The Annals of Probability
1987, Vol. 15, No. 2, 824–830

A UNIFIED APPROACH TO A CLASS OF OPTIMAL SELECTION PROBLEMS WITH AN UNKNOWN NUMBER OF OPTIONS

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In the so-called *secretary problem*, if an unknown number, N, of options arrive at i.i.d. times with a known continuous distribution, then ignorance of how many options there are becomes almost irrelevant: The optimal rule for infinitely many options is shown to be minimax with respect to all possible distributions of N, nearly optimal whenever N is likely to be large, and formal Bayes against a noninformative prior. These results hold whatever the loss function.

1. Introduction. An unknown random number, N, of *options*, will arrive at times $Z_1, \dots, Z_N$, where $Z_1, Z_2, \dots$ are i.i.d. random variables with some known continuous distribution, F, on an interval $(0, T)$, possibly infinite. The options can be ranked from best (rank one) to worst, and Z_i is the arrival time of the ith best. At any time, $t \in (0, T)$, only the relative ranks of those options which have arrived so far can be observed. The object is to find a stopping rule, τ, based only on the observed relative ranks, which minimizes some risk function, $Eq(X_\tau)$, where X_τ is the rank of the option selected by τ, and $\{q(i)\colon i = 1, 2, \dots\}$ is a prescribed nondecreasing, nonnegative loss function.

Since the distribution of N is unknown, a more interesting object is to find *robust* rules. In Bruss (1984), the special case $q(1) = 0$, $q(2) = \cdots = 1$, usually called the *best choice problem*, was studied, and the rule: *stop with the first option of relative rank one after time* $F^{-1}(e^{-1})$, *if any*, was shown to have risk smaller than $1 - e^{-1}$ for all distributions of N. Since $1 - e^{-1}$ is the well-known limiting risk for fixed N, as $N \to \infty$, this rule is, therefore, minimax. Thus, in the *best choice problem*, at least, the existence and knowledge of an arrival time distribution fully compensates for the ignorance about the number of options. This contrasts with the situation in which the distribution of N is known, but arrivals occur at fixed times $1, 2, \dots, N$; for example, in Presman and Sonin (1972), if N is uniform on the integers 1 to n, the optimal risk tends to $1 - 2e^{-2}$ as $n \to \infty$. See also Petruccelli (1983) and Yasuda (1984).

In this paper, we shall show that the same phenomenon applies for quite general loss functions. For example, when $q(i) = i$ (so the goal is to minimize the expected rank of the selected option), and N is known, the optimal risk is asymptotically $\prod_{j=1}^{\infty}(1 + 2/j)^{1/(j+1)} \approx 3.8695$, from Chow et al. (1964). If, however, N is uniform on the integers 1 to n, the risk becomes asymptotically infinite; see Gianini-Pettitt (1979). Nevertheless, as we shall show, the existence

Received September 1984; revised September 1986.
AMS 1980 *subject classification*. 60G40.
Key words and phrases. Best choice problem, secretary problem, minimax strategy, Bayes strategy, noninformative prior.

and knowledge of an arrival time distribution leads to a *single* stopping rule with risk smaller then 3.8695, whatever the distribution of N.

The proposed model, which allows for the intervention of time, is based on the idea, as argued in Bruss (1984)—see also Samuels (1985)—that it is easier to predict when options will arrive, under the hypothesis that they *do* arrive, than to predict the distribution of their number.

To obtain our results, we embed the *unknown number of options problem* in the so-called *infinite secretary problem* of Rubin (1966) and Gianini and Samuels (1976), where an *infinite* number of options arrive at times which are i.i.d. uniform on $(0,1)$, and what is known at any time $t \in (0,1)$ is the sequence, $\mathbf{Z}(t) = (Z_1(t), Z_2(t), \dots)$, of arrival times of the best, second best, etc., options among those which arrive by time t. The embedding consists of two parts: First, without loss of generality, we can transform the known, continuous arrival-time distribution to uniform on $(0,1)$. [We note that Stewart (1981), in looking at formal Bayes rules for the *best choice problem*, took the arrival-time distribution to be exponential, with known parameter; this is equivalent.] Second, we introduce N, independent of the arrival process, and the corresponding *censoring* of the observables, $\mathbf{Z}(t)$, namely $\mathbf{Z}_N(t) = (Z_1(t), \dots, Z_{N(t)}(t))$, where

$$\begin{aligned} N(t) &= \max\{n \le N \colon Z_n \le t\} \\ &= 0, \quad \text{if } Z_i > t, \forall\, i \le N. \end{aligned}$$

The class of allowable stopping rules, τ, must be adapted to the censored observables, while maintaining, as an essential feature, the possibility of not stopping (for convenience, we set $\tau = 1$ on the complement of the event $\{\tau < 1\}$). How, then, shall we define the loss for not stopping? In the *infinite* problem, it was defined to be $q(\infty) \equiv \sup_i q(i)$; but here a logical choice is the *random variable*, $q(N)$.

In Section 2, we show how the embedding yields quite general results very easily and in Section 3, we show how these results unify and generalize various special cases considered elsewhere.

2. Main results. In the *infinite secretary problem*, the optimal rules are always of the form:

> Choose a sequence of *cutoff points* $0 < t_1 \le t_2 \le \cdots \le 1$; then stop at the first time in $[t_k, t_{k+1})$ at which there is an arrival of relative rank $\le k$, if there is such a time, and if we have not already stopped before time t_k.

In Gianini and Samuels (1976), such rules were called *cutoff-point rules*. These rules are perfectly legitimate for the censored problem; the effect of censoring is simply this:

censoring delays stopping.

Specifically, let $\tau^{(\infty)}$ be any cutoff-point rule used in the infinite problem, and $\tau^{(N)}$ have the same cutoff-points, but be used when there are N options. Then $\tau^{(N)} \ge \tau^{(\infty)}$; and, moreover, strict inequality holds if and only if the option selected by $\tau^{(\infty)}$ has rank greater than N. On this event, either $\tau^{(N)}$ stops later or it does not stop at all; in either case the loss is at most $q(N)$. Thus we have proved

THEOREM 2.1. *For any increasing loss function, $q(\cdot)$, if the optimal risk in the infinite secretary problem is finite, then the optimal in infinite-secretary-problem-rule (or any other cutoff-point rule), when applied in the unknown N problem, yields a smaller risk, for every possible distribution of N.*

Thus the effect of censoring with respect to risks can be summarized as:

censoring reduces risks.

The risk is *strictly* smaller unless $q(\cdot)$ is constant, by the following argument: When $q(1) < q(\infty)$, the (finite) optimal risk in the infinite problem is strictly less than $q(\infty)$, which guarantees that the first cutoff point, t_1, is strictly less than 1, and also insures the existence of an m for which $q(m) > q(1)$ and $P(N < m) > 0$. Then, for any $s \in (t_1, 1)$, there is a positive probability that the best $m > N$ options will arrive after s, with the very best of these m arriving first, and the option of rank $m + 1$ will arrive in (t_1, s). On this event, the difference between the losses is at least $q(m) - q(1)$.

Conditions for finiteness of the optimal risk are given in Gianini and Samuels (1976); in particular, the risk is finite whenever $q(\cdot)$ grows no faster than some polynomial. See also Mucci (1973a, 1973b), and Frank and Samuels (1980). In addition, it was shown that the optimal *infinite secretary problem* risk, say $v \equiv v(q(\cdot))$, is always the limit of the optimal risks, v_n, for fixed $N = n$, which are, in fact, nondecreasing in n. Now, the optimal rule for some random N cannot do better, on each of the events $\{N = n\}$, than the optimal fixed-size rule; hence its risk, say $v^{(N)}$, must satisfy

$$\sum_{n=1}^{\infty} v_n P(N = n) \le v^{(N)} \le v. \tag{1}$$

Since $v_n \uparrow v$ as $n \to \infty$, we have

$$N \ge n \text{ a.s.} \;\Rightarrow\; v_n \le v^{(N)} \le v \tag{2}$$

and

$$N \uparrow \infty \text{ in distribution} \;\Rightarrow\; v^{(N)} \uparrow v. \tag{3}$$

To summarize:

THEOREM 2.2. *For any increasing loss function, $q(\cdot)$, if the optimal risk in the infinite secretary problem is finite, then the optimal infinite-secretary-problem-rule is nearly optimal for unknown N, whenever N is likely to be large. Specifically,* (1), (2) *and* (3) *all hold.*

2.1. *A partial ordering.* The preceding theorems tell us almost all we need to know about (*nearly*)-optimal selection with unknown N, without requiring us to know *anything* about optimal stopping rules for random N. However, Theorem 2.1 has an extension, the proof of which does require some knowledge of the structure of such optimal rules.

Consider the usual stochastic ordering of probability distributions (labeled by random variables), namely, $M \prec N$ if $\forall\, n,\ P(M > n) \le P(N > n)$. Then:

THEOREM 2.3.

$$M \prec N \;\Rightarrow\; v^{(M)} \le v^{(N)} \le v.$$

Intuitively, this theorem should be true because we can model N as $M + \Delta$—where Δ is nonnegative, so $\tau^{(M)} \ge \tau^{(N)}$—and then apply an argument analogous to the one used to prove Theorem 2.1. But there's a complication because an optimal rule for N may not be of *cutoff* type. Undoubtedly there are certain unpleasant distributions of N for which the phenomenon of *islands*, which Presman and Sonin (1972) observed in the fixed-arrival-time problem, occurs here as well; namely that a relative-rank j arrival is acceptable at an earlier time, but not at a later time (because later N is much likelier to be larger than it was earlier).

However, the argument to prove Theorem 2.1 does not require *cutoff-point rules*. All we need is that an optimal rule would never be willing to accept an option of relative rank j, but, at the same time, be unwilling to accept one of *smaller* relative rank. This much is indeed true, and not hard to see. The proof depends on essentially two facts: First, the monotonicity of $q(\cdot)$ insures that the stopping risk is increasing in the relative rank, with probability one; and, second, the risk for not stopping never depends on the relative rank of the current option. This is a consequence of the arrival times being i.i.d. uniform.

We should remark that the partial ordering result embodied in Theorem 2.3 may fail to hold in the fixed-arrival-time model. For example, as we remarked in the Introduction, Gianini-Pettitt (1979) showed that it fails in the *ranks problem*, $q(i) = i$, when M is uniform on the integers 1 to n, and $N \equiv n$, for all n sufficiently large.

Note also that *strict* inequality may fail to hold in Theorem 2.3; for example, in the best choice problem when $N = M + 1$ is constant.

3. Bayes rules. Suppose we know, a priori, the distribution of N. Then a key step in finding an optimal stopping rule and its risk, for a specified loss function, $q(\cdot)$, is to compute the posterior distribution, at time t, of the actual rank of an option which has relative rank, say j, and is one of, say k, arrivals by time t. Clearly this depends only on t, j and k, and not on the order of arrivals or the arrival times. An elementary application of *Bayes' rule* gives

$$\begin{aligned} p_t(i|j,k) &= \sum_{n=k}^{\infty} p_t(i|j,k,N=n)\,p_t(N=n|j,k) \\ &= \sum_{n=k}^{\infty} \frac{\binom{i-1}{j-1}\binom{n-i}{k-j}}{\binom{n}{k}} \frac{\binom{n}{k}(1-t)^{n-k}t^{k+1}P(N=n)}{\sum_{r=k}^{\infty}\binom{r}{k}(1-t)^{r-k}t^{k+1}P(N=r)}. \end{aligned} \tag{4}$$

This can be rewritten as

$$p_t(i|j,k) = \binom{i-1}{j-1} t^j (1-t)^{i-j} \tag{5}$$
$$\times \left[\frac{\sum_{s=k-j}^{\infty} \binom{s}{k-j} (1-t)^{s-(k-j)} t^{k-j+1} P(N = s+i)}{\sum_{r=k}^{\infty} \binom{r}{k} (1-t)^{r-k} t^{k+1} P(N=r)} \right].$$

The factor in the square brackets is a ratio of weighted averages, which can be expressed as expectations, namely $EP(N = W_{k-j+1} + i)/EP(N = W_{k+1})$, where W_d represents the waiting time until the dth success in independent Bernoulli trials with success probability t; and W is, of course, independent of N.

The posterior risk, if we select an option which arrives at time t, has relative rank j, and is the kth arrival so far, is then

$$R_{j,k}(t) = \sum_{i=j}^{\infty} q(i) p_t(i|j,k). \tag{6}$$

3.1. *Poisson prior.* Cowan and Zabczyk (1976, 1978) considered the *best choice problem* with arrival times according to a Poisson process, with known intensity, on an interval of known length (which we may, without loss of generality, take to be the unit interval); this is well known to be equivalent to the special case of our model in which N is taken to be Poisson with known parameter, say λ. In the *best choice problem*, the only relevant posterior probability is $p_t(1|1,k)$ which, after simplification, becomes

$$p_t(1|1,k) = \sum_{r=0}^{\infty} \frac{k}{k+r} \frac{[\lambda(1-t)]^r}{r!} e^{-\lambda(1-t)}.$$

Using the theory of optimal stopping of a Markov chain, which works for the *best choice problem* but not for more general loss functions, Cowan and Zabczyk derived, implicitly, a sequence $y_1, y_2, \dots$, in terms of which, the optimal rule is to stop as soon as, for some k, the kth arrival has relative rank one and its arrival time, say t_k, is late enough so that the expected number of additional arrivals, $\lambda(1-t)$, is no bigger than y_k. Later, Ciesielski and Zabczyk (1979) showed that $y_k/k \to e - 1$ as $\lambda \to \infty$. This is tantamount to showing that the optimal rules, and their risks, are, asymptotically, those of the *best choice* case of the *infinite secretary problem*, a result which, while not given in Ciesielski and Zabczyk (1979), can be derived from Theorem 2.2 by looking at the problem under a suitable time scale transformation, allowing $N \to \infty$ in distribution (for fixed λ). The resulting proof is much shorter than the original analytical proof. For details, see Bruss (1987), where various aspects of the Cowan and Zabczyk model are studied.

3.2. *Uniform (improper) prior.* Another special case was studied in Stewart (1981); this was also a *best choice problem*, but this time giving N the improper, so-called *noninformative* prior: $P(N = n) \equiv 1$. Formally, at least, this greatly simplifies the posterior distribution, because the entire square bracket factor in (5) becomes one, leaving

$$p_t(i|j, k) = \binom{i-1}{j-1} t^j (1-t)^{i-j}, \tag{7}$$

which does not depend at all on k, the number of arrivals by time t. Stewart showed that the *formal Bayes* rule is again the optimal *best choice*, *infinite secretary problem* rule. His argument seems to hinge on the lack-of-memory property of the exponential arrival times, and begs the question of whether this result is indeed the limit for priors "tending" to the *noninformative* prior.

When (7) is substituted into (6), the posterior risk becomes precisely the "risk for choosing an option arriving at time t, which has relative rank j" in the *infinite secretary problem*, namely formula (3.1) of Gianini and Samuels (1976). This fact could be used to show that, for any loss function for which the *infinite secretary problem* risk, v, is finite, the optimal *infinite secretary problem* rule is formal Bayes. But we suggest that such an argument is unnecessary, since Theorem 2.2 already gives a stronger result, namely:

COROLLARY 3.1. *For any sequence of prior distributions of N which "tend" to the noninformative "uniform" prior, the Bayes risks tend to the risk of the corresponding infinite secretary problem optimal rule whenever the latter risk is finite.*

Another way to think about the noninformative prior is this: From (4) the (formal) posterior distribution of N given that the earliest arrival is at, say σ, is

$$\begin{aligned} P(N = n|\sigma) &= p_\sigma(N = n|k, j) \\ &= \binom{n}{k} \sigma^{k+1} (1-\sigma)^{n-k} \end{aligned} \tag{8}$$

(i.e., $N + 1$ has a Pascal distribution). So, let us consider the model in which first σ is chosen according to some distribution, then N according to (8), and finally the remaining $N - 1$ arrival times are i.i.d. uniform on $(\sigma, 1)$. Because the posterior risk matches the *infinite secretary problem* risk, it could then be shown that the optimal stopping rule also matches, in the sense that it uses the same cutoff points. Hence it stops at σ if $\sigma \geq t_1$; so its risk is

$$E \min(v, R_1(\sigma)).$$

[Note: R has only one subscript here, namely $j = 1$, because the risk does not depend on k. Also we are using the fact that $R_1(\cdot)$ decreases from $q(\infty)$ to $q(1)$ and $R_1(t_1) = v$; see Gianini and Samuels (1976).] What is especially interesting is that knowledge of the distribution of σ is unnecessary! We may say that the model "tends" to the noninformative prior model if we have a sequence of σ's tending to zero in distribution.

Acknowledgment. Part of the research leading to this paper was carried out while the first author was Visiting Lecturer at the University of Strathclyde, Glasgow, from September 1984 to February 1985. He would like to thank the Department of Mathematics and Professor B. D. Ripley, in particular, for the invitation and the excellent working conditions offered to him.

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