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A CONDITIONAL LIMIT THEOREM FOR THE FRONTIER OF A BRANCHING BROWNIAN MOTION

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We prove a weak limit theorem which relates the large time behavior of the maximum of a branching Brownian motion to the limiting value of a certain associated martingale. This exhibits the minimal velocity travelling wave for the KPP–Fisher equation as a translation mixture of extreme-value distributions. We also show that every particle in a branching Brownian motion has a descendant at the frontier at some time. A final section states several conjectures concerning a hypothesized stationary "standing wave of particles" process and the relationship of this process to branching Brownian motion.

1. Introduction. *Branching Brownian motion* is the stochastic process which evolves as follows. Starting at time $t = 0$ and position $x \in \mathbb{R}$, a particle moves according to the law of a standard Brownian motion until a random time T independent of the motion, with a unit exponential distribution $P(T > t) = e^{-t}$. At this time the particle splits into two identical particles, which then begin independent branching Brownian motions emanating from the point of fission. Thus, at any time $t \geq 0$ the "state" of the process is completely described by the positions $X_1^x(t), \dots, X_{N(t)}^x$ of the particles in existence at time t, arranged in order from smallest to largest; the random process $N(t)$, $t \geq 0$, is the "Yule" or "binary fission" process [cf., for example, Athreya and Ney (1972)]. Observe that the processes $(X_i^x(t))_{1 \leq i \leq N(t)}$, $x \in \mathbb{R}$, may be constructed from a single process $(X_i(t))_{1 \leq i \leq N(t)}$ commencing at the origin by

$$X_i^x(t) = X_i(t) + x, \qquad i = 1, 2, \dots, N(t). \tag{1}$$

Interest in the branching Brownian motion has recently centered on its connection with the so-called KPP–Fisher [for Kolmogorov, Petrovsky and Piscounov (1937) and Fisher (1937)] equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u. \tag{2}$$

In particular, if $M(t) = \max_{1 \leq i \leq N(t)} X_i(t)$ and $u(t, x) = P\{M(t) \leq x\}$, then u is the (unique) solution to the KPP equation for Heaviside initial data $u(0, x) = 1\{x \geq 0\}$ [cf. McKean (1975)]. Now the main result of Kolmogorov, Petrovsky and Piscounov has it that this solution settles down to a "travelling wave" with

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velocity $\sqrt{2}$ for large t; thus,

$$\lim_{t\to\infty} P\{M(t) - m(t) \le x\} = w(x), \qquad \forall\, x \in \mathbb{R}, \tag{3}$$

where

$$\tfrac{1}{2}w'' + \sqrt{2}\,w' + w^2 - w = 0 \tag{4}$$

and

$$m(t) = \operatorname{med} M(t) \sim \sqrt{2}\,t. \tag{5}$$

By exploiting the connection between the branching Brownian motion and the KPP equation, Bramson (1978) showed that the centering term $m(t)$ satisfies $m(t) = \sqrt{2}\,t - (3/2\sqrt{2})\log t + O(1)$, and by using somewhat different techniques based on the Feynman–Kac formula, Bramson (1983) improved this by showing that

$$m(t) = \sqrt{2}\,t - (3/2\sqrt{2})\log(t) + \text{constant} + o(1). \tag{6}$$

Our interest in the branching Brownian motion began with the realization that despite the weak convergence (3) the corresponding pointwise ergodic theorem cannot possibly hold for all $x \in \mathbb{R}$, i.e.,

$$t^{-1}\int_0^t 1\{M(s) - m(s) \le x\}\,ds \nrightarrow w(x) \quad \text{a.s.} \tag{7}$$

The argument for this is very simple. Suppose that the convergence indicated in (7) *did* hold for all x. Then by (1) it would follow that

$$t^{-1}\int_0^t 1\{M^x(s) - m(s) \le x\}\,ds \to w(0) \quad \text{a.s.,}$$

where $M^x(s) = \max_{1\le i\le N(s)} X_i^x(s)$. On the other hand, if *independent* branching Brownian motions were started at 0 and x, respectively, then with positive probability the two particles would meet before either fissioned, and therefore a successful "coupling" could be achieved with positive probability. This would imply that $\lim t^{-1}\int_0^t 1\{M(s) - m(s) \le x\}\,ds$ is a constant independent of x, contradicting the fact that $w(0) \ne w(x)$ for some $x \ne 0$.

Our interest was further piqued by a representation for the travelling wave $w(x)$ given by McKean (1975), according to which

$$w(x) = E\exp\{-Ye^{-\sqrt{2}x}\}, \qquad x \in \mathbb{R},$$

where Y is the almost sure limit of the positive martingale $Y(t) = \sum_{i=1}^{N(t)}\exp\{\sqrt{2}\,X_i(t) - 2t\}$. McKean's statement is actually false [his argument fails because it depends on his proof of (3′) from (2′), cf. McKean (1976)]; in fact, we will show in Section 2 that $Y = 0$ a.s. [cf. (24)]. But the "representation" is nevertheless interesting because, if it were true, it would exhibit $w(x)$ as a mixture of translates of the extreme value distribution $\exp\{-e^{-\sqrt{2}x}\}$. This would suggest that in a typical "sample path" a "tidal wave" of particles eventually builds up, with a random "delay" of $(\log Y)/\sqrt{2}$, after which the behavior of the maximum $M(t)$ is more or less determined by the extreme-value mechanism.

The purpose of this note is to salvage this appealing intuitive picture of the sample path behavior of the branching Brownian motion. Our main result is

THEOREM 1. *Let* $Z(t) = \sum_{i=1}^{N(t)}(\sqrt{2}\,t - X_i(t))e^{\sqrt{2}X_i(t)-2t}$. *Then*

$$Z = \lim_{t\to\infty} Z(t) \tag{8}$$

exists, is finite and is positive with probability 1. *Let* $\mathscr{F}_s = \sigma\{(X_i(t))_{1\le i\le N(t)}, 0 \le t \le s\}$. *There is a constant* $C > 0$ *such that for each* $x \in \mathbb{R}$,

$$\lim_{s\to\infty}\lim_{t\to\infty} P\{M(t+s) \le m(t+s) + x|\mathscr{F}_s\} = \exp\{-CZe^{-\sqrt{2}x}\} \quad a.s., \tag{9}$$

for suitable versions of the conditional probabilities. Consequently, the travelling wave $w(x)$ *has the representation*

$$w(x) = E\exp\{-CZe^{-\sqrt{2}x}\}. \tag{10}$$

It will turn out that the constant C in (9) and (10) is determined by the tail behavior of $w(x)$: In particular, as $x \to \infty$,

$$1 - w(x) \sim Cxe^{-\sqrt{2}x}. \tag{11}$$

The pointwise ergodic behavior of $M(t) - m(t)$ is apparently much more difficult to determine. We conjecture that

$$t^{-1}\int_0^t 1\{M(s) - m(s) \le x\}\,ds \to \exp\{-CZe^{-\sqrt{2}x}\}, \tag{12}$$

almost surely for all $x \in \mathbb{R}$, but have had no success in proving this. (See Section 4 for some background for this conjecture.)

In Section 3 we will prove

THEOREM 2. *Suppose two independent branching Brownian motions* $(X_1^A(t),\dots,X_{N^A(t)}^A)$ *and* $(X_1^B(t),\dots,X_{N^B(t)}^B)$ *are started at* 0 *and* x, *respectively, where* $x < 0$. *Then with probability* 1 *there exist random times* $t_n \uparrow +\infty$ *such that*

$$M^A(t_n) < M^B(t_n) \tag{13}$$

for all n, *where*

$$M^A(t) = \max_{1\le i\le N^A(t)} X_i^A(t)$$

and

$$M^B(t) = \max_{1\le i\le N^B(t)} X_i^B(t).$$

COROLLARY. *Every particle born in a branching Brownian motion has a descendant particle in the "lead" at some future time.*

2. Proof of Theorem 1. Let $X_i^x(t)$, $i = 1,2,\dots,N(t)$, denote the positions of the particles in existence at time $t \ge 0$ in a branching Brownian motion started at x. Let $f(y)$ be a function of $y \in \mathbb{R}$ satisfying $0 \le f \le 1$, and let

$$u^*(t,x) = E\prod_{i=1}^{N(t)} f(X_i^x(t)). \tag{14}$$

Then u^* is the unique solution of the KPP equation (2) with initial data $u^*(0,x) = f(x)$. [That (14) solves (2) is easily proved by conditioning on the time of the first fission, cf. McKean (1975). Uniqueness follows from the maximum principle for parabolic equations, cf. Proposition 2.1 of Aronson and Weinberger (1975) for the argument.] Similarly,

$$\hat{u}(t,x) = E\prod_{i=1}^{N(t)} f\big(\sqrt{2}\,t - X_i^{-x}(t)\big) \tag{15}$$

is the unique solution of the modified KPP equation

$$\frac{\partial \hat{u}}{\partial t} = \frac{1}{2}\frac{\partial^2 \hat{u}}{\partial x^2} + \sqrt{2}\frac{\partial \hat{u}}{\partial x} - \hat{u}(1-\hat{u}), \tag{16}$$

for the initial data $\hat{u}(0,x) = f(x)$.

Notice that if $w(x)$ is the increasing solution to (4) with $\lim_{x\to\infty} w(x) = 1$ and $\lim_{x\to-\infty} w(x) = 0$, then the function

$$\hat{u}(t,x) = w(x)$$

is a solution to (16), hence by (15) and uniqueness of solutions

$$w(x) = E\prod_{i=1}^{N(t)} w\big(\sqrt{2}\,t - X_i^{-x}(t)\big), \tag{17}$$

for all $x \in \mathbb{R}$ and $t \ge 0$. Now the branching Brownian motion is Markovian, and conditional on everything that has happened up to time $t \ge 0$ the "future" of the process is the same as if independent branching Brownian motions were started at time t at each of the positions $X_1^{-x}(t), X_2^{-x}(t), \dots, X_{N(t)}^{-x}$. Thus, in view of (17) we have

PROPOSITION 1. *The process* $W^x(t) \triangleq \prod_{i=1}^{N(t)} w(\sqrt{2}\,t - X_i^{-x}(t))$ *is a martingale with respect to the filtration* $\mathscr{F}_t \triangleq \sigma$ (*everything that has happened up to time* t).

Since $W(t)$ is nonnegative and bounded,

$$W^x \triangleq \lim_{t\to\infty} W^x(t) \tag{18}$$

exists almost surely, with $0 \le W^x \le 1$ almost surely and

$$EW^x = w(x). \tag{19}$$

Now consider the positive martingale $\sum_{i=1}^{N(t)} e^{\sqrt{2}X_i(t)-2t}$. This converges almost surely to a finite nonnegative limit. It therefore must be the case that

$$\min_{1\le i\le N(t)} \big(\sqrt{2}\,t - X_i(t)\big) \to +\infty \quad \text{a.s.}, \tag{20}$$

because otherwise the value of $\sum_{i=1}^{N(t)} e^{\sqrt{2}X_i(t)-2t}$ would fluctuate for arbitrarily large t, contradicting the martingale convergence theorem. In view of (1) it follows from (20) that for all $x \in \mathbb{R}$,

$$\min_{1 \le i \le N(t)} \left(\sqrt{2}\,t - X_i^{-x}(t)\right) \to +\infty \quad \text{a.s.} \tag{21}$$

The convergence (21) indicates that the large time behavior of the martingale $W^x(t)$ is intimately related to the tail behavior of the distribution function $w(y)$. It is known that as $y \to \infty$

$$1 - w(y) \sim Cye^{-\sqrt{2}y} \tag{22}$$

[cf. Bramson (1983), equation 1.13. *Caution*: McKean (1975) claims that $1 - w(y) \sim Ce^{-\sqrt{2}y}$, but this is false; cf. McKean (1976).] Now it follows easily from (21) and (22) that as $t \to \infty$

$$\begin{aligned} \log W^x(t) &= \sum_{i=1}^{N(t)} \log w\left(\sqrt{2}\,t - X_i^{-x}(t)\right) \\ &\sim \sum_{i=1}^{N(t)} -C\left(\sqrt{2}\,t - X_i(t) + x\right)\exp\left\{\sqrt{2}\,X_i(t) - 2t - \sqrt{2}\,x\right\} \\ &\sim -CZ(t)e^{-\sqrt{2}x} - CY(t)xe^{-\sqrt{2}x}, \end{aligned} \tag{23}$$

where

$$Y(t) = \sum_{i=1}^{N(t)} e^{\sqrt{2}X_i(t) - 2t}$$

and

$$Z(t) = \sum_{i=1}^{N(t)} \left(\sqrt{2}\,t - X_i(t)\right)e^{\sqrt{2}X_i(t) - 2t}.$$

It is clear that $Y(t)/Z(t) \to 0$ a.s. as $t \to \infty$ by (20). Since $Y(t)$ is a positive martingale $\lim Y(t) = Y \ge 0$ exists almost surely, and, consequently, $Z(t) \to +\infty$ a.s. on the event $\{Y > 0\}$. But by (23) and (18) this implies that $W^x = 0$ a.s. on $\{Y > 0\}$ for *all* $x \in \mathbb{R}$. Therefore

$$P\{Y > 0\} = 0, \tag{24}$$

because $EW^x = w(x) \to 1$ as $x \to \infty$, and $0 \le W^x \le 1$; moreover, for every $x \in \mathbb{R}$

$$\lim_{t \to \infty} Z(t) = \left(-e^{\sqrt{2}x}/C\right)\log W^x \quad \text{a.s.,} \tag{25}$$

by (23), (24) and (18). This and (19) prove (8) and (10).

McKean (1975) shows that

$$\lim_{t \to \infty} \left\{m(t+s) - m(t) - \sqrt{2}\,s\right\} = 0, \tag{26}$$

for each $s > 0$. [This is, of course, also an immediate consequence of the Bramson (1983) result quoted above in (6).] Now for any $s \ge 0$ the "future" of the branching Brownian motion beyond time s depends on the "past" $\mathscr{F}_s$ only through the values $X_1(s), X_2(s), \dots, X_{N(s)}(s)$, by the Markov property. Moreover, the particles alive at time s give rise to *independent* branching Brownian motions with initial positions $X_1(s), \dots, X_{N(s)}(s)$. Thus, for each $x \in \mathbb{R}$

$$P\{M(t+s) \le m(t+s) + x | \mathscr{F}_s\} = \prod_{i=1}^{N(s)} u(t, x + m(t+s) - X_i(s)), \tag{27}$$

where

$$u(t,x) \triangleq P\{M(t) \le x\} \tag{28}$$

is the solution to the KPP–Fisher equation (2) with Heaviside initial data. Combining (26), (27), (28) and (3) gives

$$\begin{aligned} &\lim_{t\to\infty} P\{M(t+s) \le m(t+s) + x | \mathscr{F}_s\} \\ &\quad = \prod_{i=1}^{N(s)} w\big(x + \sqrt{2}s - X_i(s)\big) \triangleq W^x(s). \end{aligned} \tag{29}$$

In view of (18) and (25) this proves (9). □

3. Proof of Theorem 2 and the corollary. To prove the corollary it suffices to show that if independent branching Brownian motions are started at a finite number of positions $x_1, x_2, \dots, x_n$, then with probability 1 there will be a time t at which a descendant of the particle started at x_1 is in the "lead." Let "A" and "B" denote independent branching Brownian motions $X_1^A(t), \dots, X_{N^A(t)}^A$ and $X_1^B(t), \dots, X_{N^B}^B(t))$ as in Theorem 2. Then there is a nonzero probability that at time $t = 1$, the "A" process will have given rise to $(n-1)$ or more particles, all of them to the right of $\max(x_2, \dots, x_n)$, while process "B" will consist of a single particle to the left of x_1. But Theorem 2 guarantees that a descendant of "B" will eventually take the lead. Consequently, it follows from the Markov property and the translation invariance of branching Brownian motion that if independent branching BM's are started at $x_1, x_2, \dots, x_n$, then at some future time a descendant of the particle started at x_1 will be in the lead. Thus, the corollary follows from Theorem 2.

Our original proof of Theorem 2 made use of Theorem 1, but the referee and Burgess Davis independently came up with the following clever argument, which uses only (3) and basic probability.

Let

$$\begin{aligned} \mathscr{G}_t = \sigma\big(&\text{everything that happens in process "}A\text{" by time } t + \tfrac{1}{2}, \\ &\text{and everything that happens in process "}B\text{" by time } t\big). \end{aligned}$$

Let T_k be the first time in the interval $[k-1, k-\frac{1}{2})$ at which

$$M^B(t) \ge m(t) - N_1 \tag{a}$$

and

$$M^A\big(t + \tfrac{1}{2}\big) \le m\big(t + \tfrac{1}{2}\big) + N_1\big(\le m(t) + N_1 + 1\big), \tag{b}$$

where N_1 is for the moment fixed. Let $T_k^* = T_k \wedge (k - \frac{1}{2})$. Note that $T_k^* < T_{k+1}^*$, and that T_k, $M^B(T_k^*)$ and $M^A(T_k^* + \frac{1}{2})$ are all $\mathscr{G}_{T_k^*}$-measurable.

Let $\varepsilon > 0$. We will first show that

$$P\{T_k < \infty \text{ infinitely often}\} > 1 - 2\varepsilon, \tag{30}$$

if N_1 is chosen large enough. By (3) and Fubini, N_1 can be chosen so that for large s,

$$P\left[\mu\left\{t \in \left[s - 1, s - \tfrac{1}{2}\right): M^B(t) \geq m(t) - N_1\right\} \leq \tfrac{4}{10}\right] < \varepsilon \tag{31}$$

and

$$P\left[\mu\left\{t \in \left[s - 1, s - \tfrac{1}{2}\right): M^A(t) \leq m(t) + N_1\right\} \leq \tfrac{4}{10}\right] < \varepsilon, \tag{32}$$

where μ is Lebesgue measure on $\mathbb{R}$. Consequently, for large k,

$$P\left[\mu\left\{t \in \left[k - 1, k - \tfrac{1}{2}\right): \text{(a) and (b) hold}\right\} > \tfrac{3}{10}\right] > 1 - 2\varepsilon, \tag{33}$$

and hence

$$P\{T_k < \infty\} > 1 - 2\varepsilon. \tag{34}$$

Since (34) holds for all large k,

$$\begin{aligned} P\{T_k < \infty \text{ infinitely often}\} &= P\left[\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{T_k < \infty\}\right] \\ &= \lim_{n \to \infty} P\left[\bigcup_{k=n}^{\infty} \{T_k < \infty\}\right] \\ &> 1 - 2\varepsilon. \end{aligned}$$

If $T_k < \infty$, then by (a) and (b)

$$M^B(T_k^*) \geq M^A\left(T_k^* + \tfrac{1}{2}\right) - (2N_1 + 1). \tag{35}$$

Thus, we will have $M^B(T_k^* + \frac{1}{2}) > M^A(T_k^* + \frac{1}{2})$ if $T_k < \infty$ and $M^B(T_k^* + \frac{1}{2}) - M^B(T_k^*) > (2N_1 + 1)$. But the conditional distribution of $M^B(T_k^* + \frac{1}{2}) - M^B(T_k^*)$, given $\mathscr{G}_{T_k^*}$, is clearly stochastically larger than the distribution of $X(\frac{1}{2})$, where $X(t)$ is standard Brownian motion. Thus, it follows that

$$1\{T_k < \infty\} P\left\{M^A\left(T_k^* + \tfrac{1}{2}\right) \geq M^B\left(T_k^* + \tfrac{1}{2}\right) | \mathscr{G}_{T_k^*}\right\} \leq P\left\{X\left(\tfrac{1}{2}\right) \leq 2N_1 + 1\right\}. \tag{36}$$

Let j be a positive integer. By (30) and (36), the probability that $M^A(T_k^* + \frac{1}{2}) \geq M^B(T_k^* + \frac{1}{2})$ holds for all of the first m finite T_k's, $k \geq j$, is less than

$$\left[P\left\{X\left(\tfrac{1}{2}\right) \leq 2N_1 + 1\right\}\right]^m + 2\varepsilon. \tag{37}$$

Letting $m \to \infty$ shows that

$$P\left\{M^A(t) \geq M^B(t), \text{for all } t \geq j - \tfrac{1}{2}\right\} \leq 2\varepsilon \tag{38}$$

and hence

$$P\left\{\exists\, t_n \uparrow \infty: M^A(t_n) < M^B(t_n)\right\} \geq 1 - 2\varepsilon. \tag{39}$$

Since $\varepsilon > 0$ was arbitrary, we are done. □

4. Conjectures. What does a branching Brownian motion look like in the vicinity of its frontier? In this final section we will elaborate on the "tidal wave" story mentioned in Section 1. This "tidal wave" story leads to the conjecture (12).

Start off with a Poisson point process with intensity $e^{-\lambda x}$ on $\mathbb{R}$, where λ is a positive constant. Then start independent branching Brownian motions at these points, but where the Brownian motions have drift $-\mu$, with μ another positive constant. Let $f(t, x)$ be the total expected particle density at time t and at position x, so that the expected number of particles in the measurable set $A \subset \mathbb{R}$ at time t is $\int_A f(t, x)\,dx$. Then, of course, $f(0, x) = e^{-\lambda x}$, and $f(t, x)$ evolves according to the PDE

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \mu\frac{\partial f}{\partial x} + f. \tag{40}$$

If we take $\mu = \mu(\lambda) = (\lambda^2/2 + 1)/\lambda$, then $\partial/\partial t f(t, x) = 0$, so that $f(t, x) = e^{-\lambda x}$ for all $t \geq 0$. Thus, the expected number of particles in a set remains constant over time. From now on, use $\mu = \mu(\lambda) = (\lambda^2/2 + 1)/\lambda$. Note that $\mu(\lambda)$ is minimized by $\lambda = \sqrt{2}$, and that $\mu(\sqrt{2}) = \sqrt{2}$.

Let $\tilde{M}_\lambda(t)$ equal the position at time t of the rightmost particle in the above process. What does this stochastic process look like? First of all,

$$\tilde{M}_\lambda(t) =_D \tilde{M}^\lambda(0) + a(t, \lambda),$$

i.e., the distribution of $\tilde{M}_\lambda(t)$ is a location shift of the distribution of $\tilde{M}_\lambda(0)$, which is easily seen to be the extreme-value distribution with c.d.f. $\exp(-e^{-\lambda x})$. Furthermore $a(t, \lambda) < 0$ for $t > 0$. The argument for all this is as follows. Each original particle has a rightmost descendant at time t. As in Section 1, let $u(t, x) = P\{M(t) \leq x\}$, where $M(t)$ is the maximum at time t of a standard (driftless) branching Brownian motion started at position 0. Let $\tilde{u}_\lambda(t, x) = u(t, x + \mu t)$. Then $\tilde{u}_\lambda(t, x)$ is the c.d.f. for the difference in position between a rightmost descendant at time t and the corresponding original particle. Since the original particles came from a Poisson point process with intensity $e^{-\lambda x}$ and since the branching Brownian motion starting at different points are independent, the point process of "all rightmost descendants at time t" is also a Poisson point process with intensity

$$\begin{aligned} g(t, x) &= \int_{-\infty}^{\infty} e^{-\lambda(x-y)}\tilde{u}_\lambda(t, dy) \\ &= e^{-\lambda x}\int_{-\infty}^{\infty} e^{\lambda y}\tilde{u}_\lambda(t, dy) \\ &= \exp\{-\lambda(x - a(t, \lambda))\}, \end{aligned} \tag{41}$$

where

$$a(t, \lambda) = \lambda^{-1}\log\left\{\int_{-\infty}^{\infty} e^{\lambda y}\tilde{u}_\lambda(t, dy)\right\}. \tag{42}$$

Since $\int_0^\infty f(t, x)\,dx > \int_0^\infty g(t, x)\,dx$ (the expected number of particles to the right of 0 at time t is greater than the expected number of rightmost descendants to the right of 0 at time t) we must have $a(t, \lambda) < 0$ for $t > 0$.

It is not hard to show that $\tilde{M}_\lambda(t) \to -\infty$ a.s. if $\lambda < \sqrt{2}$. Indeed, suppose that we start with a Poisson point process with intensity $e^{-\lambda x}$, $\lambda < \sqrt{2}$, and then start independent *standard* branching Brownian motions (with no drift) at each of the points. Let

(43)
$$\tilde{Y}(t) \triangleq \sum_{i=1}^{\infty} e^{\sqrt{2} X_i(t) - 2t},$$

where the summation is over all particles present at time t. Then $\tilde{Y}(0)$ is almost surely finite [although $E\tilde{Y}(0) = \infty$], and $\tilde{Y}(t)$ is a positive martingale if one conditions on the value of $\tilde{Y}(0)$. As in (20) of Section 2, it follows that

(44)
$$\min\{\sqrt{2}\, t - X_i(t)\} \to +\infty \quad \text{a.s.},$$

since otherwise $\tilde{Y}(t)$ would fluctuate for arbitrarily large t. Thus, $\tilde{M}_\lambda(t)$ goes to $-\infty$ at least as fast as $\{\sqrt{2} - \mu(\lambda)\}t$ if $\lambda < \sqrt{2}$. It follows that $a(t, \lambda) \to -\infty$ as $t \to \infty$ for $\lambda < \sqrt{2}$.

For $\lambda \geq \sqrt{2}$, we conjecture that $a(t, \lambda)$ converges (probably monotonically) down to a finite limit $a(\infty, \lambda)$ as $t \to \infty$. We further conjecture that the entire (point-process valued) stochastic process converges rapidly in total variation norm to a limiting stationary (point-process valued) stochastic process which is strongly mixing. It would follow that

(45)
$$t^{-1} \int_0^t 1\{\tilde{M}_\lambda(x) \leq x + a(\infty, \lambda)\}\, ds \to \exp\{e^{-\lambda x}\} \quad \text{a.s.}$$

This limiting stochastic process would be a sort of "standing wave" of particles.

A similar but much simpler "standing wave of particles" can be obtained if one eliminates the branching: If independent Brownian motions with drift $-\lambda/2$ are started at the points arising from a Poisson point process with intensity $e^{-\lambda x}$, then the distribution of the ensemble of particles at time t is still that of a Poisson point process with intensity $e^{-\lambda x}$.

Now, what does all this have to do with our original problem? We conjecture that a "wave of particles" builds up behind the frontier of a branching Brownian motion. If one subtracts off the "position" of this wave, then the resulting point-process valued stochastic process converges in distribution (in the proper sense) to the $\lambda = \sqrt{2}$ "standing wave" process. More specifically, if $X_1(t), \dots, X_{N(t)}$ are the positions at time t of the particles in a branching Brownian motion, look at the values of

$$X_i(t) - m(t) + a(\infty, \sqrt{2}) - 2^{-1/2} \log CZ(t)$$

which fall inside an interval $[b, \infty)$, $b \in \mathbb{R}$ fixed. This "point process on $[b, \infty)$" valued stochastic process should converge (in total variation norm, say) to the "point process on $[b, \infty)$" valued stochastic process obtained from the $\lambda = \sqrt{2}$ "standing wave" process, for every $b \in \mathbb{R}$. Our conjecture (12) in Section 1 is a consequence of this and (45).

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mentioned earlier, this more elementary proof of Theorem 2 was also discovered by Burgess Davis, who also pointed out an error in the original version of Theorem 1.

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