The Annals of Probability
1987, Vol. 15, No. 4, 1524–1526

A STOPPED BROWNIAN MOTION FORMULA WITH TWO SLOPING LINE BOUNDARIES

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We find the moment generating function of the time at which Brownian motion exits from a region bounded by two nonconvergent straight lines, using the martingale stopping theorem.

1. Introduction. Let $W(t)$, $t \geq 0$, be a standard Brownian motion starting at zero. Define $T_1 = \inf(t \geq 0 | W(t) \geq a + bt)$, $T_2 = \inf(t \geq 0 | W(t) \leq m + nt)$, for $a > 0 > m$ and $b \geq n$. In this paper, we determine the moment generating function of $T = \min(T_1, T_2)$. Doob (1949) found the probability that Brownian motion always leaves the two-sided region $|x| < a + bt$, using an inclusion–exclusion argument. Another method is given in Breiman (1968). Karlin and Taylor (1975) applied the martingale stopping theorem to get the probability that a Brownian motion with drift $\mu \neq 0$ reaches the level $a > 0$ before hitting $m < 0$. Darling and Siegert (1953) obtained the Laplace transform for the case of two parallel lines, that is for $b = n$, and Harrison (1985) derives the same formula, also using the martingale stopping theorem. Anderson (1960), Theorem 4.1, gives the probability that Brownian motion touches an upper straight-line boundary before a lower one, or equivalently, $P(T = T_1 < \infty)$. The following result generalizes the ones mentioned above.

2. Theorem.

$$\begin{aligned} &E(\exp[zT]; T = T_1 < \infty) \\ &\quad = \sum_{k=1}^{\infty} \exp\Big[2(b-n)m(k-1)^2 - 2(b-n)a(k-1)k - ab \\ &\qquad\qquad + (b^2 - 2z)^{1/2}(2(m-a)(k-1) - a)\Big] \\ &\qquad \times \Big(1 - \exp\Big[2m\big((b-n)(2k-1) + (b^2-2z)^{1/2}\big)\Big]\Big); \\ &E(\exp[zT]; T = T_2 < \infty) \\ &\quad = \sum_{k=1}^{\infty} \exp\Big[2(b-n)m(k-1)k - 2(b-n)a(k-1)^2 - mn \\ &\qquad\qquad + (n^2 - 2z)^{1/2}(2(m-a)(k-1) + m)\Big] \\ &\qquad \times \Big(1 - \exp\Big[-2a\big((b-n)(2k-1) + (n^2-2z)^{1/2}\big)\Big]\Big), \end{aligned}$$

for all $z \leq 0$, except for $z = b = n = 0$.

Received August 1986.
AMS 1980 *subject classifications*. Primary 60G40; secondary 60J65.
Key words and phrases. Brownian motion, stopping time, moment generating function, martingale.

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PROOF. Consider, for any real c, the exponential martingale $M_c(t) = \exp[cW(t) - c^2t/2]$, $t \geq 0$. If $c \geq \max(0, 2b)$, $M_c(T \wedge t)$ can be majorized, uniformly in t, by $\exp(ca)$; similarly, if $c \leq \min(0, 2n)$ we have $M_c(T \wedge t) \leq \exp(cm)$ for all $t \geq 0$. In both cases, the optional stopping theorem can be applied to conclude: $EM_c(0) = E(M_c(T);\ T < \infty) = E(M_c(T);\ T = T_1 < \infty) + E(M_c(T);\ T = T_2 < \infty)$. Let $G_i(z) = E(\exp(zT);\ T = T_i < \infty)$, $i = 1, 2$. Then we obtain, when $c \geq \max(0, 2b)$ or $c \leq \min(0, 2n)$,

$$1 = \exp(ca)G_1[c(b - c/2)] + \exp(cm)G_2[c(n - c/2)]. \tag{1}$$

For fixed $z_1 \leq 0$, define c_1 by $c_1 = b + (b^2 - 2z_1)^{1/2}$, so that $c_1(b - c_1/2) = z_1$ and $c_1 \geq \max(0, 2b)$. Then from (1) we get, for $z_2 = c_1(n - c_1/2)$,

$$1 = \exp(c_1 a)G_1(z_1) + \exp(c_1 m)G_2(z_2). \tag{2}$$

Define now $c_2 = 2n - c_1$, so that $c_2(n - c_2/2) = z_2$ and $c_2 \leq \min(0, 2n)$. Again, from (1) we get, for $z_3 = c_2(b - c_2/2)$,

$$1 = \exp(c_2 a)G_1(z_3) + \exp(c_2 m)G_2(z_2). \tag{3}$$

Solving (2) and (3) for $G_1(z_1)$, we obtain

$$\begin{aligned} G_1(z_1) &= \exp(-c_1 a) - \exp(c_1(m - a) - c_2 m) \\ &\quad + \exp((c_1 - c_2)(m - a))G_1(z_3). \end{aligned} \tag{4}$$

Now we have $z_3 \leq 0$, and by the same procedure as before we derive

$$\begin{aligned} G_1(z_3) &= \exp(-c_3 a) - \exp(c_3(m - a) - c_4 m) \\ &\quad + \exp((c_3 - c_4)(m - a))G_1(z_5), \end{aligned} \tag{5}$$

where $c_3 = b + (b^2 - 2z_3)^{1/2} = 2b - c_2$, $c_4 = 2n - c_3$ and $z_5 = c_4(b - c_4/2)$. If $G_1(z_3)$ is replaced in (4) by its expression (5), we can put $G_1(z_1)$ in terms of $G_1(z_5)$. Thus we have $z_5 \leq 0$ and we can repeat the procedure, getting an expression for $G_1(z_5)$ in terms of $G_1(z_7)$; then, it is possible to express $G_1(z_1)$ in terms of $G_1(z_7)$ and so on. By carrying this procedure on, we arrive at

$$G_1(z_1) = \sum_{k=1}^{r} \left[\exp(-c_{2k-1}a)Q_{k-1} - \exp(-c_{2k}a)Q_k\right] + Q_r G_1(z_{2r+1}), \tag{6}$$

for $r \geq 1$, where

$$Q_k = \exp\left[(m - a)\sum_{i=1}^{k}(c_{2i-1} - c_{2i})\right],$$

if $k \geq 1$, $Q_0 = 1$ and

$$c_{2k-1} = 2b - c_{2k-2}, \qquad k \geq 2, \tag{7}$$

$$c_{2k} = 2n - c_{2k-1}, \qquad k \geq 1, \tag{8}$$

$$z_{2k+1} = c_{2k}(b - c_{2k}/2), \qquad (k \geq 1).$$

We note that $c_{2k-1} \geq \max(0, 2b) \geq \min(0, 2n) \geq c_{2k}$ and $z_{2k+1} \leq 0$ for $k \geq 1$.

Hence, for $k \geq 1$,

$$\exp(-c_{2k-1}a)Q_{k-1} - \exp(-c_{2k}a)Q_k = Q_{k-1}\exp(-c_{2k-1}a)\left[1 - \exp((c_{2k-1} - c_{2k})m)\right] \geq 0.$$

Furthermore, from (7) and (8) we derive

$$c_{2k-1} = 2(k-1)(b-n) + c_1, \qquad k \geq 1, \tag{9}$$

and then

$$\begin{aligned} Q_r &= \exp\left[(m-a)\sum_{i=1}^{r}(2c_{2i-1} - 2n)\right] \\ &= \exp\left[2(m-a)\left(r(r-1)(b-n) + r(c_1 - n)\right)\right]. \end{aligned} \tag{10}$$

So, $\lim_{r\to\infty}Q_r = 0$, except for $c_1 = b = n = z_1 = 0$. For any $z \leq 0$ we have $0 \leq G_1(z) \leq 1$; then $\lim_{r\to\infty}Q_rG_1(z_{2r+1}) = 0$. From this and (6) we get

$$G_1(z_1) = \sum_{k=1}^{\infty} Q_{k-1}\exp(-c_{2k-1}a)\left(1 - \exp\left[(c_{2k-1} - c_{2k})m\right]\right).$$

Using (8)–(10) and substituting $c_1 = b + (b^2 - 2z_1)^{1/2}$, we complete the proof for $E(\exp[zT];\ T = T_1 < \infty)$. The second formula can be obtained in a similar way, or merely from the first one by replacing a, b, m, n by $-m, -n, -a, -b$, respectively. □

3. Remarks. (a) The case $z = b = n = 0$, excluded in the theorem, reduces to find $P(W(T_{am}) = a)$ and $P(W(T_{am}) = m)$, with T_{am} being the first time the process reaches $a > 0$ or $m < 0$. The solution is well known.

(b) One may easily extend the former result to a Brownian motion with a drift parameter μ and variance parameter σ^2. Using conventional techniques, we can also compute the moments of the stopping time and other related quantities.

(c) The case $b < n$ cannot be handled in the same way, even though the stopping time is bounded and (1) holds for any c. The iterative formula (6) remains valid, but Q_r does not have limit 0 as $r \to 0$ and the series diverges.

REFERENCES

ANDERSON, T. W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Ann. Math. Statist.* **31** 165–197.

BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, Mass.

DARLING, D. and SIEGERT, A. J. F. (1953). The first passage problem for a continuous Markov process. *Ann. Math. Statist.* **24** 624–639.

DOOB, J. L. (1949). Heuristic approach to the Kolmogorov–Smirnov theorems. *Ann. Math. Statist.* **20** 393–403.

HARRISON, J. M. (1985). *Brownian Motion and Stochastic Flow Systems*. Wiley, New York.

KARLIN, S. and TAYLOR, H. M. (1975). *A First Course in Stochastic Processes*. Academic, New York.

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