t(x, y)\,dt &\le \frac{1}{2\pi}\int_{b^2/4t_0}^1 \frac{1}{s}\,ds + \frac{1}{2\pi}\int_1^\infty e^{-s}\,ds \\ &= -\frac{1}{\pi}\ln\|x - y\| + \frac{1}{2\pi}\ln\frac{4\lambda(D)}{\pi} + \frac{1}{2\pi}. \end{aligned} \tag{16}$$

If, on the other hand, $b^2/4t_0 > 1$, then

$$\int_0^{2t_0} p_t(x, y)\,dt \le \frac{1}{2\pi}\int_1^\infty e^{-s}\,ds = \frac{1}{2\pi}. \tag{17}$$

On combining (15), (16) and (17) we see that we may take

$$c_1 = \frac{1}{2\pi}\ln\frac{4\lambda(D)}{\pi} + \frac{1}{\pi}$$

and

$$c_2 = \frac{1}{\pi}.$$

This completes the proof of the lemma. □

In the following result, which will be needed in the proof of Theorem 2, D may be an arbitrary domain in $\mathbb{R}^d$.

LEMMA 2. *Suppose C is a connected open subset of D such that $K = D \setminus C$ is compact. Let F be a compact subset of C. Then*

$$\inf_{y \in F,\, h \in H} {}_hP^y\{\tau_c = \tau\} > 0. \tag{18}$$

PROOF. We have

$${}_hP^y\{\tau_c = \tau\} = \frac{h(y) - f(y, h)}{h(y)},$$

where $f(y, h) = E^y\{h(X(\tau_c));\ \tau_c < \tau\}$. But note that $f(\cdot, h) = P_K h$, the réduite of h over K, which is a potential since K is compact. From this and from the connectedness of C, it follows that $h - f(\cdot, h)$ is strictly positive in C. Next, H is compact when given the topology of uniform convergence on compact subsets of D. This well-known fact follows from Harnack's inequality and Theorem 2.18 in [10]. To finish the proof, it suffices to show that f is continuous on $C \times H$. Suppose $y_i \to y$ in C and $h_i \to h$ in H. Then for each $x \in C$, $f(x, h_i) \to f(x, h)$. But the functions $f(\cdot, h_i)$ are positive and harmonic in C, so it follows that $f(\cdot, h_i) \to f(\cdot, h)$ uniformly on compact subsets of C. Hence $f(y_i, h_i) \to f(y, h)$. This completes the proof of the lemma. □

THEOREM 2. *Suppose D is rapidly exhaustible and $q \in L^p(D)$ for some $p > d/2$. Let $e_q(\tau) = \exp[\int_0^\tau q(X_s)\, ds]$, which is defined a.s. because of (5). If ${}_{h_0}E^{x_0}\{e_q(\tau)\} < \infty$ for some $(x_0, h_0) \in D \times H$, then*

$$\sup_{x \in D,\, h \in H} {}_hE^x\{e_q(\tau)\} < \infty.$$

PROOF. First note that we can construct an open connected set $C \subseteq D$ whose measure $\lambda(C)$ is as small as we like and such that $D \setminus C$ is compact. Indeed, $\partial_\varepsilon D$ is open, $D \setminus \partial_\varepsilon D$ is closed in $\mathbb{R}^d$ and bounded [otherwise D would contain an unbounded sequence of balls of radius ε, which would contradict the finiteness of $\lambda(D)$], and $\lambda(\partial_\varepsilon D) \to 0$ as $\varepsilon \downarrow 0$. The only trouble is that $\partial_\varepsilon D$ need not be connected. However, it has only countably many connected components. [In fact, only finitely many, since $\lambda(D) < \infty$.] By joining together the components of $\partial_\varepsilon D$ with thin tubes through D of small measure, a set C of the desired sort can be constructed. We shall take $\lambda(C)$ to be small enough so that

$$b_1 = \sup_{x \in D,\, h \in H} {}_hE^x\left\{\int_0^{\tau_c} |q(X_s)|\, ds\right\} < 1. \tag{19}$$

This can be done, by Theorem 1. It follows from (19) that

$$(20)\qquad {}_hE^x\left\{\exp\left[\int_0^{\tau_c}|q(X_s)|\,ds\right\}\le\frac{1}{1-b_1}.$$

(To see this, expand the exponential, apply the Markov property and compare with the series $1+b_1+b_1^2+\cdots$.) Now let U be an open, relatively compact subset of D such that $D\setminus C\subseteq U$. Then ∂U is a compact subset of C so by Lemma 2,

$$(21)\qquad b_2=\inf_{y\in\partial U,\,h\in H}{}_hP^y\{\tau_c=\tau\}>0.$$

(The connectedness of C is vital here.) Now we claim that there are constants $c_1>0$ and $c_2<\infty$ such that

$$(22)\qquad c_1\le{}_hE^x\{e_q(\tau);\ \tau_c=\tau\}\le c_2.$$

By (20), we can take $c_2=1/(1-b_1)$. To find a suitable value for c_1, we apply (19) in conjunction with Jensen's inequality,

$$\begin{aligned}{}_hE^x\{e_q(\tau)\,|\tau_c=\tau\}&\ge{}_hE^x\{e_{-|q|}(\tau)\,|\tau_c=\tau\}\\&\ge\exp\left\{-{}_hE^x\left[\int_0^{\tau}|q(X_s)|\,ds\,\middle|\tau_c=\tau\right]\right\}\\&\ge e^{-b_1/b_2}.\end{aligned}$$

Thus we can take $c_1=b_2e^{-b_1/b_2}$. Now let $T_0=0$ and for $n=1,2,3,\dots,$ let

$$T_{2n-1}=T_{2n-2}+\tau_U\circ\theta_{T_{2n-2}},$$
$$T_{2n}=T_{2n-1}+\tau_C\circ\theta_{T_{2n-1}}.$$

For ${}_hP^x$ almost all ω, $\tau(\omega)<\infty$ (by Theorem 1) and $(X_t(\omega))_{0\le t<\tau(\omega)}$ is ultimately outside every compact subset of D. For such an ω, there exists an integer $n\ge1$ such that $T_{2n}(\omega)=\tau(\omega)<\infty$ while $T_{2n+1}(\omega)=\infty$. Therefore

$$(23)\qquad\begin{aligned}{}_hE^x\{e_q(\tau)\}&=\sum_{n=1}^{\infty}{}_hE^x\{e_q(\tau);\ T_{2n}=\tau\}\\&=\sum_{n=1}^{\infty}{}_hE^x\{{}_hE^{X(T_{2n-1})}[e_q(\tau_c);\ \tau_c=\tau]\,e_q(T_{2n-1});\ T_{2n-1}<\tau\},\end{aligned}$$

where the second step follows by a straightforward calculation based on the strong Markov property. From (22) and (23) we obtain

$$(24)\qquad c_1\phi(x,h)\le{}_hE^x\{e_q(\tau)\}\le c_2\phi(x,h),$$

where $\phi(x,h)=\sum_{n=1}^{\infty}\phi_n(x,h)$ and

$$\begin{aligned}\phi_n(x,h)&={}_hE^x\{e_q(T_{2n-1});\ T_{2n-1}<\tau\}\\&=\frac{1}{h(x)}E^x\{e_q(T_{2n-1})h(X(T_{2n-1}));\ T_{2n-1}<\tau\}.\end{aligned}$$

By Harnack's inequality, there are constants $c_3 > 0$ and $c_4 < \infty$ such that for all $h \in H$,

$$c_3 \le h \le c_4 \quad \text{on } \overline{U}. \tag{25}$$

By assumption,

$$_{h_0}E^{x_0}\{e_q(\tau)\} < \infty.$$

We may suppose that U was chosen so that $x_0 \in U$. Hence, by (24) and (25),

$$\begin{aligned} E^{x_0}\{e_q(\tau)\} &\le c_2\phi(x_0, 1) \\ &\le \frac{c_2c_4}{c_3}\phi(x_0, h_0) \\ &\le \frac{c_2c_4}{c_1c_3}\,{}_{h_0}E^{x_0}\{e_q(\tau)\} \\ &< \infty. \end{aligned}$$

Hence

$$c_5 = \sup_{x \in U} E^x\{e_q(\tau)\} < \infty.$$

This follows from the "Harnack inequality" proved in [1], Theorem 3.10. (For a simpler proof, see [13].) Then by (24) and (25), for $x \in U$ we have

$$\begin{aligned} {}_hE^x\{e_q(\tau)\} &\le c_2\phi(x, h) \\ &\le \frac{c_2c_4}{c_3}\phi(x, 1) \\ &\le \frac{c_2c_4c_5}{c_1c_3}. \end{aligned} \tag{26}$$

Finally, for $x \in D \setminus U$,

$$\begin{aligned} {}_hE^x\{e_q(\tau)\} &= {}_hE^x\{e_q(\tau_c);\ \tau_c = \tau\} + {}_hE^x\{{}_hE^{X(\tau_c)}[e_q(\tau)]e_q(\tau_c);\ \tau_c < \tau\} \\ &\le \left[1 + \frac{c_2c_4c_5}{c_1c_3}\right]\left[\frac{1}{1 - b_1}\right], \end{aligned}$$

by (20) and (26). This completes the proof of Theorem 2. □

REMARK. The conclusion of Theorem 2 also holds for an arbitrary domain D of finite area in $\mathbb{R}^2$, provided q is bounded. This follows from the proof of Theorem 2 together with the remark which follows the proof of Theorem 1.

EXAMPLE. We now describe a domain D in $\mathbb{R}^3$ which is rapidly exhaustible even though it does not satisfy a (uniform) interior cone condition. The boundary of this domain is very rough: Its Hausdorff dimension is $\ln 20/\ln 3 \approx 2.7$. To construct D, start with a unit cube. Then bore square holes of side $1/3$ through this cube in each of the three obvious perpendicular directions. What is left of the unit cube may then be viewed as consisting of 20 cubes of side $1/3$. Through each of these 20 cubes, bore square holes of side $1/9$ in each of the same three perpendicular directions. And so on. After the nth step, what is left of the original unit cube consists of 20^n cubes of side 3^{-n}. (For a picture of what is left at the fourth stage, see [12], page 145.) After this procedure has been repeated infinitely many times, what remains of the original cube is a compact connected set M which in [12] is called the Menger sponge. By the self-similarity of M, the Hausdorff dimension of M may be easily computed (see [8], Section 8.3). It comes out to be $\ln 20/\ln 3$. Now the domain D consists of the interior of what was removed from the original cube. The boundary of D also has Hausdorff dimension $\ln 20/\ln 3$, since it consists of M together with a certain subset of the boundary of the original cube. Let $I = \{1, \dots, 20\}$. Note that what is removed from the cube at the nth step breaks up naturally into 20^{n-1} pieces U_s, $s \in I^{n-1}$, each piece having volume $(7/27)27^{1-n}$. Thus the total volume of what is removed at the nth stage decreases geometrically. Let x_s be the "center" of U_s. Clearly there is a positive integer l such that for any $n \geq l$, any $s \in I^{n-1}$, and any $i \in I$, there is a dilatable chain of length l from x_s to $x_{s,i}$. Using these observations, and the similarity of the pieces U_s, one can readily check that D is rapidly exhaustible. Indeed, $\lambda(E_n) \to 0$ geometrically fast.

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