The Annals of Probability
1987, Vol. 15, No. 3, 885–896

THE CONTRIBUTION TO THE SUM OF THE SUMMAND OF MAXIMUM MODULUS[1]

BY WILLIAM E. PRUITT

University of Minnesota

Let X_k be i.i.d., $S_n = X_1 + \cdots + X_n$, and $X_n^{(1)}$ the term of maximum modulus among $\{X_1, \dots, X_n\}$. Let $u_k = P\{2^k < |X_1| \le 2^{k+1} \,|\, |X_1| > 2^k\}$. The main result is that $X_n^{(1)}/S_n \to 1$ a.s. iff $\Sigma u_k^2 < \infty$. Furthermore, for any positive integer r, $\liminf_{n\to\infty} |X_n^{(1)}/S_n| = r^{-1}$ a.s. iff $\Sigma_k u_k^r = \infty$ and $\Sigma_k u_k^{r+1} < \infty$. If $\Sigma_k u_k^r = \infty$ for all r then $\liminf_{n\to\infty}|X_n^{(1)}/S_n| = 0$ a.s.

1. Introduction. Let $X, X_1, X_2, \dots$ be a sequence of independent, identically distributed nondegenerate real valued random variables having common distribution function F. Let $S_n = X_1 + \cdots + X_n$ and let $X_n^{(1)}$ be the term of maximum modulus among $\{X_1, \dots, X_n\}$, i.e., $|X_n^{(1)}| \ge |X_k|$ for $k = 1, \dots, n$. Feller (1946) observed that $X_n^{(1)}$ could be used as a tool for studying the behavior of the *large* values of $|S_n|$ whenever $E|X| = \infty$. This idea has since been exploited in a variety of ways by numerous authors. It has also been observed that $X_n^{(1)}$ may be used to study S_n *throughout its range* if the tail of F is sufficiently fat. Darling (1952) showed that with nonnegative summands, if $1 - F(x)$ is slowly varying then $X_n^{(1)}/S_n \to 1$ in probability. This was extended to general summands by Arov and Bobrov (1960) and the converse was obtained by Maller and Resnick (1984). Maller and Resnick also gave separate necessary and sufficient conditions for $X_n^{(1)}/S_n \to 1$ a.s. and proved their conditions were equivalent under a supplementary hypothesis. In the present paper we will show that the two conditions of Maller and Resnick are equivalent in general. In order to completely answer the question of when $X_n^{(1)}$ contains essentially all the information about S_n one wants to know when $|X_n^{(1)}/S_n|$ is bounded above and below. We will answer this on the lower side by evaluating $\liminf|X_n^{(1)}/S_n|$ in general; it may surprise the reader that the only possible values of this lim inf are zero and r^{-1} where r is a positive integer. With nonnegative summands, $X_n^{(1)}/S_n \le 1$, so the ratio is automatically bounded above. But with general summands the question of when $\limsup|X_n^{(1)}/S_n| < \infty$ a.s. remains. More information including an explanation of why the lim inf is of the form r^{-1} is given in the statement of the theorems. Examples are given at the end of the paper.

2. Results. We need to introduce a little notation. Let $X_n^{(r)}$ denote the rth largest in modulus of $\{X_1, \dots, X_n\}$. In case of ties, they may be broken in any reasonable manner. (Ties may present a problem if the support of X is bounded; see the first part of the proof of Theorem 1.) Next ${}^{(r)}S_n$ will be the trimmed sum with the r largest summands discarded, i.e.,

$${}^{(r)}S_n = S_n - X_n^{(1)} - \cdots - X_n^{(r)}.$$

Received July 1985; revised December 1985.
[1]This work was partially supported by NSF Grant MCS 83-01080.
AMS 1980 *subject classifications*. 60F15, 60G50.
Key words and phrases. Random walk, trimmed sums, dominance of maximal summand, slowly varying tails.

Also for $x > 0$, define

$$G(x) = P\{|X| > x\}, \qquad H(x) = 1 - G(x) = P\{|X| \le x\}$$
$$M(x) = x^{-1}E|X|1\{|X| \le x\}, \qquad K(x) = x^{-2}EX^2 1\{|X| \le x\}.$$

Finally, for $\varepsilon \in (0,1)$, let

$$u_k = P\{2^k < |X| \le 2^{k+1} \mid |X| > 2^k\},$$
$$v_k = v_k(\varepsilon) = P\{\varepsilon^{-k} < |X| \le \varepsilon^{-k-1} \mid |X| > \varepsilon^{-k}\}.$$

These are not well defined if G ever vanishes; in this case we make the convention that $u_k, v_k = 1$ whenever the condition has probability 0. Now we can state the first result.

THEOREM 1. *Let r be a positive integer. Then the following are equivalent:*

$$\int \left(\frac{M(x)}{G(x)}\right)^r \frac{dH(x)}{G(x)} < \infty, \tag{1}$$

$$\int \left(\frac{K(x)}{G(x)}\right)^r \frac{dH(x)}{G(x)} < \infty, \tag{2}$$

$$\int \left(\frac{G(\varepsilon x) - G(x)}{G(x)}\right)^r \frac{dH(x)}{G(x)} < \infty \quad \textit{for every (for some) } \varepsilon \in (0,1), \tag{3}$$

$$\sum_k (v_k(\varepsilon))^{r+1} < \infty \quad \textit{for every (for some) } \varepsilon \in (0,1), \tag{4}$$

$$\sum_k u_k^{r+1} < \infty, \tag{5}$$

$$\frac{{}^{(r)}S_n}{X_n^{(1)}} \to 0 \quad a.s., \tag{6}$$

$$\frac{X_n^{(r+1)}}{X_n^{(1)}} \to 0 \quad a.s., \tag{7}$$

$$\limsup_{n\to\infty} \frac{X_n^{(r+1)}}{X_n^{(1)}} < 1 \quad a.s., \tag{8}$$

$$\liminf_{n\to\infty} \frac{|X_n^{(1)}|}{|S_n|} \ge \frac{1}{r} \quad a.s., \tag{9}$$

$$\liminf_{n\to\infty} \frac{|X_n^{(1)}|}{|S_n|} > \frac{1}{r+1} \quad a.s. \tag{10}$$

REMARK 1. Condition (1) with $r = 1$ is Maller and Resnick's sufficient condition while condition (3) with $r = 1$ and the quantifier for every is their necessary condition. Condition (5) is simply condition (4) for $\varepsilon = \frac{1}{2}$.

REMARK 2. All the conditions except (6) and (8)–(10) are in terms of the $|X_k|$ and we will see in the proof that (6) and (8) could be formulated that way as well. Thus much of the theorem is a result about nonnegative summands. But the implication (10) ⇒ (1) seems to require more work for general summands.

PROOF. We will start by showing the equivalence of the analytic conditions by proving (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1). Then we will complete the proof by showing (5) ⇒ (6) ⇒ (7) ⇒ (8), (6) ⇒ (9) ⇒ (10), and finally either (8) or (10) ⇒ (4). Note that the integrals in (1)–(3) are well defined but will diverge if G ever vanishes. In conjunction with our convention about v_k this shows that none of (1)–(4) holds in this case. Also it is clear that (except as noted below)

$$\limsup_{n\to\infty}\left|\frac{{}^{(r)}S_n}{X_n^{(1)}}\right| = \infty, \qquad \limsup_{n\to\infty}\frac{X_n^{(r+1)}}{X_n^{(1)}} = 1,$$

$$\liminf_{n\to\infty}\frac{|X_n^{(1)}|}{|S_n|} = 0 \quad \text{a.s.}$$

when the distribution of X has compact support. (The second statement may fail if $|X| \le a$ and $P\{X = a\} > 0$, $P\{X = -a\} > 0$. If one breaks the ties in a capricious manner, one might arbitrarily take $X_n^{(1)} = a$, $X_n^{(2)} = -a$ for all large n. However, if the ties are broken according to the order of occurrence, as usual, then one will have $\limsup_{n\to\infty} X_n^{(r+1)}/X_n^{(1)} = 1$ with positive probability in this case and so (8) will fail.) Thus for the remainder of the proof we assume that $G(x) > 0$ for all x.

(1) ⇒ (2): This is clear since $K(x) \le M(x)$.

(2) ⇒ (3): This follows from

$$K(x) \ge x^{-2}\int_{\varepsilon x < |y| \le x} y^2\,dF(y) \ge \varepsilon^2(G(\varepsilon x) - G(x)).$$

(3) ⇒ (4): Here we assume (3) for some $\varepsilon > 0$ and we will show that $\sum_k (v_k(\varepsilon))^{r+1} < \infty$ for that ε. We let

$$A_k = \{x \in \mathbb{R}^1 : \varepsilon^{-k} < x \le \varepsilon^{-k-1}\}.$$

Then

$$\begin{aligned}
\int_{A_k}&\left(\frac{G(\varepsilon x) - G(x)}{G(x)}\right)^r \frac{dH(x)}{G(x)} \\
&\ge \left(G(\varepsilon^{-k})\right)^{-r-1}\int_{A_k}(G(\varepsilon x) - G(x))^r\,dH(x) \\
&= \left(G(\varepsilon^{-k})\right)^{-r-1} P\{\varepsilon|X_1| < |X_j| \le |X_1|,\ j = 2,\dots,r+1;\ |X_1| \in A_k\} \\
&\ge \left(G(\varepsilon^{-k})\right)^{-r-1} P\{\varepsilon^{-k} < |X_j| \le |X_1|,\ j = 2,\dots,r+1;\ |X_1| \in A_k\} \\
&\ge \frac{1}{r+1}\left(G(\varepsilon^{-k})\right)^{-r-1} P\{|X_j| \in A_k,\ j = 1,\dots,r+1\} \\
&= \frac{1}{r+1}\left(v_k(\varepsilon)\right)^{r+1}.
\end{aligned}$$

(4) $\Rightarrow$ (1): First note that $v_k(\varepsilon) \to 0$ which implies $G(\varepsilon^{-k-1}) \sim G(\varepsilon^{-k})$. Since G is monotone, this means that G is slowly varying. Choose $\rho \in (\varepsilon, 1)$ and find N so that

$$\frac{G(\varepsilon^{-k-1})}{G(\varepsilon^{-k})} > \rho \quad \text{for } k \geq N.$$

Let $x_0 = \varepsilon^{-N}$. Now by Fubini or integrating by parts

$$M(x) = \int_0^1 (G(ux) - G(x))\, du.$$

For $\varepsilon^{-k} < x \leq \varepsilon^{-k-1}$ and $k \geq N$ we have

$$\begin{aligned} M(x) &\leq \frac{x_0}{x} + \sum_{j=N}^{k} \int_{\varepsilon^{-j}x^{-1}}^{\varepsilon^{-j-1}x^{-1}} (G(ux) - G(x))\, du \\ &\leq \frac{x_0}{x} + \sum_{j=N}^{k} \varepsilon^{-j-1}x^{-1}\left(G(\varepsilon^{-j}) - G(\varepsilon^{-k-1})\right) \\ &= \frac{x_0}{x} + \sum_{j=N}^{k} \varepsilon^{-j-1}x^{-1} \sum_{i=j}^{k} v_i G(\varepsilon^{-i}) \\ &\leq \frac{x_0}{x} + \sum_{j=N}^{k} \varepsilon^{-j-1}x^{-1} \sum_{i=j}^{k} v_i \rho^{i-k-1} G(\varepsilon^{-k-1}) \\ &\leq \frac{x_0}{x} + G(x) \sum_{i=N}^{k} v_i \rho^{i-k-1} \sum_{j=N}^{i} \varepsilon^{k-j-1} \\ &\leq \frac{x_0}{x} + G(x) \sum_{i=N}^{k} v_i \rho^{i-k-1} \varepsilon^{k-i-1} (1-\varepsilon)^{-1}. \end{aligned}$$

Thus for $\varepsilon^{-k} < x \leq \varepsilon^{-k-1}$, $k \geq N$, we have

$$\frac{M(x)}{G(x)} \leq \frac{x_0}{xG(x)} + C \sum_{i=1}^{k} v_i \left(\varepsilon\rho^{-1}\right)^{k-i}. \tag{2.1}$$

The first term in the bound causes no difficulty since it leads to an integrand of $Cx^{-r}(G(x))^{-r-1} \to 0$ as $x \to \infty$ since G is slowly varying and $r \geq 1$. Since the integrand in (1) is bounded on any fixed interval we may start the integration at x_0 and use the second term of the bound in (2.1). This leads to a bound for the integral of

$$\begin{aligned} &C^r \sum_k \frac{G(\varepsilon^{-k}) - G(\varepsilon^{-k-1})}{G(\varepsilon^{-k-1})} \left(\sum_{i=1}^{k} v_i \left(\varepsilon\rho^{-1}\right)^{k-i} \right)^r \\ &\quad \leq C_1 \sum_k \sum_{i_1 \leq k} \cdots \sum_{i_r \leq k} v_k v_{i_1} \cdots v_{i_r} \left(\varepsilon\rho^{-1}\right)^{kr - i_1 - \cdots - i_r} \\ &\quad \leq C_1 \sum_k \sum_{i_1 \leq k} \cdots \sum_{i_r \leq k} \left(v_k^{r+1} + v_{i_1}^{r+1} + \cdots + v_{i_r}^{r+1}\right)\left(\varepsilon\rho^{-1}\right)^{kr - i_1 - \cdots - i_r}. \end{aligned}$$

For the v_k^{r+1} term we sum first on $i_1, \dots, i_r$ and since $\varepsilon < \rho$, this will lead to an estimate of $C_2 \Sigma_k v_k^{r+1}$. For $v_{i_1}^{r+1}$, sum first on $i_2, \dots, i_r$ to obtain a bound of

$$C_3 \sum_k \sum_{i_1 \le k} v_{i_1}^{r+1} (\varepsilon\rho^{-1})^{k-i_1},$$

and then summing over k leads to a bound of $C_4 \Sigma_{i_1} v_{i_1}^{r+1}$. The other v_{i_j} terms are handled the same way. Thus the integral in (1) converges.

(5) ⇒ (6): This one is easier when F is continuous. We will assume this for now and explain how to complete the proof at the end. First we will prove another analytic condition:

$$\text{(0)} \qquad \iint_{0<x\le y<\infty} \frac{x}{y} \frac{M(x)}{G(x)} \left(\frac{G(x) - G(y)}{G(x)} \right)^{r-2} \frac{dH(y)\,dH(x)}{G^2(x)} < \infty.$$

Then we will show that (0) ⇒ (6). (If F is not assumed continuous then the $G(x)$ in $G(x) - G(y)$ in (0) should be replaced by $G(x^-)$.) We will use the bound (2.1) for $M(x)/G(x)$. As above, G is slowly varying so the first term in the bound leads to an integrand which approaches zero as $x \to \infty$ and so can be ignored as above. We will consider

$$\text{(2.2)} \qquad 2^j < x \le 2^{j+1}, \qquad 2^k x < y \le 2^{k+1} x,$$

and then sum over j and k. Since

$$G(x) - G(y) \le G(2^j) - G(2^{k+j+2}) = \sum_{i=j}^{k+j+1} u_i G(2^i)$$
$$\le G(2^j) \sum_{i=j}^{k+j+1} u_i \le CG(x) \sum_{i=j}^{k+j+1} u_i,$$

the contribution to the integral in (0) for the region in (2.2) is at most

$$C2^{-k} \left(\sum_{l=1}^{j} u_l (2\rho)^{l-j} \right) \left(\sum_{i=j}^{k+j+1} u_i \right)^{r-2} (u_{k+j} + u_{k+j+1}) u_j,$$

the last two terms coming from the y and x integrations. Thus we must bound

$$\sum_j \sum_k \sum_{l \le j} \sum_{i_1=j}^{k+j+1} \cdots \sum_{i_{r-2}=j}^{k+j+1} 2^{-k} (2\rho)^{l-j} u_j (u_{k+j} + u_{k+j+1}) u_l u_{i_1} \cdots u_{i_{r-2}}.$$

Since the u_{k+j} and u_{k+j+1} terms are similar we will only do the former. Then we must consider

$$\sum_j \sum_k \sum_{l \le j} \sum_{i_1=j}^{k+j+1} \cdots \sum_{i_{r-2}=j}^{k+j+1} 2^{-k} (2\rho)^{l-j} \left(u_j^{r+1} + u_{k+j}^{r+1} + u_l^{r+1} + u_{i_1}^{r+1} + \cdots + u_{i_{r-2}}^{r+1} \right).$$

We now consider each of the u terms separately:

u_j^{r+1}: Sum first on $i_1, \dots, i_{r-2}$ and l. Recall that $2\rho > 1$. This leads to a bound of $C(k+2)^{r-2} 2^{-k} u_j^{r+1}$. Now sum on k.

u_{k+j}^{r+1}: Sum on $i_1, \ldots, i_{r-2}$, and l to obtain a bound of $C(k+2)^{r-2}2^{-k}u_{k+j}^{r+1}$. Now change variables letting $k+j=m$ and sum on k.

u_l^{r+1}: Sum on $i_1, \ldots, i_{r-2}$ and then j to obtain a bound of $C(k+2)^{r-2}2^{-k}u_l^{r+1}$. Then sum on k.

$u_{i_\nu}^{r+1}$: Sum on the other i's and l to obtain a bound of $C(k+2)^{r-3}2^{-k}u_{i_\nu}^{r+1}$. Now sum on j. The bounds $j \le i_\nu \le k+j+1$ become $i_\nu - k - 1 \le j \le i_\nu$ so there are at most $k+2$ values of j that occur for fixed i_ν and k. Thus the bound is $C(k+2)^{r-2}2^{-k}u_{i_\nu}^{r+1}$ and now sum on k as before.

Since all the terms lead to bounds of the form $\Sigma_i u_i^{r+1}$ the integral in (0) converges. Now we need to prove that (0) $\Rightarrow$ (6). We will actually prove that $(\Sigma_{k=r+1}^n |X_n^{(k)}|)/|X_n^{(1)}| \to 0$ a.s. Since this ratio is increasing on intervals where $|X_n^{(1)}|$ is constant we only need to consider those n for which $|X_n^{(1)}| < |X_{n+1}|$. Next observe that if $X_n^{(1)}$ and $X_n^{(r)}$ are given then the $n-r$ summands smaller than $|X_n^{(r)}| = x$, say, are i.i.d. with distribution $dF(z)1\{|z| \le x\}(1-G(x))^{-1}$. This is where the continuity of F is used. Using Markov's inequality, we obtain

$$P\left\{\sum_{k=r+1}^{n} |X_n^{(k)}| > \varepsilon|X_n^{(1)}|, |X_n^{(1)}| < |X_{n+1}|\right\}$$

$$\le \iint_{0<x\le y<\infty} \frac{(n-r)xM(x)}{\varepsilon y(1-G(x))} G(y)\, dP\{|X_n^{(r)}| = x, |X_n^{(1)}| = y\}$$

$$= \iint_{0<x\le y<\infty} \frac{(n-r)xM(x)}{\varepsilon y(1-G(x))} G(y) \frac{(n)_r}{(r-2)!}$$

$$\times (1-G(x))^{n-r}(G(x)-G(y))^{r-2}\, dH(x)\, dH(y)$$

$$\le \varepsilon^{-1}n^{r+1} \iint_{0<x\le y<\infty} \frac{x}{y}\frac{M(x)}{1-G(x)} G(y)(1-G(x))^{n-r}$$

$$\times (G(x)-G(y))^{r-2}\, dH(x)\, dH(y).$$

Now we sum on n in order to use Borel–Cantelli. This yields

$$\sum_n P\left\{\sum_{k=r+1}^{n} |X_n^{(k)}| > \varepsilon|X_n^{(1)}|, |X_n^{(1)}| < |X_{n+1}|\right\}$$

$$\le C\iint_{0<x\le y<\infty} \frac{x}{y}\frac{M(x)}{1-G(x)} G(y)(G(x))^{-r-2}$$

$$\times (G(x)-G(y))^{r-2}\, dH(x)\, dH(y).$$

After replacing $G(y)$ by $G(x)$ this is the integral in (0) except for the factor $1-G(x)$. This does not affect convergence of the integral as the integral clearly converges for $x \le C$ for any C.

It remains to prove that the continuity of F is not needed. We introduce a new sequence of independent, identically distributed uniform $[0,1]$ random variables $\{U_k\}$ which are independent of the $\{X_k\}$ and let $Y_k = X_k + U_k$. Since

$$G(x+1) \le P\{|Y_1| > x\} \le G(x-1)$$

and since (5) implies that G is slowly varying we have $P\{|Y_1| > x\} \sim G(x)$. Also for $k \geq 1$,

$$P\{2^k < |Y_1| \leq 2^{k+1}\} \leq G(2^{k-1}) - G(2^{k+2}),$$

so that

$$P\{2^k < |Y_1| \leq 2^{k+1} \,|\, |Y_1| > 2^k\} \leq C(u_{k-1} + u_k + u_{k+1}).$$

Thus the condition (5) holds for the sequence $\{Y_k\}$ which has a continuous distribution. By what we have already proved, $(\sum_{k=r+1}^{n} |Y_n^{(k)}|)/|Y_n^{(1)}| \to 0$ a.s. Finally

$$\left| |X_n^{(k)}| - |Y_n^{(k)}| \right| \leq 1$$

for all n, k so

$$\left| \sum_{k=r+1}^{n} |X_n^{(k)}| - \sum_{k=r+1}^{n} |Y_n^{(k)}| \right| \leq n,$$

and since G is slowly varying, it is easy to check that $n^{-1}|X_n^{(1)}| \to \infty$ a.s. Putting these facts together yields (6).

(6) $\Rightarrow$ (7): We consider the (random) subsequence n_k defined by $n_0 = 1$ and for $k \geq 1$

$$n_k = \min\left\{n > n_{k-1}\colon |X_n^{(r+1)}| > |X_{n_{k-1}}^{(r+1)}|\right\}.$$

Then for $n_{k-1} \leq n < n_k$ we have

$$\frac{|X_n^{(r+1)}|}{|X_n^{(1)}|} \leq \frac{|X_{n_{k-1}}^{(r+1)}|}{|X_{n_{k-1}}^{(1)}|},$$

so it is enough to consider the behavior of this ratio along the subsequence. Furthermore,

$$|X_{n_k}^{(r+1)}| = |{}^{(r)}S_{n_k} - {}^{(r)}S_{n_k - 1}|$$

and then (7) follows from (6).

(7) $\Rightarrow$ (8): Trivial.

(6) $\Rightarrow$ (9): We have for large n

$$|S_n| \leq |X_n^{(1)}| + \cdots + |X_n^{(r)}| + |{}^{(r)}S_n| \leq (r + \varepsilon)|X_n^{(1)}|.$$

(9) $\Rightarrow$ (10): Trivial.

(8) or (10) $\Rightarrow$ (4): With nonnegative summands there is an easy argument that (10) $\Rightarrow$ (8) since $(r+1)X_n^{(r+1)} \leq S_n$. However, we can see no connection between them in general without using the analytic conditions. Thus we will prove the contrapositive of both statements. We assume that $\sum_k (v_k(\varepsilon))^{r+1} = \infty$ for $\varepsilon < 1$ and will prove that

$$\limsup_{n\to\infty} \frac{X_n^{(r+1)}}{X_n^{(1)}} = 1, \qquad \liminf_{n\to\infty} \frac{|X_n^{(1)}|}{|S_n|} \leq \frac{1}{r+1} \quad \text{a.s.} \tag{2.3}$$

For now we also assume that G is slowly varying. The remaining case will be handled at the end of the proof. We fix ε and define

$$\xi_k = \begin{cases} 1, & \text{if } P\{\varepsilon^{-k} < X \le \varepsilon^{-k-1}\} \ge \frac{1}{2}P\{\varepsilon^{-k} < |X| \le \varepsilon^{-k-1}\}, \\ -1, & \text{otherwise,} \end{cases}$$

and for given $\delta > 0$ let

$$C_{n,k} = A_{n,k} \cap B_{n,k}, \qquad B_{n,k} = \{|^{(r+1)}S_n| \le \delta\varepsilon^{-k}\},$$

$$A_{n,k} = \{\varepsilon^{-k} < \xi_k X_n^{(j)} \le \varepsilon^{-k-1},\ j \le r+1;\ |X_n^{(r+2)}| \le \varepsilon^{-k};\ |X_{n+1}| > \varepsilon^{-k-1}\}.$$

Since G is slowly varying, $M(x)/G(x) \to 0$ as $x \to \infty$ (see Lemma 2.5 of [5]). Thus we find k_0 such that

$$(2.4) \qquad M(\varepsilon^{-k}) \le \frac{\delta}{3}G(\varepsilon^{-k}) \quad \text{and} \quad G(\varepsilon^{-k}) \le \frac{1}{3r} \quad \text{for all } k \ge k_0.$$

Now we define

$$I = \{(n,k): k \ge k_0,\ nG(\varepsilon^{-k}) \le 1\}.$$

Note that

$$P(A_{n,k}) = \binom{n}{r+1}\left(P\{\varepsilon^{-k} < \xi_k X \le \varepsilon^{-k-1}\}\right)^{r+1}\left(1 - G(\varepsilon^{-k})\right)^{n-r-1}G(\varepsilon^{-k-1})$$

and

$$P(A_{n,k}B_{n,k}^c) = \binom{n}{r+1}\left(P\{\varepsilon^{-k} < \xi_k X \le \varepsilon^{-k-1}\}\right)^{r+1}G(\varepsilon^{-k-1})P(D_{n,k}),$$

where

$$D_{n,k} = \{|X_j| \le \varepsilon^{-k},\ j \le n-r-1,\ |S_{n-r-1}| > \delta\varepsilon^{-k}\}.$$

By Markov's inequality, for $(n,k) \in I$ using (2.4),

$$\begin{aligned} P(D_{n,k}) &\le \delta^{-1}\varepsilon^k \int_{\{|X_j| \le \varepsilon^{-k},\ j \le n-r-1\}} |S_{n-r-1}|\,dP \\ &\le \delta^{-1}\varepsilon^k n\left(1 - G(\varepsilon^{-k})\right)^{n-r-2}\int_{|X| \le \varepsilon^{-k}} |X|\,dP \\ &= \delta^{-1} n\left(1 - G(\varepsilon^{-k})\right)^{n-r-2} M(\varepsilon^{-k}) \le \left(1 - G(\varepsilon^{-k})\right)^{n-r-2}\tfrac{1}{3}nG(\varepsilon^{-k}) \\ &\le \tfrac{1}{2}\left(1 - G(\varepsilon^{-k})\right)^{n-r-1}. \end{aligned}$$

Thus we have

$$\tfrac{1}{2}P(A_{n,k}) \le P(C_{n,k}) \le P(A_{n,k}) \quad \text{for all } (n,k) \in I.$$

Letting

$$I_1 = \{(n,k): k \ge k_0,\ \tfrac{1}{2} \le nG(\varepsilon^{-k}) \le 1\},$$

we have

$$\sum_{(n,k)\in I} P(C_{n,k}) \geq \tfrac{1}{2} \sum_{(n,k)\in I_1} \binom{n}{r+1} \left(P\{\varepsilon^{-k} < \xi_k X \leq \varepsilon^{-k-1}\}\right)^{r+1}$$

$$\times \left(1 - G(\varepsilon^{-k})\right)^{n-r-1} G(\varepsilon^{-k-1})$$

$$\geq c \sum_{(n,k)\in I_1} n^{r+1} \left(G(\varepsilon^{-k}) - G(\varepsilon^{-k-1})\right)^{r+1} G(\varepsilon^{-k-1})$$

$$\geq c_1 \sum_{k \geq k_0} \left(G(\varepsilon^{-k})\right)^{-r-2} \left(G(\varepsilon^{-k}) - G(\varepsilon^{-k-1})\right)^{r+1} G(\varepsilon^{-k-1})$$

$$\geq c_2 \sum_{k \geq k_0} \left(v_k(\varepsilon)\right)^{r+1} = \infty.$$

Next we will obtain a bound for $P(C_{m,j}C_{n,k})$ which is good enough to allow us to use a generalized Borel–Cantelli lemma [see page 317 of Spitzer (1976)]. Since the $C_{n,k}$ are disjoint for fixed n, we consider $m < n$. Let $Z^{(k)}$ be the kth largest in modulus among $X_{m+2}, \dots, X_n$. Then for $(m, j), (n, k) \in I$

$$P\left\{C_{m,j}C_{n,k};\, X_{m+1} \notin \left\{X_n^{(1)}, \dots, X_n^{(r+1)}\right\}\right\}$$

$$\leq P\left\{A_{m,j};\, \varepsilon^{-k} < \xi_k Z^{(i)} \leq \varepsilon^{-k-1}, i \leq r+1;\, |Z^{(r+2)}| \leq \varepsilon^{-k};\, |X_{n+1}| > \varepsilon^{-k-1}\right\}$$

$$= P\left(A_{m,j}\right) P\left(A_{n-m-1,k}\right)$$

$$\leq 4P\left(C_{m,j}\right) P\left(C_{n-m-1,k}\right).$$

Also we have

$$\sum_{\{(m,j)\in I:\, m<n\}} P\left\{C_{m,j}C_{n,k};\, X_{m+1} \in \left\{X_n^{(1)}, \dots, X_n^{(r+1)}\right\}\right\}$$

$$= \sum_{\{(m,j)\in I:\, m<n\}} \sum_{l=1}^{r+1} P\left\{C_{m,j}C_{n,k};\, X_{m+1} = X_n^{(l)}\right\}$$

$$= \sum_{l=1}^{r+1} \sum_{\{(m,j)\in I:\, m<n\}} P\left\{C_{m,j}C_{n,k};\, X_{m+1} = X_n^{(l)}\right\}$$

$$\leq (r+1) P\left(C_{n,k}\right)$$

since with n and l fixed the events $C_{m,j}\{X_{m+1} = X_n^{(l)}\}$ are disjoint as j and m vary. These two bounds take care of the supplementary Borel–Cantelli condition. Thus we have

$$P\left(C_{n,k} \text{ i.o.}, (n,k) \in I\right) > 0.$$

Since the $C_{n,k}$ are disjoint for fixed n, this means infinitely many n will be involved. Thus

$$\{C_{n,k} \text{ i.o.}\} \subset \left\{\frac{X_n^{(j)}}{X_n^{(1)}} > \varepsilon,\, j \leq r+1;\, |{}^{(r+1)}S_n| < \delta |X_n^{(1)}| \text{ i.o.}\right\}$$

and for those n for which the event on the right occurs,

$$|S_n| \geq |X_n^{(1)} + \cdots + X_n^{(r+1)}| - |{}^{(r+1)}S_n| > [(r+1)\varepsilon - \delta]|X_n^{(1)}|.$$

By taking ε near 1 and δ near 0 we see that with positive probability

$$\limsup_{n\to\infty} \frac{X_n^{(r+1)}}{X_n^{(1)}} \geq 1 - \eta, \qquad \liminf_{n\to\infty} \frac{|X_n^{(1)}|}{|S_n|} \leq \frac{1+\eta}{r+1}.$$

By Hewitt–Savage, this probability is 1. Since η is arbitrary this proves (2.3). We must still deal with the case when G is not slowly varying. Fix $\varepsilon \in (0,1)$. Then there is a sequence $x_k \to \infty$ and $C > 1$ such that $G(\varepsilon x_k) > CG(x_k)$. Let $n_k = [(G(\varepsilon x_k))^{-1}]$. Then very much as above, we have with ξ_k now defined in terms of the interval $(\varepsilon x_k, x_k]$,

$$\begin{aligned} &P\left\{\varepsilon x_k < \xi_k X_{n_k}^{(j)} \leq x_k,\, j \leq r+1\right\} \\ &\quad\geq \binom{n_k}{r+1} 2^{-r-1}\left(G(\varepsilon x_k) - G(x_k)\right)^{r+1}\left(1 - G(\varepsilon x_k)\right)^{n_k - r - 1} \\ &\quad\geq c\left(n_k(1 - C^{-1})G(\varepsilon x_k)\right)^{r+1} \sim c_1 > 0. \end{aligned}$$

Thus $X_n^{(r+1)}/X_n^{(1)} > \varepsilon$ i.o. with probability 1 by using Hewitt–Savage, so the first statement in (2.3) is valid. The other statement in (2.3) is clear if

$$P\left\{|S_{n_k}| > (r+1)\varepsilon x_k, |X_{n_k}^{(1)}| \leq \varepsilon x_k\right\} \geq (2e)^{-1} \quad \text{i.o.}$$

by a similar argument, so we assume this probability is $< (2e)^{-1}$. Then, since

$$P\left\{|X_{n_k}^{(1)}| \leq \varepsilon x_k\right\} = \left(1 - G(\varepsilon x_k)\right)^{n_k} \sim e^{-1},$$

we have for large k

$$P\left\{|S_{n_k}| \leq (r+1)\varepsilon x_k, |X_{n_k}^{(1)}| \leq \varepsilon x_k\right\} \geq (3e)^{-1}.$$

Letting $m_k = n_k + 2r + 3$,

$$\begin{aligned} &P\left\{\varepsilon x_k < \xi_k X_{m_k}^{(j)} \leq x_k,\, j \leq 2r+3;\, |X_{m_k}^{(2r+4)}| \leq \varepsilon x_k,\, |{}^{(2r+3)}S_{m_k}| \leq (r+1)\varepsilon x_k\right\} \\ &\quad\geq \binom{m_k}{2r+3}\left(\tfrac{1}{2}\left(G(\varepsilon x_k) - G(x_k)\right)\right)^{2r+3} P\left\{|X_{n_k}^{(1)}| \leq \varepsilon x_k, |S_{n_k}| \leq (r+1)\varepsilon x_k\right\} \\ &\quad\geq c. \end{aligned}$$

Whenever this event occurs, we have

$$\begin{aligned} |S_{m_k}| &\geq |X_{m_k}^{(1)} + \cdots + X_{m_k}^{(2r+3)}| - |{}^{(2r+3)}S_{m_k}| \\ &\geq (2r+3)\varepsilon x_k - (r+1)\varepsilon x_k = (r+2)\varepsilon x_k \geq (r+2)\varepsilon |X_{m_k}^{(1)}|. \end{aligned}$$

Again using Hewitt–Savage, this will occur infinitely often with probability 1 and so if $\varepsilon > (r+1)/(r+2)$ we have completed the proof of (2.3). □

The main result is now an immediate consequence of Theorem 1.

THEOREM 2. *The series $\sum u_k^2$ converges iff*

$$\frac{X_n^{(1)}}{S_n} \to 1 \quad a.s. \tag{2.5}$$

If r is an integer greater than 1, *then*

$$\sum_k u_k^r = \infty, \qquad \sum_k u_k^{r+1} < \infty \tag{2.6}$$

iff

$$\liminf_{n\to\infty} \frac{|X_n^{(1)}|}{|S_n|} = \frac{1}{r} \quad a.s. \tag{2.7}$$

If $\sum_k u_k^r = \infty$ for all r then the lim inf *is* 0.

REMARK 3. The series $\sum u_k$ always diverges.

PROOF. The first statement is because (2.5) is equivalent to ${}^{(1)}S_n/X_n^{(1)} \to 0$ a.s. The second condition in (2.6) implies that the lim inf in (2.7) is at least r^{-1} by (9) while the first condition implies that it is at most r^{-1} since (10) must fail. The final statement follows since (10) fails for all r.

There is an analogue of (2.5) for general r with nonnegative summands that is worth pointing out. However, unlike (2.5), it is not valid with general summands. If, for example, X has mass $k^{\alpha-1}\exp\{-k^\alpha\}$ at $\pm 2^k$ for large k with the remaining mass at 0 where $\frac{1}{2} \le \alpha < \frac{2}{3}$, then $\sum u_k^3 < \infty$ as in the examples below, but $X_n^{(1)} = -X_n^{(2)}$ i.o. as in the proof of (2.3) so that (2.8) fails. For a continuous example, spread the mass at 2^k uniformly over $[2^k, 2^k + 1]$, etc. Then (2.8) still fails since $\{S_n\}$ is transient. The converse is true (by Theorem 1) without assuming $X \ge 0$. □

THEOREM 3. *Suppose $X \ge 0$ a.s. Then $\sum_k u_k^{r+1} < \infty$ iff*

$$\frac{X_n^{(1)} + \cdots + X_n^{(r)}}{S_n} \to 1 \quad a.s. \tag{2.8}$$

PROOF. (2.8) is equivalent to

$${}^{(r)}S_n\left(X_n^{(1)} + \cdots + X_n^{(r)}\right)^{-1} \to 0 \quad \text{a.s.},$$

and with nonnegative summands,

$$X_n^{(1)} \le X_n^{(1)} + \cdots + X_n^{(r)} \le rX_n^{(1)}.$$

Now use Theorem 1. □

EXAMPLES. Let $G(x) = \exp\{-(\log x)^\alpha\}$, $x \ge 1$, where $0 < \alpha < 1$. By the mean value theorem, there is a $\xi \in (x, 2x)$ such that

$$G(x) - G(2x) = -xG'(\xi) = \alpha(\log \xi)^{\alpha-1}\xi^{-1}xG(\xi) \le \alpha(\log x)^{\alpha-1}G(x)$$

so that G is slowly varying. Then a similar lower bound shows that

$$\frac{G(x) - G(2x)}{G(x)} \approx (\log x)^{\alpha-1}.$$

Thus $u_k \approx k^{\alpha-1}$ and so by Theorem 2,

$$\frac{X_n^{(1)}}{S_n} \to 1 \quad \text{a.s.} \quad \text{iff} \quad 0 < \alpha < \tfrac{1}{2};$$

this was already observed by Maller and Resnick (1984). But we can also say that for $r \geq 2$,

$$\liminf_{n\to\infty} \frac{X_n^{(1)}}{S_n} = \frac{1}{r} \quad \text{iff} \quad \frac{r-1}{r} \leq \alpha < \frac{r}{r+1}.$$

REFERENCES

[1] Arov, D. Z. and Bobrov, A. A. (1960). The extreme terms of a sample and their role in the sum of independent variables. *Theory Probab. Appl.* **5** 377–396.

[2] Darling, D. A. (1952). The influence of the maximum term in the addition of independent random variables. *Trans. Amer. Math. Soc.* **73** 95–107.

[3] Feller, W. (1946). A limit theorem for random variables with infinite moments. *Amer. J. Math.* **68** 257–262.

[4] Maller, R. A. and Resnick, S. I. (1984). Limiting behaviour of sums and the term of maximum modulus. *Proc. London Math. Soc.* **49** 385–422.

[5] Pruitt, W. E. (1981). General one-sided laws of the iterated logarithm. *Ann. Probab.* **9** 1–48.

[6] Spitzer, F. (1976). *Principles of Random Walk*, 2nd ed. Springer, New York.

School of Mathematics
University of Minnesota
206 Church Street S.E.
Minneapolis, Minnesota 55455