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GAUSSIAN PROCESSES AND MIXED VOLUMES

BY V. D. MILMAN AND G. PISIER

Tel Aviv University and Université Paris VI

We prove the following conjecture of Dudley: If the volume exponent of a compact convex and symmetric subset K of a Hilbert space is less than -1, then K is a G.C. set.

Let K be a compact subset of a real Hilbert space H. We denote by $N(K, \varepsilon)$ the smallest number of open balls of radius ε which cover K.

The exponent of entropy of K is defined by

$$r(K) = \limsup_{\varepsilon \to 0} \frac{\log \log N(K, \varepsilon)}{\log 1/\varepsilon}.$$

On the other hand, we denote

$$V_n(K) = \sup\{\text{vol}(P(K))\},$$

where the supremum runs over all the orthogonal projections P: $H \to H$ of rank n, and where vol( ) denotes the n-dimensional volume of a set in $P(H)$ (with the normalization determined by the metric of H).

Let $(X_t)_{t \in H}$ be any Gaussian process, indexed by H, and such that

$$\forall\ t, s \in H, \qquad \|X_t - X_s\|_2 = \|t - s\|. \tag{1}$$

The set K is called a G.C. set (resp. G.B. set) if the process $(X_t)_{t \in K}$ has a version with continuous (resp. bounded) paths on K.

It is known that if this happens for one process satisfying (1), then it also does for all such processes.

In his 1967 paper, Dudley proved that if

$$\int_0 (\log N(K, \varepsilon))^{1/2}\, d\varepsilon < \infty,$$

then K is a G.C. set. This happens in particular if $r(K) < 2$. In the converse direction, he conjectured that $r(K) \le 2$ is necessary for K to be a G.C., or more generally a G.B. set. This was later proved by Sudakov (1971). In the same paper, Dudley introduced the exponent of volume of K,

$$\text{EV}(K) = \limsup_{n \to \infty} \frac{\log V_n(K)}{n \log n}.$$

He proved that $\text{EV}(K) \le -1$ for any G.B. set (see below for a proof) and he conjectured that $\text{EV}(K) < -1$ implies that K is a G.C. set for K convex and symmetric. He also conjectured the identity $(*)$ below. We will prove this conjecture in the following equivalent formulation.

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THEOREM 1. *Let K be a convex, symmetric and compact subset of a Hilbert space H.*

Suppose that there are numbers $\delta > 0$ and C such that

$$V_n(K)^{1/n} \le Cn^{-1-\delta}, \quad \textit{for all } n. \tag{2}$$

Then K is a G.C. set.

Moreover, under the assumption (2), *we have*

$$\mathrm{EV}(K) = -\frac{1}{r(K)} - \frac{1}{2}. \tag{$*$}$$

In terms of random processes, it is easy to see that this can be also reformulated as follows.

COROLLARY 2. *Let T be a compact metric space, and let $X = (X_t)_{t\in T}$ be any Gaussian process on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$.*

We assume that $t \to X_t$ is a continuous map from T into $L_2(\Omega, \mathscr{A}, \mathbb{P})$. Let $V_n(X) = \sup\{\mathrm{vol}(\overline{\mathrm{cos}}\{PX_t | t \in T\})\}$ where the supremum runs over all projections P: $L_2(\Omega) \to L_2(\Omega)$ of rank n, and where $\overline{\mathrm{cos}}$ denotes the closed convex and symmetric hull of a set.

Assume that $\limsup_{n\to\infty} n^{1+\delta} V_n(X)^{1/n} < \infty$ *for some $\delta > 0$. Then $(X_t)_{t\in T}$ has a version with continuous sample paths.*

As is well known to specialists, Corollary 2 follows from Theorem 1 by considering $H = L_2(\Omega)$ and taking for K the closed convex and symmetric hull of the compact set $\{X_t | t \in T\}$. For the proof, we will yet consider a third equivalent formulation of Theorem 1, in another language.

Let u: $l_2 \to E$ be a bounded linear operator with values in a Banach space E. Let (e_n) be an orthonormal basis of l_2, and let (g_n) be an i.i.d. sequence of Gaussian standard random variables (i.e., normal and with mean zero).

We define

$$l(u) = \sup_n \mathbb{E}\left\| \sum_1^n g_i u(e_i) \right\|.$$

It is well known that (by the rotational invariance of Gaussian measures) this does not depend of the choice of the orthonormal basis (e_n). We will be interested in the particular operators u for which $l(u) < \infty$.

This notation will be convenient in the proofs below. We could also phrase our results in the framework of abstract Wiener spaces in the sense of Gross [see Badrikian and Chevet (1974)].

Let us denote by B_E the unit ball of E, and by u^* the adjoint operator. Let $K = u^*(B_{E^*}) \subset l_2$.

Then, by well-known results [for an account of the theory see, e.g., Badrikian and Chevet (1974) or Fernique (1975)], it can be shown that K is a G.B. set iff $l(u) < \infty$ and K is a G.C. set iff

$$\begin{cases} \text{for any } \varepsilon > 0, \text{ there is a finite rank orthogonal projection } P \text{ on } l_2 \\ \text{such that } l(u - uP) < \varepsilon. \end{cases} \tag{3}$$

In other words K is G.C. iff u lies in the closure of the finite rank operators in the sense of the l-norm.

Actually the point of view of linear operators can be used for any compact convex symmetric subset K of l_2. Indeed, given such a set K, let p_K be the semi-norm defined by

$$\forall x \in l_2, \qquad p_K(x) = \sup_{y \in K} |\langle x, y\rangle|.$$

Then $p_K(x) \leq D\|x\|$ for $D = \sup_K \|y\|$.

Let E_K be the Banach space obtained from l_2 equipped with p_K after taking the quotient by the kernel of p_K and completing. We denote by $u_K\colon l_2 \to E_K$ the operator associated naturally to the identity operator from $(l_2, \| \ \|)$ into (l_2, p_K). By the bipolar theorem, we have $u_K^*(B_{E_K^*}) = K$. Therefore K is a G.B. set (resp. G.C. set) iff $l(u_K) < \infty$ [resp. u_K satisfies (3)]. This will allow us to work in the sequel with this condition (3) instead of the G.C. condition. Let (e_n) be an orthonormal basis of l_2. Let us denote by P_n the orthogonal projection onto the span of $\{e_1, \ldots, e_n\}$, identified with $\mathbb{R}^n$. Let $K_n = P_n(K)$. To relate $l(u)$ with the numbers $V_n(K)$, we first observe that

$$\mathbb{E}\left\| \sum_1^n g_i u(e_i) \right\| = c_n \int \|x\|_{K_n} \, d\sigma_n(x), \tag{4}$$

where σ_n is the normalized invariant measure on the unit sphere of $\mathbb{R}^n$ and where

$$\|x\|_{K_n} = \sup_{t \in K} \left| \left\langle \sum_1^n x_i e_i, t \right\rangle \right| = \sup_{s \in K_n} |\langle x, s\rangle|,$$

and $c_n = \mathbb{E}(\sum_1^n g_i^2)^{1/2}$ so that

$$c_n/\sqrt{n} \to 1, \quad \text{when } n \to \infty. \tag{5}$$

The identity (4) follows immediately from an integration in polar coordinates.

We will denote by B_n the unit ball of l_2^n (i.e., the Euclidean unit ball in $\mathbb{R}^n$). Let us recall here a classical inequality of Urysohn (1924): For any convex body C in $\mathbb{R}^n$ we have

$$\left(\frac{\operatorname{vol}(C)}{\operatorname{vol}(B_n)} \right)^{1/n} \leq \int \sup_{y \in C} |\langle x, y\rangle| \, d\sigma_n(x) \tag{6}$$

[cf., e.g., Milman (1985)]. Applying this to K_n, we find

$$\left\{ \frac{\operatorname{vol}(P_n(K))}{\operatorname{vol}(B_n)} \right\}^{1/n} \leq \int \|x\|_{K_n} \, d\sigma_n(x). \tag{6$'$}$$

Moreover, it is well known that $\operatorname{vol}(B_n) = \pi^{n/2}(\Gamma(n/2 + 1))^{-1}$ so that we have

$$a n^{-1/2} \leq \operatorname{vol}(B_n)^{1/n} \leq b n^{-1/2}, \tag{7}$$

for some numerical positive constants a and b. It follows from (4), (5), (6) and (7) that

$$\mathrm{vol}(P_nK)^{1/n} \le b'l(u)n^{-1},$$

for some numerical constant b'. Therefore [since the orthonormal basis (e_n) is arbitrary]

$$(8) \qquad V_n(K)^{1/n} \le b'l(u)n^{-1}.$$

Going back to the terminology of Dudley, this means that $\sup_n nV_n(K)^{1/n} < \infty$ is necessary for K to be a G.B. set, as was first proved in Dudley (1967).

Conversely, we will show below that if for some constants C and $\delta > 0$ we have

$$(9) \qquad V_n(K)^{1/n} \le Cn^{-1-\delta}, \quad \text{for all } n,$$

then $l(u) < \infty$ and in fact (3) holds so that K is a G.C. set.

REMARK 3. In the proof below, it will suffice to assume that there is a sequence $a_n > 0$ such that

$$(10) \qquad \sum_{n\ge 1} a_n n^{-1/2}\log n < \infty,$$

and for which there is a constant C_1 such that

$$(11) \qquad \left\{\sum_{i=1}^{n}\binom{n}{i}\left[\frac{i^{1/2}V_i(K)^{1/i}}{a_n}\right]^i\right\}^{1/n} \le C_1,$$

for all $n \ge 1$.

It will be worthwhile to observe here that (2) implies the existence of such a sequence (a_n). Indeed, let us assume (2). We can take $a_n = n^{-1/2-\delta'}$ for $0 < \delta' < \delta$. Then (10) clearly holds. On the other hand, the left-hand side of (11) is majorized by

$$\left\{\sum_{i=1}^{n}\binom{n}{i}\left(Ci^{-1/2-\delta}n^{1/2+\delta'}\right)^i\right\}^{1/n}.$$

The latter sum can be split into two terms

$$\mathrm{I} = \sum_{i\le n^\alpha} \quad \text{and} \quad \mathrm{II} = \sum_{i>n^\alpha},$$

where $\alpha = (\frac{1}{2} + \delta')(\frac{1}{2} + \delta)^{-1} < 1$. An easy calculation shows that

$$\mathrm{I} \le \left\{\left(Cn^{1/2+\delta'}\right)^{n^\alpha}\cdot 2^n\right\}^{1/n}$$

and this is uniformly bounded since $\alpha < 1$. Also

$$\mathrm{II} \le \left\{\sum\binom{n}{i}C^i\right\}^{1/n} \le 1 + C.$$

In conclusion, we have checked that (2) implies (11) for $a_n = n^{-1/2-\delta'}$, with $0 < \delta' < \delta$.

For the proof of Theorem 1, we will use the following classical formula from the theory of mixed volumes. Consider a convex set C in l_2^n (i.e., $\mathbb{R}^n$ equipped with its Euclidean structure). Then

$$\frac{\operatorname{vol}(C+\varepsilon B_n)}{\operatorname{vol}(B_n)} = \sum_{i=0}^{n} \binom{n}{i} \varepsilon^{n-i} \alpha_i(C),$$

where $\alpha_i(C)$ is the average over all possible orthogonal projections $Q\colon \mathbb{R}^n \to \mathbb{R}^n$ with rank i of the quantity $[\operatorname{vol}(Q(C))]/[\operatorname{vol}(Q(B_n))]$ (here the volume is i-dimensional).

More precisely, we have

$$\alpha_i(C) = \int \frac{\operatorname{vol}(Q_F(C))}{\operatorname{vol}(B_i)}\, d\mu(F), \tag{12}$$

where we have denoted by $d\mu(F)$ the normalized canonical measure on the Grassmann manifold of all i-dimensional subspaces F of $\mathbb{R}^n$. Note that the normalization leads to $\alpha_i(B_n) = 1$ for $i = 0, 1, \dots, n$.

The definition (12) of $\alpha_i(C)$ clearly implies that $\alpha_i(C) \le V_i(C)(\operatorname{vol}(B_i))^{-1}$, hence by (7)

$$\alpha_i(C) \le (a^{-1}\sqrt{i})^i V_i(C).$$

From (12), it is easy to see that

$$\alpha_1(C) = \int \sup_{y \in C} |\langle x, y\rangle|\, d\sigma_n(x),$$

which is a known identity [cf., e.g., Badrikian and Chevet (1974)], showing that Gaussian integrals coincide [recall (4)] with a certain mixed volume.

We will make heavy use of the following lemma, which originates in the work of Kašin, for which we refer to Szarek (1978) and Szarek and Tomczak-Jaegermann (1980). We use it in the form already put forward in Milman (1986a). In the sequel, we denote by $|\ |$ the Euclidean norm on $\mathbb{R}^n$.

LEMMA 4. *Let C be a symmetric convex body in $\mathbb{R}^n$. Assume that C is the unit ball of a norm, denoted by $|\ |_C$ on $\mathbb{R}^n$, and assume that*

$$\left\{\frac{\operatorname{vol}(C+\varepsilon B_n)}{\operatorname{vol}(B_n)}\right\}^{1/n} \le A\varepsilon,$$

for some constants A and $\varepsilon > 0$.

Then, for any λ with $0 < \lambda < 1$, there are a subspace F of $\mathbb{R}^n$ of dimension $[\lambda n]$ such that

$$\forall\, x \in F, \qquad |x| \le \varepsilon C(\lambda, A)|x|_C,$$

where $C(\lambda, A)$ is a constant depending only on λ and A.

REMARK. Let E be a Banach space and let $u\colon l_2^n \to E$ be any operator. It follows from Lemma 4 that there is a subspace F of l_2^n of dimension $[\lambda n]$ such that

$$\|u^*|_{u^{*-1}(F)}\| \leq C_\lambda \frac{l(u)}{\sqrt{n}},$$

for some constant C_λ depending only on λ. Indeed, let $K = u^*(B_{E^*})$, let $\delta > 0$ and let $C = K + \delta B_n$, so that C is a convex body in $\mathbb{R}^n$.

We have then by (6) and (4)

$$\left(\frac{\operatorname{vol}(C + \varepsilon B_n)}{\operatorname{vol}(B_n)}\right)^{1/n} \leq \int [\|u(x)\| + \varepsilon + \delta]\, d\sigma_n(x)$$
$$\leq \frac{\beta}{\sqrt{n}} l(u) + \varepsilon + \delta,$$

for some numerical constant β.

Taking $\delta = \varepsilon = l(u)/\sqrt{n}$ we obtain the assumptions of Lemma 4 with $A = \beta + 2$ and the preceding claim follows since, if $F_1 = u^{*-1}(F)$, (with F as in Lemma 4) we find, with $C_\lambda = C(\lambda, \beta + 2)$,

$$\forall\, x \in F, \qquad |x| \leq C_\lambda \frac{l(u)}{\sqrt{n}} |x|_C;$$

hence

$$\forall\, y \in F_1, \qquad |u^*(y)| \leq C_\lambda \frac{l(u)}{\sqrt{n}} |u^*(y)|_C,$$

but obviously $|u^*(y)|_C \leq \|y\|$, so that

$$\|u^*|_{F_1}\| \leq C_\lambda \frac{l(u)}{\sqrt{n}}$$

as claimed above.

Note that codim $F_1 \leq n - [\lambda n]$.

For a more delicate study of the function C_λ, we refer the reader to Milman (1986b), and to a forthcoming paper of Pajor and Tomczak. The main idea in the sequel is to use Lemma 4 under the assumption (2) in the case of an operator u: $l_2^n \to E$ for $K = u^*(B_{E^*})$ and to obtain a subspace F of l_2^n of dimension $[n/2]$ (say) such that

$$\|u^*|_{u^{*-1}(F)}\| \leq \frac{\text{Constant}}{\sqrt{n}} \frac{1}{n^{\delta'}},$$

for some $\delta' > 0$. From this last fact (applied for every restriction of u) we then finally prove that $l(u)$ itself can be majorized by a constant.

We now turn to the proof of Theorem 1. We will assume (2) and will show that (3) follows. We now give the details.

We will denote by $\mathcal{P}$ (resp. $\mathcal{P}_n$) the set of all finite rank orthogonal projection (resp. of rank n) on l_2. Let Q_1, Q be elements of $\mathcal{P}$. The notation $Q_1 \leq Q$ means that the range of Q_1 is included in the range of Q. By a projection on l_2, we always mean an orthogonal one. We will prove below the following.

LEMMA 5. *Under the assumption* (2), *there are a constant* C_1 *and a number* $\delta_1 > 0$ *such that, for any* n, *for any* Q *in* $\mathscr{P}_{2n}$ *there is a* Q_1 *in* $\mathscr{P}_n$ *such that* $Q_1 \leq Q$ *and* $l(uQ_1) \leq C_1 n^{-\delta_1}$.

To prove this lemma, we will use two more facts. The first one is well known and easy to prove.

LEMMA 6. *Let* v: $l_2 \to E$ *be an operator of rank* n *with values in an arbitrary Banach space* E. *We have*

$$l(v) \leq \sqrt{n}\|v\|.$$

PROOF. Clearly v factors as $v = v_1 v_2$ with v_2: $l_2 \to l_2^n$ and v_1: $l_2^n \to E$ such that

$$\|v_1\|\,\|v_2\| = \|v\|.$$

Hence $l(v) \leq l(v_1)\|v_2\|$, and if $(e_1, \ldots, e_n)$ is the canonical basis of l_2^n

$$l(v) \leq \|v_2\|\mathbb{E}\left\|\sum_1^n g_i v_1(e_i)\right\| \leq \|v_2\|\,\|v_1\|\mathbb{E}\left(\sum_1^n g_i^2\right)^{1/2} \leq \sqrt{n}\|v\|. \qquad \square$$

The next lemma is more involved. It has been quite useful in recent work in the geometry of Banach spaces, but has not been used yet for Gaussian r.v.'s.

LEMMA 7. *Let* E_1 *be a Banach space of dimension not more than* $2n$, *let* $E_2 \subset E_1$ *be a subspace of* E_1 *and let* σ: $E_1 \to E_1/E_2$ *be the quotient map. Then, for any* v: $l_2 \to E_1/E_2$, *there is an operator* $\tilde{v}$: $l_2 \to E_1$ *such that* $\sigma\tilde{v} = v$ (*i.e.*, $\tilde{v}$ *is a "lifting" of* v) *and*

$$l(\tilde{v}) \leq K \log(2n+1) l(v),$$

where K *is an absolute constant.*

This lemma is an easy consequence of certain basic facts concerning the notion of K-convexity, which is studied in Pisier (1982). This notion is usually developed for the Rademacher functions, but it can be developed identically for an i.i.d. sequence (g_n) of standard Gaussian r.v.'s. on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Let us denote by P the orthogonal projection from $L^2(\Omega, \mathscr{A}, \mathbb{P})$ onto the closed linear span of such a sequence $(g_n)_n$. A Banach space X is K-convex if the operator $\tilde{P} = P \otimes Id_X$ defines a bounded linear operator from $L_2(\Omega, \mathbb{P}; X)$ into itself. It is easy to see that this definition is equivalent to the usual one of Pisier (1982a) [see Figiel and Tomczak-Jaegermann (1979) for details]. When this operator $\tilde{P}$ is bounded, it defines a bounded linear projection from $L_2(\Omega, \mathbb{P}; X)$ onto the subspace of all the convergent series of the form $\sum_1^\infty g_n x_n$, with coefficients x_n in X.

We define the (Gaussian) K-convexity constant $\tilde{K}(X)$ as equal to the norm of $\tilde{P}$ on $L_2(\Omega, \mathbb{P}; X)$. We will use below the fact that if X is finite dimensional, then

$$\tilde{K}(X) \leq K \log(1 + \dim X)$$

for some numerical constant K.

This fact first appeared in Pisier (1980) in the Rademacher case. The Gaussian case is entirely similar as explained in Pisier (1982a); the only difference is that Hermite polynomials have to be substituted for Walsh functions. Another proof appeared in Pisier (1980/81) [cf. also Milman and Schechtman (1986)] and Pisier (1982b) contains a related lifting theorem.

PROOF OF LEMMA 7. Let $x_n = v(e_n)$, denoting here by (e_n) the canonical basis of l_2. Then, the series $\sum_1^\infty g_n x_n$ converges in $L_2(\Omega, \mathbb{P}; E_1/E_2)$. Clearly there is for any $\varepsilon > 0$ a random variable Φ in $L_2(\Omega, \mathbb{P}; E_1)$ such that

$$\sigma(\Phi) = \sum_1^\infty g_n x_n$$

and

$$\|\Phi\|_{L_2(E_1)} \leq (1+\varepsilon)\left\|\sum g_n x_n\right\|_{L_2(E_1/E_2)} = (1+\varepsilon) l(v)$$

(to obtain Φ, just compose the r.v. $\omega \to \sum g_n(\omega) x_n$ with a measurable lifting φ: $E_1/E_2 \to E_1$ such that $\|\varphi(x)\| \leq (1+\varepsilon)\|x\|$ for all x).

Now if we apply the operator $\tilde{P}$ to Φ we obtain a series of the form

$$\tilde{P}(\Phi) = \sum_1^\infty g_n \tilde{x}_n,$$

for some $\tilde{x}_n$ in E_1, which must necessarily satisfy $\sigma(\tilde{x}_n) = x_n$. Moreover, we have

$$\begin{aligned}\|\tilde{P}(\Phi)\|_{L_2(E_1)} &\leq K \log(\dim E_1 + 1)\|\Phi\|_{L_2(E_1)}\\ &\leq K \log(\dim E_1 + 1) l(v)(1+\varepsilon).\end{aligned}$$

Let now $\tilde{v}$: $l_2 \to E_1$ be the operator defined by $\tilde{v}e_n = \tilde{x}_n$. We have

$$\begin{aligned}l(\tilde{v}) &= \|\tilde{P}(\Phi)\|_{L_2(E_1)}\\ &\leq K \log(2n+1) l(v)(1+\varepsilon).\end{aligned}$$ □

PROOF OF LEMMA 5. Recall $K = u^*(B_{E^*})$. Let Q be an element of $\mathscr{P}_{2n}$. Let H be the range of Q. Clearly, the set $Q(K) \subset H$ still satisfies our assumption (2) in H.

By Remark 3 and the remark following it, we can find $\delta' > 0$ and C' (independent of n) such that, if $\varepsilon_n = (2n)^{-1/2-\delta'}$, we have

$$\left(\frac{\operatorname{vol}(Q(K) + \varepsilon_n B_{2n})}{\operatorname{vol}(B_{2n})}\right)^{1/2n} \leq C'\varepsilon_n.$$

Therefore, we may apply Lemma 4 to the convex body $C = Q(K) + \frac{1}{2}\varepsilon_n B_{2n}$.

This implies that there is a subspace $F \subset H$ of dimension n such that

$$\forall x \in F, \qquad |x| \le D\varepsilon_n |x|_C, \tag{13}$$

for some constant D, independent of n. Let S be the subspace of E^* defined by $S = (Qu^*)^{-1}(F)$. We have

$$|Qu^*(y)| \le D\varepsilon_n \|y\|, \qquad \forall y \in S. \tag{14}$$

Indeed, if y is in B_{E^*}, $Qu^*(y)$ is in $Q(K)$, hence $\|Qu^*(y)\|_C \le \|y\|$ and (14) follows from (13).

Observe that the codimension of S in E^* satisfies $\operatorname{codim}(S) \le n$. Indeed, let $q\colon H \to H/F$ be the quotient map. Then $S = \operatorname{Ker}(qQu^*)$, hence

$$\operatorname{codim} S = \dim qQu^*(E^*) \le \dim H/F = n.$$

Let us denote by $S^\perp$ the subspace of E which is the annihilator of S, and by $\pi\colon E \to E/S^\perp$ the quotient map.

By (14), $\|Qu^*|_S\| \le D\varepsilon_n$. Dualizing, we immediately find

$$\|\pi u Q\| \le D\varepsilon_n, \tag{15}$$

and $\dim S^\perp = \operatorname{codim} S \le n$. Let E_1 be the range of the operator uQ. Then $\dim E_1 \le 2n$. Moreover $S^\perp \subset E_1$. Indeed, $E_1^\perp = \operatorname{Ker}(Qu^*) \subset S$ hence $E_1 = E_1^{\perp\perp} \supset S^\perp$. (Here the annihilators are with respect to the duality between E and E^*.) Let us denote by $\sigma\colon E_1 \to E_1/S^\perp$ the restriction of π to E_1, so that we may rewrite (15) as $\|\sigma u Q\| \le D\varepsilon_n$.

By Lemma 6, $l(\sigma u Q) \le D\varepsilon_n\sqrt{2n}$ and by Lemma 7, there is an operator $\tilde{v}\colon l_2 \to E_1$ such that $\sigma\tilde{v} = \sigma u Q$ and $l(\tilde{v}) \le KD\varepsilon_n\sqrt{2n}\log(2n+1)$. Since $\sigma(\tilde{v} - uQ) = 0$, the operator $w = (\tilde{v} - uQ)Q$ takes its values into $S^\perp$, hence

$$\operatorname{codim} \operatorname{Ker} w \le \dim S^\perp \le n.$$

Note that $\operatorname{Ker} w \supset \operatorname{Ker} Q$.

Let Q' be the orthogonal projection onto $(\operatorname{Ker} w)^\perp \subset (\operatorname{Ker} Q)^\perp = \operatorname{Im} Q$.

We have $Q' \le Q$, $\operatorname{rank} Q' \le n$, and $w = wQ'$. Hence

$$\tilde{v}Q - uQ = \tilde{v}Q' - uQ',$$

so that

$$\begin{aligned} l(u(Q - Q')) &= l(\tilde{v}(Q - Q')) \\ &\le l(\tilde{v}) \\ &\le KD\varepsilon_n\sqrt{2n}\log(2n+1). \end{aligned}$$

Now if $0 < \delta_1 < \delta'$, there is a constant C_1 independent of n such that $KD\varepsilon_n\sqrt{2n}\log(n+1) \le C_1 n^{-\delta_1}$. Finally, we note that $\operatorname{rank}(Q - Q') \ge n$. Therefore we may take for Q_1 any projection of rank n such that $Q_1 \le Q - Q'$ and we find $l(uQ_1) \le l(u(Q - Q')) \le C_1 n^{-\delta_1}$, which concludes the proof of Lemma 5. □

We can now deduce Theorem 1 from Lemma 5.

Proof of Theorem 1. We first prove that $l(u)$ is finite. Let $\lambda(m) = \sup\{l(uQ) | Q \in \mathscr{P}_{2^m}\}$. By Lemma 5, for any Q in $\mathscr{P}_{2^m}$, there is a $Q_1 \le Q$ in $\mathscr{P}_{2^{m-1}}$

such that $l(uQ_1) \le C_1(2^{m-1})^{-\delta_1}$. Hence

$$l(uQ) \le l(uQ_1) + l(u(Q - Q_1)) \le C_1(2^{m-1})^{-\delta_1} + \lambda(m-1),$$

so that

$$\lambda(m) \le C_1(2^{m-1})^{-\delta_1} + \lambda(m-1),$$

and clearly this implies

$$\sup_m \lambda(m) < \infty,$$

so that $l(u) < \infty$. Moreover, applying Lemma 5 again with $Q - Q_1$ in the place of Q, we obtain $Q_2 \le Q - Q_1$ of rank 2^{m-2} such that $l(uQ_2) \le C_1[2^{m-2}]^{-\delta_1}$. We can now apply Lemma 5 with $Q - Q_1 - Q_2$ in the place of Q, and so on. This yields a sequence of mutually orthogonal projections $Q_1, Q_2, \dots$ with rank $(Q_j) = 2^{m-j}$ and such that $l(uQ_j) \le C_1(2^{m-j})^{-\delta_1}$.

We claim that this implies that for each $\varepsilon > 0$ there is an integer $r(\varepsilon)$ with the following property: for any projection Q of finite rank there is a projection $\tilde{Q} \le Q$ such that $l(uQ - u\tilde{Q}) < \varepsilon$ and $\text{rank}(\tilde{Q}) < r(\varepsilon)$. Indeed, if we let, with the above notation, for $1 \le k \le m$, $\tilde{Q} = Q - (Q_1 + \cdots + Q_{m-k})$ we find $\text{rank}(\tilde{Q}) = 2^k$ and

$$\begin{aligned} l(uQ - u\tilde{Q}) &\le l(uQ_1) + \cdots + l(uQ_{m-k}) \\ &\le C_1\left[(2^{m-1})^{-\delta_1} + \cdots + (2^k)^{-\delta_1}\right] \\ &\le C_1' 2^{-k\delta_1}, \end{aligned}$$

for some constant C_1', and this leads immediately to the preceding claim. (Note that we can always replace Q by a larger projection the rank of which is a power of 2.)

Finally let us check that u is in the closure in the sense of the l-norm of the finite rank operators. Note that u is necessarily compact since $l(u) < \infty$ implies that $u^*(B_{E^*})$ is a G.B. set and G.B. sets are compact [cf. Dudley (1967)]. Let $\varepsilon > 0$ and $\xi > 0$ be arbitrary. Since u is compact, there is a finite rank projection P on l_2 such that $\|u - uP\| < \xi$. Now, let Q be any finite rank projection such that $Q \le 1 - P$. By the preceding claim, there is a projection $\tilde{Q} \le Q$ of rank less than $r(\varepsilon)$ such that $l(uQ - u\tilde{Q}) < \varepsilon$. By Lemma 6, we have

$$\begin{aligned} l(uQ) &\le l(u\tilde{Q}) + l(uQ - u\tilde{Q}) \\ &\le \sqrt{r(\varepsilon)}\,\|u\tilde{Q}\| + \varepsilon \\ &\le \xi\sqrt{r(\varepsilon)} + \varepsilon. \end{aligned}$$

This shows that $l(u(1-P)) \le \xi\sqrt{r(\varepsilon)} + \varepsilon$. We have thus shown that for any $\varepsilon' > 0$ there is a projection P of finite rank on l_2 such that $l(u - uP) < \varepsilon'$. Therefore (3) holds, so that $K = u^*(B_{E^*})$ is a G.C. set.

We now come to the proof of the identity $(*)$. The inequality

$$\text{EV}(K) \le -\frac{1}{r(K)} - \frac{1}{2} \qquad (16)$$

is proved in Dudley (1967) as follows: Assume that $P \in \mathscr{P}_n$ and that PK is covered by N Euclidean balls of radius ε with $N \leq N(PK; \varepsilon) \leq N(K; \varepsilon)$. Then

$$\mathrm{vol}(PK) \leq N\varepsilon^n \mathrm{vol}(B_n);$$

hence

$$V_n(K)^{1/n} \leq N(K; \varepsilon)^{1/n} \varepsilon \frac{b}{\sqrt{n}}. \tag{17}$$

Let us now assume that $r(K) < r$ so that $N(K; \varepsilon) \leq \exp(1/\varepsilon)^r$ for all $\varepsilon > 0$ small enough. We have then

$$N\left(K; \frac{1}{n^{1/r}}\right) \leq \exp n, \quad \text{for all } n \text{ large enough.}$$

From (17) we deduce

$$V_n(K)^{1/n} \leq ebn^{-1/r-1/2},$$

for all n large enough, so that

$$\mathrm{EV}(K) \leq -\frac{1}{r} - \frac{1}{2},$$

which establishes (16).

To go conversely, we proceed as follows. We assume that $\mathrm{EV}(K) < \gamma < -1$ so that (2) holds. We will then prove that $-(1/r(K)) - \frac{1}{2} < \gamma$. A close look at the preceding proof shows that we have proved that for any k and any Q in $\mathscr{P}$ there is a projection $\tilde{Q}$ of rank at most k such that $\tilde{Q} \leq Q$ and

$$l(uQ - u\tilde{Q}) \leq C_2 k^{\gamma+1}, \tag{18}$$

for some constant C_2.

Let us introduce a convenient notation,

$$e_n(u^*) = \inf\{\varepsilon > 0 | N(u^*(B_{E^*}), \varepsilon) \leq 2^n\}.$$

With this notation, Sudakov's minorization [cf. Sudakov (1971) or Fernique (1975)] can be stated as

$$\sup \sqrt{n}\, e_n(u^*) \leq C_3 l(u), \tag{19}$$

for some absolute constant C_3. Moreover, it is immediately checked that

$$-\frac{1}{r(K)} = \limsup_{n \to \infty} \frac{\log e_n(u^*)}{\log n}.$$

We recall that if S is a subset of the unit ball of a k-dimensional Euclidean space then

$$N(S, \varepsilon) \leq \left(1 + \frac{2}{\varepsilon}\right)^k \tag{20}$$

[cf., e.g., the proof of Lemma 2.4 in Figiel, Lindenstrauss and Milman (1977)]. Let us assume $\|u\| \leq 1$. Then if $\tilde{Q}$ is in $\mathscr{P}_k$ we deduce from (20) that

$$\left(1 + \frac{2}{e_n(\tilde{Q}u^*)}\right)^k \geq 2^n,$$

so that

$$e_n(\tilde{Q}u^*) \leq \frac{2}{2^{n/k} - 1}.$$

We will choose $n = [k^{1+\xi}]$ for some $\xi > 0$.

Then $e_n(\tilde{Q}u^*)$ tends to zero faster than any negative power of n. We can write by an elementary reasoning

$$e_{3n}(u^*) \leq e_n(u^* - Qu^*) + e_n(Qu^* - \tilde{Q}u^*) + e_n(\tilde{Q}u^*).$$

Since u is compact, we can always choose Q so that $e_n(u^* - Qu^*) \leq 2^{-n}$ (say).

Then by (19) and (18) we have

$$\begin{aligned} e_n(Qu^* - \tilde{Q}u^*) &\leq C_2 C_3 n^{-1/2} k^{\gamma+1} \\ &\leq C_4 n^{(\gamma+1)/(1+\xi) - 1/2}. \end{aligned}$$

Hence we find

$$\limsup \frac{\log e_{3n}(u^*)}{\log n} \leq (\gamma + 1)\frac{1}{1+\xi} - \frac{1}{2},$$

and since $\xi > 0$ is arbitrary this yields

$$-\frac{1}{r(K)} \leq \gamma + \frac{1}{2},$$

which is the announced result,

$$-\frac{1}{r(K)} \leq \mathrm{EV}(K) + \frac{1}{2}. \qquad \square$$

We refer the reader interested in mixed volumes to Burago and Zalgaler (1980) and to Santalo (1976).

NOTE ADDED IN PROOF. After this paper was accepted, we observed that a simpler proof can be obtained by using the main result of Milman (1986a) instead of the above Lemma 4. One can then prove more directly the following refined version of Lemma 5: For any $n > 1$ and any Q in $\mathscr{P}_{4n}$, there is a Q_1 in $\mathscr{P}_{2n}$ such that $Q_1 \leq Q$ and

$$l(uQ_1) \leq CV_n(K)^{1/n} n \log n,$$

for some numerical constant C. In particular, the condition $\Sigma V_n(K)^{1/n} \log n < \infty$ is sufficient for K to be a G.C. set.

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I.H.E.S. AND
TEL AVIV UNIVERSITY
RAMAT AVIV, ISRAEL

I.H.E.S.
AND EQUIPE D'ANALYSE
UNIVERSITÉ PARIS VI
4 PLACE JUSSIEU
75230 PARIS CEDEX 05
FRANCE