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THE INFLUENCE OF MARK KAC ON PROBABILITY THEORY

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Kac's work covers a wide spectrum, from classical analysis through number theory, potential theory, pure probability theory, ergodic theory to statistical physics. One of the fascinating aspects of his work is that it demonstrates the interplay between these fields, and, despite the title of this paper, it would not do Kac's work justice to restrict ourselves here to its purely probabilistic aspects. Kac's enormous influence on statistical physics is discussed by Thompson in the companion article in this issue, and we shall therefore say essentially nothing about his work in that field. Nevertheless, it should be pointed out that, apart from the direct results obtained, Kac's activity in statistical physics also had the indirect influence on probability theory of convincing probabilists that there are exciting and important probability problems in statistical mechanics. Through his writings, lectures, and personal propaganda, Kac was instrumental in no small measure in the development of the very strong international group of probabilists who presently work on the borderline of statistical mechanics and probability theory.

An excellent survey of most of Kac's work was given in the introductory Commentary to the collection of selected reprints of Kac [1]. In addition Kac has written several illuminating autobiographical notes and a book ([2], [K165], [K182]).[1] I owe many of my remarks to these sources. In most cases I have not repeated all the references given in [1]; when possible I have added later references. Clearly all my remarks have been influenced by my tastes and limited knowledge. I apologize for misrepresentations and omissions of much work influenced by Kac.

For the purpose of this article it is convenient to divide Mark Kac's papers into the following (somewhat arbitrary) categories:

1. Probabilistic aspects of gap series and probabilistic number theory.
2. Interplay between probability theory and analysis.
3. Potential theory.
4. Limit theorems and invariance principles.
5. Feynman–Kac formula.

1. Probabilistic aspects of gap series and probabilistic number theory. Under this heading fall the limit theorems for lacunary series and number theoretic functions, which resemble the classical limit theorems (such as the three series criterion and the central limit theorem) both in form and in technique. Kac was brought to questions of this kind by his teacher Steinhaus

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[1] Reference citations preceded by K refer to references listed in Publications of Mark Kac, which appears later in this issue, pages 1149–1154.

who had tried to model independent random variables by "independent functions," i.e., functions f_n on $[0,1]$ which satisfy

$$\begin{aligned}&|\{t \in [0,1]: \alpha_i < f_i(t) \leq \beta_i, 1 \leq i \leq n\}|\\ &\quad = \prod_{i=1}^{n} |\{t \in [0,1]: \alpha_i < f_i(t) \leq \beta_i\}|,\end{aligned} \tag{1}$$

where $|\cdot|$ denotes Lebesgue measure. Steinhaus [3] had started to put (countably many) independent random variables on a sound analytical footing and Kac and Steinhaus [K5]–[K9] derived a number of properties of independent functions which are now familiar to every student who takes even one course in probability. For instance in [K5] it is proved that two functions are independent if and only if their joint characteristic function is the product of the two separate characteristic functions. But remember that Steinhaus' work on these functions started some 10 years before Kolmogorov's fundamental monograph [4]. It is also of some historical interest (as pointed out by Kac himself in [2]) that [K5] was the first place where Kronecker's lemma was used to prove the strong law of large numbers. See Kac's note [K165] for further background material and historical remarks. Kolmogorov's work came to overshadow these early papers of Steinhaus and Kac completely, but the latter papers had the important effect of making Kac search for independence in traditional areas of analysis. He found this in "gap series" and in additive number theory. (A survey of Kac's work in these areas can be found in [1] and [K50]; see also the Introduction and Chapter 12 of [5].)

Various well known analysts such as Kolmogorov, Paley, Zygmund, Sidon, and Banach discovered that convergence of series of the form

$$\sum_{k=1}^{\infty} c_k f(n_k t), \tag{2}$$

for certain periodic f of period 1 and integral n_k satisfying Hadamard's gap condition

$$n_{k+1}/n_k \geq q > 1, \tag{3}$$

is governed by Khinchine and Kolmogorov's three series criterion. (See [K50]; [6], volume I, lacunary series appear on pages 202–212, 215, 230, 247–250, 379–380; or [7] for references.) For instance,

$$\sum c_k \sin(2\pi n_k t) \tag{4}$$

converges for almost all t if and only if

$$\sum c_k^2 < \infty. \tag{5}$$

Kac found a number of other results about the convergence and divergence of (2) (see [K26] and [K32]), but it seems more interesting that he realized that these phenomena were due to the fact that the summands of the gap series behave like independent random variables, and that it was therefore natural to look for a central limit theorem. In [K15] Kac proved by the method of moments that for t

uniformly distributed on [0, 1],

$$\left\{\tfrac{1}{2}\sum_{1}^{n} c_k^2\right\}^{-1/2} \sum_{1}^{n} c_k \sin(2\pi n_k t) \tag{6}$$

converges in law to a standard normal distribution, provided

$$|c_n| = o\left(\left\{\sum_{1}^{n} c_k^2\right\}^{1/2}\right), \qquad n \to \infty, \qquad \sum_{1}^{\infty} c_k^2 = \infty, \tag{7}$$

and the integers n_k satisfy the stronger gap condition

$$n_{k+1}/n_k \to \infty. \tag{8}$$

This work was continued by Ferrand and Fortet [8] and Salem and Zygmund [9]. The latter authors removed the restrictions that the n_k be integers. They also showed that the gap condition (3) suffices for the normal limit distribution of (4).[2] More generally one can consider sums

$$\left\{\sum_{1}^{n} c_k^2\right\}^{-1/2} \sum_{1}^{n} c_k f(n_k t). \tag{9}$$

For integers n_k satisfying (8), c_k satisfying (7), and "smooth" periodic functions f, a central limit theorem also holds for these sums (see [K50, Section 3]). However, in this case, the situation becomes quite different when the gap condition (8) is relaxed to the Hadamard gap condition (3). The limit law (if it exists at all) of (9) can now be nonnormal, even for bounded smooth f. It depends strongly on the arithmetic structure of the n_k. This was demonstrated by examples of Erdös, Fortet, and Kac (see [K50]). The central limit theorem holds nicely for (9) though, when one takes $n_k = a^k$ for an integer a ([K37], [10], and [11]). This even continues to hold for some nonintegral $a > 1$ and certain functions f, but that is much harder to establish (see [12]). Other choices for n_k were investigated by several people. For instance, following a problem posed by Bellman [13], the present author investigated the limit law of

$$S_n := \sum_{1}^{n} \left\{\tilde{I}_{[a,b]}(kx + y) - (b - a)\right\}, \tag{10}$$

where $\tilde{I}_{[a,b]}$ is the indicator function of $[a, b] \subset [0, 1]$, periodically extended. This corresponds to $n_k = k$, which is the extreme opposite of a gap series. If x and y are independent uniform variables on [0, 1], a second moment calculation suggests that (10) should also have central limit behavior; the summands are orthogonal and

$$E\{S_n\} = 0, \qquad E\{S_n^2\} = n(b - a)(1 - b + a).$$

It therefore came as a bit of a surprise that in fact $(\log n)^{-1} S_n$ has a Cauchy

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[2] In footnote 3 of [K50] a paper by Erdös, Ferrand, Fortet, and Kac is promised which will prove this same fact. It seems that the paper was never published.

limit law with a scale factor which depends only on the length $b - a$, but in a discontinuous and complicated way (see [14]; see also footnote 3).

From our vantage point it is natural to continue and try to prove invariance principles and a law of the iterated logarithm for sums of the form (9). A simple special invariance principle of this kind with $n_k = 2^k$ is in [15] and an almost sure invariance principle (which implies a law of the iterated logarithm) for (4) was proved by Philipp and Stout [16] by using Skorokhod imbedding techniques. Even though Gaposhkin [17] takes a slightly different point of view, this major survey discusses a large number of articles on lacunary series which follow Kac's work. It also discusses a number of new questions not discussed above. We also refer the reader to Kahane's review [7] for further work on lacunary series. Today investigations of (9) do not seem to be an active research topic. There is, however, enormous activity in the study of sums of the form

$$\sum_{1}^{n} f(T^k t) \tag{11}$$

for T a transformation of $[0,1]$ into itself. Of course $Tx = ax \pmod 1$ is a special case of this and Kac himself already moved in the direction of looking at sums (11) in [K79]; a central limit theorem for some cases of (11) can also be found in [10].

A second major area where Kac found independence was *additive number theory*. For any set A of integers set

$$P_n\{A\} = \frac{1}{n}\{\#\text{ of integers } k \text{ in } A \cap [1, n]\}$$

and

$$D\{A\} = \lim_{n\to\infty} P_n\{A\} \quad \text{if this limit exists.}$$

If $D\{A\}$ exists it is called the density of A. Also define

$$\delta_p(n) = \begin{cases} 1, & \text{if } p|n, \\ 0, & \text{otherwise.} \end{cases}$$

Then for distinct primes $p_1, \dots, p_k$,

$$P_n\{m\colon \delta_{p_i}(m) = 1, 1 \le i \le k\} = P_n\{m\colon p_i|m, 1 \le i \le k\} = \lfloor n/p_1 \cdots p_k \rfloor,$$

where $\lfloor a \rfloor$ is the largest integer $\le a$. Therefore,

$$D\{m\colon \delta_{p_i}(m) = 1, 1 \le i \le k\} = (p_1 \cdots p_k)^{-1} = \prod_{i=1}^{n} D\{m\colon \delta_{p_i}(m) = 1\}.$$

[3] The result for $\tilde{I}_{[a,b]}$ is atypical for the case $n_k = k$, because the function $\tilde{I}_{[a,b]}$ is not smooth. If f has a sufficiently rapidly convergent Fourier series and $\int_0^1 f(t)\,dt = 0$, then

$$\frac{1}{\gamma_n}\sum_{0}^{n} f(kt) \to 0 \quad \text{in probability}$$

for any sequence γ_n tending to ∞.

Thus, asymptotically the δ_{p_i} act like independent Bernoulli random variables with

$$D\{\delta_{p_i} = 1\} = 1/p_i.$$

"Primes play a game of chance" as Kac put it in his delightful Carus monograph [K178]. It is this independence property which will lead to a central limit theorem. There is, however, a major difficulty in proving such a theorem: the density $D\{\cdot\}$ is only finitely additive and not countably additive, even on sets for which it exists. Nevertheless, a central limit theorem with respect to $D\{\cdot\}$ was proved by Erdös and Kac for strongly additive functions. A function f on the positive integers is called *additive* if

$$f(mn) = f(m) + f(n)$$

whenever m and n are relatively prime. If n has the prime factorization

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

and f is additive, then

$$f(n) = \sum_1^k f(p_i^{\alpha_i}).$$

A function f is called *strongly additive* if it is additive and in addition

$$f(p^\alpha) = f(p), \quad p \text{ prime}.$$

Thus an additive function is determined by its values on powers of primes and a strongly additive function by its values on the primes. In the latter case we can write

$$f(n) = \sum_{p|n} f(p) = \sum_{\text{all } p} f(p)\delta_p(n). \tag{12}$$

For example if one takes $f(p) \equiv 1$, the corresponding strongly additive function is

$$\nu(n) := \text{number of primes dividing } n.$$

Another strongly additive function is $(\log \phi(n))/n$, where ϕ is Euler's function. This corresponds to $f(p) = \log(1 - 1/p)$. Hardy and Ramanujan [18] proved a "tightness result" for the function $\nu(\cdot)$. Specifically they showed that

$$\frac{1}{n}\left\{\#\text{ of integers } m \le n \text{ for which } |\nu(m) - \log\log n| > \gamma_n(\log\log n)^{1/2}\right\} \tag{13}$$

tends to 0 for any sequence γ_n tending to ∞. Their proof used complicated number-theoretic estimates but no probability. Turán [19] simplified their proof by using Chebyshev's inequality, but still without realizing that this was a standard probabilistic method (see [5], volume II, pages 18, 19). It was Kac who realized the full relation with probability theory and tried to prove the following theorem.

(14) THEOREM. *Let f be a strongly additive function which satisfies*

(15) $$|f(p)| \leq 1, \quad p \text{ prime}.$$

Let

$$A(n) = \sum_{p \leq n} \frac{f(p)}{p}, \qquad B(n) = \sum_{p \leq n} \frac{f^2(p)}{p}.$$

If $B(n) \to \infty$, then

(16) $$\lim_{n \to \infty} P_n\{m\colon f(m) - A(n) \leq xB(n)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt.$$

Since D is only finitely additive, a proof requires truncation of the sum (12). Kac could not carry out the necessary number-theoretic estimates of the error due to such a truncation. As related in [K163] and [K182], Erdös completed the proof during a lecture by Kac at Princeton. This resulted in the joint publication [K24] which contains Theorem 14, now known as the Erdös–Kac theorem. In the special case $f(p) \equiv 1$, one obtains the result

$$\lim_{n \to \infty} P_n\left\{m\colon \nu(m) \leq \log\log n + x\{\log\log n\}^{1/2}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt,$$

which sharpens the Hardy–Ramanujan result (13).

One reason why Erdös could supply the missing estimates so quickly was that he found in [20] conditions for an additive function f to have a distribution. We say that f has distribution F if the density of the set

$$A(x) := \{m\colon f(m) \leq x\}$$

exists and equals $F(x)$ at each continuity point x of F. The conditions are very similar to the three series criterion, but the probabilistic content of [20] probably was not realized until the joint paper [21] with Wintner.

Once the basic result (14) had appeared, the whole machinery of probability theory was brought to bear on additive number theory. In his thesis, written under Kac and Rosser jointly, LeVeque [22] estimated the speed of convergence in (16). In analogy with the usual rates of convergence in the central limit theorem he conjectured that the limit in (16) is achieved at the rate $O((\log\log n)^{1/2})$ uniformly in x. He obtained a weaker estimate only, but his conjecture was later proved by Rényi and Turán [23] using Dirichlet series. A simpler proof of the Erdös–Kac theorem was given by Billingsley [24], [25]. LeVeque [22] generalized the Erdös–Kac theorem and showed that the joint limiting distribution of $f(m), f(m+1), \dots, f(m+r)$ (when suitably normalized) exists and is multidimensional normal with independent components (this generalization was first stated by Erdös [26] without proof). Erdös [26] gives a law of the iterated logarithm. Kubilius [27] obtained the full analogue of the Lévy–Khinchine theory for convergence of a triangular array to an infinitely divisible distribution. He found conditions for the convergence of the left-hand side of (16) to any infinitely divisible law. Invariance principles started with

Kubilius ([27], Theorem 7.3) and other versions were proved by several people (see [25], Sections 4, 5, 12).

A good introduction to probabilistic number theory is Kac's Carus monograph [K178]. The elegant Wald lectures by Billingsley [25] are easily accessible and show how much can be done relatively easily with the tools of probability. Kubilius [27] has much of the earlier material as well as his own important contributions. The latest complete collection of results as well as many historical remarks are the two volumes of Elliott [5]. One only has to look at the wealth of material in these books to see how the subject has grown since the Erdös–Kac theorem.

2. Interplay between probability theory and analysis. It goes without saying that the ratio of analysis to probability varies greatly over Kac's papers. We shall include in this section some papers which are "pure probability". Other such papers are included in Section 3.

We already discussed in Section 1 Kac's work on gap series which came out of Fourier analysis; in this area analysis and probability were really intertwined. In the papers of Kac which deal with the characterization of distributions through special properties [K16], [K17], [K73], there was more influence of analysis on probability than vice versa. [K16] proves that if X and Y are two random variables such that $X\cos\theta + Y\sin\theta$ and $X\sin\theta - Y\cos\theta$ are independent for each θ, then X and Y are independent normal variables with mean zero and the same variance. The study of the characterization of the normal distribution by independence properties of various statistics is of course still being pursued (see [28], Chapters 5, 6 for some references). Analysis comes in here because one usually reduces these questions to problems of functional equations and special properties of characteristic functions which have to be handled by means of complex variable theory. It may be worth mentioning here that in [K12] Kac has a very neat proof that the only measurable solution of $f(x + y) = f(x) + f(y)$ is the linear function. It is not clear whether characterization problems or anything in probability led Kac to this paper.

One would expect that the problem of finding the average number of real roots of a polynomial with random coefficients was dreamed up by a probabilist under the motto "if you can't solve the problem exactly, then randomize" (which I have heard attributed to Kac). In fact it originated with Bloch and Polya [29] and Littlewood and Offord [30]. The latter gave an upper bound of order $(\log n)^2$ for the expected number of real roots, N_n, of an nth degree polynomial whose coefficients are uniformly distributed on $[-1,1]$ or standard normal or take the values 1 and -1 with probability $\frac{1}{2}$. They also showed that N_n is at least of the order of $\log n/\log\log\log n$ with high probability. Kac ([K30], Lemma 1 and [K31], formula (6)) gave a formula for the number of roots in an interval of a nice function. Taking expectations in this formula leads to a proper proof of what is known as Rice's formula. Rice [31] first gave such a formula on heuristic grounds for the number of maxima per unit time of nice processes and applied it to a stationary, twice differentiable Gaussian process. He later used it for the number of zeros of a Gaussian process. In [K30] Kac used the formula to show that for an nth degree polynomial whose coefficients are independent with a standard normal distribution one has

$$EN_n \sim \frac{2}{\pi} \log n.$$

In [K49] this was extended to the case where the coefficients are uniform on $[-1,1]$, but Kac's method did not cover coefficients which are 1 or -1 with probability $\frac{1}{2}$. Various authors have continued this work. Probably the latest paper on roots of random equations is [32]. Rice's formula has also been proven under more general conditions; formulas for the higher moments of the number of curve crossings per unit time have also been derived. These results are important tools in the study of high excursions of Gaussian processes (see [33], especially Chapters 10, 13, 14 for some references and applications; also [K180], Chapter 3).

A paper in which the interplay between analysis and probability was much stronger, and which had a very stimulating influence on probability is [K63]. This paper (with its successors [K61] and [K96]) deals with Toeplitz matrices, i.e., matrices with entries of the form $c_{i,j} = c_{j-i}$. Such matrices arise naturally in probability theory as transition probability matrices of a random walk. This fact figured prominently in Kac's probabilistic proof of a theorem of Szegö which gave a very sharp result about the asymptotic size of the determinant of C_n, where C_n is the matrix with entries

$$c_{i,j} = c_{j-i}, \qquad 0 \le i, j \le n,$$

with

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\delta}\, d\theta$$

for some $f \in L^1$. Kac proved an analogue for integral equations of Szegö's formula and writes in [K63] "the way [this analogue] was discovered and proved seems sufficiently interesting to be summarized here. By a suitable reinterpretation of Szegö's result we obtained the following theorem:

Let $\{X_j\}$ be i.i.d. random variables capable of assuming integral values only. Let furthermore

$$P\{X_j = n\} = P\{X_j = -n\} = c_n = c_{-n}$$

and $S_n = X_1 + \cdots + X_n$. Then, if $\Sigma_1^\infty n c_n' < \infty$ we have

$$P\{S_n = 0\} E\{\max(0, S_1, \dots, S_{n-1}) | S_n = 0\}$$
$$= \frac{n}{2} \sum_{j=1}^{\infty} j \sum_{k=1}^{\infty} \frac{P\{S_k = j\} P\{S_{n-k} = -j\}}{k(n-k)}. \tag{17}$$

If $P\{S_n = 0\} = 0$, the left-hand side of (17) is set to be equal to 0.

...Although the remarkable identity (17) was discovered through Szegö's theorem it was desirable to have a direct and elementary proof. Using a reduction suggested by K. L. Chung, G. A. Hunt gave such a proof. The heart of this proof is the following extremely curious combinatorial identity:

Let $a_1, \ldots, a_n$ be real numbers and

$$\sigma = \begin{pmatrix} 1, 2, \ldots, n \\ \sigma_1, \sigma_2, \ldots, \sigma_n \end{pmatrix}$$

a permutation. Let finally $N(\sigma)$ be the number of nonnegative terms in the sequence $a_{\sigma_1}, a_{\sigma_1} + a_{\sigma_2}, \ldots, a_{\sigma_1} + \cdots + a_{\sigma_n}$. Then

$$\sum_{\sigma} \max(0, a_{\sigma_1}, a_{\sigma_1} + a_{\sigma_2}, \ldots, a_{\sigma_1} + \cdots + a_{\sigma_n}) = \sum_{\sigma} N(\sigma) a_{\sigma_1}, \tag{18}$$

where the summations are extended over all permutations of $1, 2, \ldots, n$.

Hunt's original proof of (18) was somewhat lengthy and we shall reproduce in the sequel a short and elegant proof due to F. J. Dyson."

Somewhat earlier Sparre Andersen proved a number of remarkable identities for the distribution of N_n, the number of (strictly) positive terms in the sequence $S_1, \ldots, S_n$, as well as for the indices where the max and min of $0, S_1, \ldots, S_n$ occur. For instance in [34] (see also its references to earlier work of Sparre Andersen) he derived, by purely combinatorial means, the completely general identity

$$P\{N_n = k\} = P\{N_k = k\} P\{N_{n-k} = 0\}, \qquad 0 \le k \le n$$

(with $N_0 = 0$), and for $|t| \le 1$,

$$\sum_0^{\infty} P\{N_j = j\} t^j = \sum_0^{\infty} P\{S_i > 0, 1 \le i \le j\} t^j = \exp\left\{ \sum_1^{\infty} \frac{t^k}{k} P\{S_k > 0\} \right\}, \tag{19}$$

plus a similar generating function for $P\{N_j = 0\} = P\{S_i \le 0, 1 \le i \le j\}$. He also derived—under symmetry conditions only—the limiting arcsine law for N_n/n.[4] When Spitzer saw the combinatorial identity (18) as well as the combinatorial results of Sparre Andersen he felt that there should be a unifying combinatorial principle behind these results. This led him to the paper [35] with the following identity for the generating function of the joint characteristic function of S_n and $M_n := \max(0, S_1, \ldots, S_n)$,

$$\phi_k(\alpha, \beta) = E \exp\{i\alpha M_k + i\beta(M_k - S_k)\}, \qquad \phi_0(\alpha, \beta) = 1.$$

For

$$|t| < 1, \qquad \operatorname{Im}(\alpha) \ge 0, \qquad \operatorname{Im}(\beta) \ge 0,$$

one has

$$\sum_0^{n} \phi_n(\alpha, \beta) t^n = \exp\left\{ \sum_1^{\infty} \frac{1}{k} (u_k(\alpha) + (v_k(\beta) - 1)) t^k \right\}, \tag{20}$$

where

$$u_k(\alpha) = E \exp\{i\alpha S_k^+\}, \qquad v_k(\beta) = E \exp\{i\beta S_k^-\}.$$

[4] The first paper of Sparre Andersen on the arcsine law was in 1949, before [K55]. Thus the comments on page 8 of [1], which give the impression that Sparre Andersen's work on the arcsine law was an outgrowth of [K55], are somewhat misleading.

One can retrieve (19) from this (see [35]). This paper led to a great number of further papers in what is now called "fluctuation theory." In addition to the random variables introduced above, in fluctuation theory one derives expressions for the (Laplace transform of the) time, T, of first entry into the positive half-line by S_n and the size of S_T. The passage from Kac [K63] cited above and the comments following it tied in an analyst, a large number of probabilists, and a mathematical physicist with fluctuation theory in the mid-fifties. As a matter of fact its history goes back further. It had long been realized that the distribution of the waiting time of the nth customer in a simple queueing system is equivalent to the distribution of M_n. In this context Pollaczek [36] (see formula 7.16)) had also obtained the case $\beta = 0$ of the identity (20). In [37] he even obtained double generating functions for all the order statistics of $0, S_1, \dots, S_n$. However, Pollaczek made heavy use of contour integration and does not seem to have realized the purely combinatorial nature of (20). The identity (20) is known as "Spitzer's identity" or the "Pollaczek–Spitzer identity." Too many different people have reproved and extended fluctuation theory to list them all. We merely mention Kemperman [38] and Baxter [39] (see [38] and [40] for further references). It is of some interest that in some proofs one almost returns to analysis; the identity (20) is then derived from the Wiener–Hopf factorization (see [38], Section 13ff.; in [41] Baxter closes the circle by using Wiener–Hopf methods to generalize the theorem of Szegö which motivated Kac in [K63]). An analogue exists when one replaces the random walk by a continuous time process with stationary independent increments. This continuous time analogue was first studied by Rogozin, with the principal later work by Fristedt and by Prabhu (see [42]).

To come back to Kac's work on Toeplitz matrices, the probability arguments also suggested some purely analytic statements about the asymptotic behavior of the extreme eigenvalues of the Toeplitz matrices C_n above. These facts were proven in one case in [K61], and further conjectures were settled by Widom and by Parter. For these and further references, results, and applications, see [43].

We now turn to the *the principle of not feeling the boundary* which is a case of doing analysis using probabilistic tools. The question is how much can one say about the geometry of a domain $\Omega \subset \mathbb{R}^d$ from the eigenvalues of the Laplacian on Ω with Dirichlet boundary conditions. More specifically, for a "nice" domain $\Omega \subset \mathbb{R}^d$ let λ_j be the eigenvalues and $u_j(x)$ the corresponding normalized eigenfunctions of the following Dirichlet problem:

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \qquad x \in \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{21}$$

For $d = 2$ Weyl [44] and Carleman [45] had proven the following asymptotic relations:

$$\sum_{\lambda_j < \lambda} 1 = \left(\#\text{ of eigenvalues } \lambda_j < \lambda\right) \sim \frac{|\Omega|}{2\pi}\lambda, \qquad \lambda \to \infty \qquad \text{(Weyl)}$$

and

$$\sum_{\lambda_j < \lambda} u_j^2(x) \sim \frac{\lambda}{2\pi}, \qquad \lambda \to \infty,\ x \in \Omega \qquad \text{(Carleman)},$$

where $|\Omega|$ is the Lebesgue measure of Ω. Weyl also treats the case $d = 3$. (Some interesting remarks about the history of this problem can be found in [44], footnote on page 442, and in [K101].) Kac ([K55] and [K101]) noticed that these relations follow from the simple probabilistic idea that a Brownian motion particle which starts at $x \in \Omega$ will not "feel the boundary of Ω" for a little time. To formulate this rigorously let $p(t, x, y)$ be the density at y at time t of an unrestricted Brownian motion starting at x, and $q(t, x, y)$ the corresponding density of the Brownian motion absorbed at $\partial\Omega$. Then for $x \in \Omega$ and y sufficiently close to x one has

$$\lim_{t \downarrow 0} \frac{p(t, x, y)}{q(t, x, y)} = 1. \tag{22}$$

Kac proved (22) essentially by explicit computation for $d = 2$ and y close to x. Ciesielski [46] proved that under a mild condition on Ω (22) holds if and only if the line segment from x to y is contained in Ω. In any case, (22) holds for $y = x \in \Omega$, and using Mercer's theorem one has

$$q(t, x, x) = \sum e^{-\lambda_j t} u_j^2(x) \sim p(t, x, x) \sim \frac{1}{2\pi t}, \qquad t \downarrow 0,$$

from which Carleman's result follows via a standard Tauberian theorem. Some more work involving estimation of

$$\int_\Omega q(t, x, x)\, dx$$

gives Weyl's result as well. In particular one can obtain $|\Omega|$ from a knowledge of the eigenvalues λ_j. A major advantage of this probabilistic approach is that it immediately suggests extensions. For a nice Ω we can expect that we can sharpen (22) by approximating the boundary by a line H (or hyperplane if $d > 2$) through the boundary point closest to x, and then taking into account absorption of the Brownian particle in H. For $d = 2$ and convex Ω this leads to

$$\sum e^{-\lambda_j t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}}, \qquad t \downarrow 0,$$

where L is the length of the boundary of Ω. Pleijel [47] proved a closely related result by purely analytic means. He obtained an asymptotic expansion for another abelian expression in the eigenvalues, namely for

$$\sum \frac{1}{\lambda_j(\lambda_j + \omega)} \quad \text{as } \omega \to \infty.$$

Thus for convex $\Omega \subset \mathbb{R}^2$ one can even find the length of the boundary, and as

Kac argued in [K101], one may even guess

$$\sum e^{-\lambda_j t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4\sqrt{2\pi t}} + \frac{1}{6}(1-r)$$

for a smooth domain Ω with r holes; Kac then speculated whether one can determine the Euler characteristic of Ω from the eigenvalues λ_j.

In [K119] Kac "proved" (he admitted to some gaps in his argument; these gaps were filled by Stroock in [48]) that one can almost determine a solid of revolution in $\mathbb{R}^3$ from the associated spectrum. The inverse problem has been generalized to that of finding the geometry of a manifold (of any dimension) from the eigenvalues of its associated Laplace–Beltrami operator. Many people have contributed to this; the problem seems to have moved back to analysis or differential geometry, though. We refer the reader to [1] for a partial summary of the progress up till 1975. More detailed information can be found in [49], especially in the contributions of Berger, Duistermaat and Guillemin, Gilkey and Gilkey, and Sacks. For a discrete version of the problem see [50].

3. Potential theory. The Laplacian is of course intimately connected with classical potential theory. Relations between random walk and harmonic functions had been known for a long time (see [K55], Section 10 for some references), but since $\frac{1}{2}\Delta$ is the generator of Brownian motion it is more natural to relate Brownian motion directly to harmonic functions. It seems that Kakutani was the first to make explicit use of this relationship. In [51] he expressed the harmonic measure relative to x of a subset E of $\partial\Omega$ for a nice domain Ω as the probability that a Brownian particle starting at x hits $\partial\Omega$ first in E. More generally, Doob [52] considered the (interior) Dirichlet problem: For a given domain $\Omega \subset \mathbb{R}^d$ and function f on $\partial\Omega$, find a function u on Ω such that

$$\Delta u(x) = 0, \qquad x \in \Omega, \tag{23}$$

$$\lim_{\substack{x \to y \\ x \in \Omega}} u(x) = f(y), \qquad y \in \partial\Omega. \tag{24}$$

Doob wrote the solution as

$$u(x) = E_x\{f(B_{\sigma_{\partial\Omega}})\}, \tag{25}$$

where $\{B_t\}$ is a Brownian motion, E_x denotes expectation with respect to the measure P_x governing the Brownian motion when it starts at x (thus B_0 is not necessarily 0 in our notation), and

$$\sigma_A = \inf\{t > 0\colon B_t \in A\}.$$

If

$$\sigma_{\partial\Omega} < \infty \quad \text{almost surely and} \quad E_x\{|f(B_{\sigma_{\partial\Omega}})|\} < \infty, \qquad x \in \Omega \tag{26}$$

(i.e., if f is integrable with respect to the harmonic measure; this condition does not depend on x), then (25) is indeed harmonic in Ω. If f is bounded and continuous on $\partial\Omega$ and every point of $\partial\Omega$ is regular for $\Omega^c :=$ the complement of

Ω, then u even has a continuous extension to the closure of Ω, which satisfies the boundary condition (24). (A point y is regular for a set A if

$$P_y\{B_t \in A \text{ for some } 0 < t < \delta\} = 1 \quad \text{for all } \delta > 0$$

i.e., the Brownian motion hits A immediately from y). For more general f the boundary condition (24) only holds in a weaker sense, but u agrees with the classical Perron–Wiener–Brelot solution under (26); see [52], Theorem 6.2, and [53], Section 2.IX.13.

In [K55] Kac considered another problem from classical potential theory; he looked for the capacitary potential of a bounded set $\Omega \subset \mathbb{R}^d$, $d \geq 3$, with positive Lebesgue measure. To be more precise, Kac was investigating the distribution of the functional

$$T = T_\Omega := \int_0^\infty I_\Omega[B_t]\, dt = \text{the occupation time of } \Omega.$$

He was led to this because his work on the Feynman–Kac formula (see Section 5 below) gave him the following expression for the Laplace transform of T when $d = 3$:

$$E_x\{e^{-uT}\} = 1 - u \sum_{j=1}^{\infty} \frac{a_j}{1 + \lambda_j u}, \tag{27}$$

where

$$a_j = \frac{1}{2\pi} \int_\Omega \varphi_j(y)\, dy \int_\Omega \frac{\varphi_j(z)\, dz}{|y - z|},$$

and λ_j and φ_j run through the eigenvalues and corresponding normalized eigenfunctions of the integral equation

$$\frac{1}{2\pi} \int_\Omega \frac{\varphi(y)}{|z - y|}\, dy = \lambda \varphi(z), \qquad z \in \Omega.$$

By taking the limit as $u \to \infty$ one obtains

$$P_x\{T > 0\} = \lim_{\delta \downarrow 0} \frac{1}{2\pi} \sum \frac{1}{\delta + \lambda_j} \int_\Omega \varphi_j(y)\, dy \int_\Omega \frac{\varphi_j(z)\, dz}{|y - z|}. \tag{28}$$

If one defines—as in [54]—the "penetration time of Ω" as

$$\tau = \tau_\Omega := \inf\left\{t\colon \int_0^t I_\Omega[B_s]\, ds > 0\right\},$$

then

$$U(x) := P_x\{T > 0\} = P_x\{\tau < \infty\}.$$

In [84] (see also [K180], Chapter 1) Kac showed for compact $\Omega \subset \mathbb{R}^3$ that $U(\cdot)$ is the potential of a measure concentrated on $\partial\Omega$, is harmonic on Ω^c, and equals 1 on the interior of Ω. Finally if y is a point of $\partial\Omega$ which is regular in the sense of Poincaré (this is stronger than y is regular for Ω), then

$$\lim_{\substack{x \to y \\ x \notin \Omega}} U(x) = 1.$$

This implies that for a nice region Ω, U equals the capacitary potential of Ω. The capacitary potential of Ω can be characterized as the "swept out" function

$$W(x) = \liminf_{y \to x} R(y),$$

where

(29) $R(y) = \inf\{v(y) \colon v \text{ is superharmonic on } \Omega,\, v \geq 1 \text{ on } \Omega \text{ and } v \geq 0 \text{ on } \mathbb{R}^d\}.$

R is harmonic on Ω^c and equals 1 on "most" points of Ω. It is now known ([55], Theorem 5.19) that W can be expressed probabilistically as

$$W(x) = P_x\{\sigma_\Omega < \infty\}.$$

Kac [84] raised the question of determining for which sets Ω it is true that U is the capacitary potential, or equivalently for what Ω is

$$P_x\{\tau_\Omega < \infty\} = P_x\{\sigma_\Omega < \infty\} \quad \text{for all } x.$$

He called such Ω *semiclassical*. Clearly this hinges on the relation between the penetration time τ_Ω and the hitting time σ_Ω. In [K55] Kac already conjectured (falsely as it turned out despite the footnote on page 211) that if $x \in \partial\Omega$ is regular for Ω then $P_x\{\tau_\Omega < \infty\} = 1$. Erdös and Dvoretzky gave a counterexample of a compact Ω which equals the closure of its interior and for which there exist $x \in \partial\Omega$ which are regular for Ω but with $P_x\{\tau_\Omega < \infty\} < 1$. These questions were the stimulus for Ciesielski's work [56] on potential theory. Among other results he gives the following analogue of (29) to characterize U. For compact Ω

$$\begin{aligned} U(y) = \inf\{v(y) \colon\ & v \text{ is superharmonic on } \Omega, \\ & v \geq 1 \text{ a.e. on } \Omega \text{ and } v \geq 0 \text{ on } \mathbb{R}^d\}. \end{aligned} \tag{30}$$

Note that this differs from (29) only in the addition of "a.e." Stroock [54] further developed Kac's potential theory and compared it with classical Newtonian potential theory. He considered for instance the exterior Dirichlet problem, that is the boundary problem (23), (24) for $\Omega = K^c$, K compact, with the additional requirement

$$\lim_{x \to \infty} u(x) = 0.$$

Stroock notes that one can write

$$T_K = \int_0^\infty I_{[\tau_{K,t} < \infty]}\, dt,$$

where

$$\tau_{K,t} = \text{inverse function of } T_{K,t} := \int_0^t I_K[B_s]\, ds.$$

He then proceeds to show that

$$u_0(x) := E_x\{f(B_{\tau_0});\ \tau_0 < \infty\}$$

corresponds to the solution of the exterior Dirichlet problem in Kac's approach. Note that, for $f \equiv 1$, u_0 becomes U. Stroock also shows that one will return to the classical solution by allowing for $\tau_{K,t}$ the inverses of more general additive functionals than $T_{K,t}$.

In the meantime much has happened in probabilistic potential theory. In particular since Hunt's pathbreaking papers [57] in 1957–58 this area has grown into a separate subfield of probability which deals with potentials of (almost) any Markov process, not just Brownian motion. We merely list the books [53] and [58] as a starting point for a reader to become acquainted with the work in probabilistic potential theory. It has become more accepted to work with the hitting time σ_Ω than with the penetration time τ_Ω. Perhaps the advantage of Kac's approach is that one can write down analytic expressions such as (28). If the set Ω is semiclassical this gives us an expression for the solution of the exterior Dirichlet problem which can be manipulated.

Kac also investigated the analogue of the logarithmic potential in $\mathbb{R}^2$. Apparently he started this work in the mid-fifties but did not publish it until [K180]. Traces of it can be found in [K71] and [K116]. In $\mathbb{R}^2$, B_t is recurrent and $T_\Omega = \infty$ for any set Ω with nonempty interior, so that the above U is identically 1 and useless. One has to replace U by the Green function at ∞, i.e., by

$$V(x) = \lim_{y \to \infty} \int_0^\infty q(t, y, x)\, dt,$$

where, in Kac's approach,

$$(31) \qquad q(t, y, x)\, dx = P_y\{T_{K,t} = 0, B_t \in dx\} = P_y\{\tau_K > t, B_t \in dx\}.$$

K is assumed compact throughout here.[5] Kac developed much of the recurrent potential theory using the penetration time in [K180], Chapter 2. The Dirichlet problem is treated, Robin's constant is defined, and expressions for V similar to (28) are derived. Historically the most interesting is the limit theorem

$$\begin{aligned} P_y\{T_{K,t} = 0\} &= P_y\{K \text{ has not been penetrated by time } t\} \\ &\sim \frac{2\pi V(y)}{\log t}, \qquad t \to \infty, \end{aligned}$$

for compact K. In the mid-fifties Kac did not have a proof of this but stated it as a conjecture (possibly suggested by his calculations in [K71]) to Hunt, who subsequently proved this result in the classical setting in [59]. That is, for compact K, Hunt proves that

$$(32) \qquad P_y\{\sigma_K > t\} \sim \frac{2\pi H(y)}{\log t}, \qquad t \to \infty,$$

where $H(y)$ is defined by replacing τ_K by σ_K in (31). In fact this is the special case $f \equiv 1$ of Hunt's asymptotic formula for

$$E_y\{f(B_{\sigma_K}); \sigma_K > t\}.$$

[5] Note that in [K180] the function U in Section II.7 is π times the function U of Section II.9.

The paper [59] in which Hunt proves Kac's conjecture (as well as the strong Markov property for processes with stationary independent increments) seems to have been Hunt's first paper in potential theory and may therefore have influenced him to write his famous papers [57]. The work of Kac and Hunt also started Spitzer on his investigations of recurrent potential theory for arbitrary random walks on $\mathbb{R}^d$ which culminated in his book [40]. Many detailed results can be derived for the special case of random walks which are not in the potential theory for general Markov processes. A similar comment applies to the potential theory for processes of stationary independent increments which was worked out by Port and Stone [60]. Also the potential theory of random walks and independent increment processes on groups has been investigated by many people; we list in [61] an early and a recent contribution together with a useful book.

We postpone to Section 4 discussion of [K71], which is closely related to potential theory.

4. Limit theorems and invariance principles. Undoubtedly the most influential limit theorems proved by Kac (with several coauthors) are those of the series [K38], [K39], [K41], [K47], [K48], and [K52]. Most of these papers deal with limit distributions for quantities of the form

$$\sum_1^n V(S_k), \tag{33}$$

where the $S_k = \Sigma_1^k X_i$ are the partial sums of independent identically distributed random variables X_i, or of the form

$$\int_0^t V(B_s)\,ds \tag{34}$$

for a Wiener process B_s. For instance, if one takes $V = I_{(0,\infty)}$ then (33) becomes the number of positive partial sums among the S_i, $1 \le i \le n$; for $V(x) = |x|$ one obtains

$$\sum_1^n |S_k|, \tag{35}$$

which played an important role in Kac's path to the Feyman–Kac formula. Not only were these papers important in the development of this important formula (discussed in Section 5), they also proved special cases of the "invariance principle" and even formulated explicitly (see [K38], [K39], and [K41]) the idea that many limit distributions for sums of the form (33) are the same for a wide choice of the distribution F of the underlying X_i. This idea was inspired by a remark of Uhlenbeck to Kac (see [1]). As a true physicist, Uhlenbeck seems to have liked "universal laws" and he suggested to Kac that the limit distribution of the absorption time of a random walk with two barriers should be independent of the distribution of the individual steps of the walk. Apparently it went unnoticed that Kolmogorov had already proved a special case of such a result in 1931 (see [62]). Kac used this idea to prove various limit laws by calculating the limit distribution for some special choice of F. The precise limit distribution obtained in the above examples is in many cases of less importance than the method. The name "invariance principle" seems to have been coined in [K41]. In [K39] it is also made explicit that the limit distribution should be the distribution of a functional of a continuous time process (a Gaussian one in the early examples and a Cauchy process in [K52]) and that the problem therefore was one of proving convergence of the distribution of a certain functional of the S_k to that of the same functional for a limit process of the (normalized) S_k. Actually this was also apparent in Kolmogorov's paper [62]; it appears once more in Doob's paper [63] on Kolmogorov–Smirnov statistics and in [K48] and [K52]. Doob could not prove the required functional limit theorem for his result, but Donsker [64], inspired by the papers [K38] and [K41] of Erdös and Kac, proved general functional limit theorems, which generalize [62] and also justify Doob's heuristics in [63]. With the later work of Prohorov and Skorokhod (see [65] for references) this exploded into a whole new field. Nowadays when proving a limit theorem, a functional limit theorem or invariance principle is the limit theorem of choice, and every graduate student in probability theory or operations research learns about these. There are excellent textbooks on the subject [65]. The techniques for proving invariance principles have become enormously sophisticated (see for instance [16] and [66], Chapter 11).

Two papers dealing with limit theorems not so much related to the invariance principle are [K54] and [K70]. The first deals with the number of changes of sign or small values among the partial sums, in the case where the X_i belongs to the domain of attraction of a stable law. The invariance principle does not apply in this case.[6] [K70] is concerned with the limit distribution of additive functionals of Markov chains X_n or Markov processes X_t of the form

$$\sum_1^n V(X_k) \quad \text{or} \quad \int_0^t V(X_s)\,ds, \tag{36}$$

respectively. If X is positive recurrent the limit distribution is usually normal (see for instance [68], Section I.16). The situation is quite different for a null recurrent process such as a random walk or Brownian motion. This is demonstrated by Darling and Kac with the following result in [K70].

THEOREM. *Let X_t be a Markov process with stationary transition probabilities*

$$p(t, x, E) = P\{X_{s+t} \in E \mid X_s = x\}$$

[6]Section 7 of [K70] is not correct. Let X_t have characteristic function $\phi(t) \sim 1 - |t|^\gamma$, and let X_t' equal aX_t with probability p and $X_t' = 0$ with probability $(1-p)$. Then for $pa^\gamma = 1$ also $\phi'(t) \sim 1 - |t|^\gamma$, where ϕ' is the characteristic function of X_t'. If $S_n' = \Sigma_1^n X_k'$ and N_n (N_n') is the number of changes in sign of $S_0, \dots, S_n$ ($S_0', \dots, S_n'$) then $n^{-1/\gamma} S_{nt}$ and $n^{-1/\gamma} S_{nt}'$ both converge to the same stable process, but $N_n' \sim N_{pn}$. The error in [K70] is that condition A was only checked for $x = 0$ and not uniformly for $x \in \{\xi\colon V(\xi) > 0\}$.

The result of [K54] for $\alpha = 1$ is also incorrect both in the original version as well as in the correction; see [67].

and assume that $V \geq 0$ *and that there exist a constant* $C > 0$ *and a function* $h(s)$ *tending to infinity as* $s \downarrow 0$, *such that*

$$\frac{1}{h(s)} \int_0^\infty dt \int e^{-st} p(t, x, dy) V(y) \to C \tag{37}$$

as $s \downarrow 0$, *uniformly on the set* $\{x\colon V(x) > 0\}$.

If

$$h(s) = \frac{s^\alpha}{L(1/s)} \quad \text{for some } 0 \leq \alpha < 1 \text{ and slowly varying } L, \tag{38}$$

then

$$\begin{aligned} &\lim P\left\{ \frac{1}{Ch(1/t)} \int_0^t V(X_s)\, ds < x \right\} \\ &= g_\alpha(x) := \frac{1}{\pi\alpha} \int_0^x \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j!} \sin(\pi\alpha j) \Gamma(\alpha j + 1) y^{j-1}\, dy. \end{aligned} \tag{39}$$

In fact Darling and Kac showed that under condition (37) either the limit distribution is concentrated on one point (corresponding to a law of large numbers with $h(s) = 1/s$) or h has to be of the form (38) and the limit distribution has to be as in (39). Of course this is again an invariance principle of some kind. The usual functional limit theorem for (36) was proved by Bingham [69]. Kasahara [70] considered the delicate case when $\alpha = 0$ in (38); this case requires introduction of a new time scale since otherwise the limit process is constant in time. Perforce most of the extensive further literature dealing with additive functionals of Markov processes is concerned with the case where V can be both positive and negative, in which case a whole new class of limit distributions can arise. The first paper in this direction was perhaps by Dorbushin [71] on the difference of the number of visits to two integers by a simple symmetric random walk. For a sample of other results in this direction see [72]; a number of these references also treat invariance principles even for a V which takes both signs.

In [K71] the tail of the distribution of (36) is studied when X_k is the sum of k i.i.d. random variables in the domain of (normal) attraction of a symmetric stable law of index γ, $1 \leq \gamma \leq 2$, and V is the indicator function of a bounded set Ω. For $1 < \gamma \leq 2$ Kac proves that

$$\lim n^{1-1/\gamma} P\left\{ \sum_1^n I_\Omega(X_k) = j \right\} \tag{40}$$

exists, and for $\gamma = 1$,

$$\lim \log n P\left\{ \sum_1^n I_\Omega(X_k) = j \right\} \tag{41}$$

exists. Since two-dimensional Brownian motion behaves (as far as recurrence times are concerned) very much like a Cauchy process, (41) for $j = 0$ may well have been the basis for the conjecture (32). Of course one can formulate analogues of (40) and (41) for any recurrent random walk and such analogues were proved in [73].

It is worth mentioning that [K48] uses the trick of simplifying the calculation of a limit distribution by using a Poisson distributed number of summands rather than a nonrandom number of summands. I do not know whether this is actually the first appearance of the trick, but by now its use is so standard (see [74] for a recent case) that by the definition of Polya and Szegö ([75], page VI) it has long since become a method instead of a trick.

5. Feynman–Kac formula. If one were to choose a single result for which Kac is most famous, it would surely be the Feynman–Kac formula. This formula in its original form (see [K47], [K55]) expresses the double Laplace transform of the additive functional

$$\int_0^t V(B_s)\,ds \tag{42}$$

for a Brownian motion B_t and nonnegative V (plus some other conditions on V) as

$$\int_0^\infty dt\, e^{-vt} E_0\left\{\exp\left[-u\int_0^t V(B_s)\,ds\right];\, a \le B(t) \le b\right\} = \int_a^b \psi(x)\,dx, \tag{43}$$

where ψ is the fundamental solution of the differential equation

$$\tfrac{1}{2}\psi''(x) - (v + uV(x))\psi(x) = 0, \qquad x \neq 0, \tag{44}$$

subject to the conditions

$$\psi(x) \to 0, \qquad |x| \to \infty, \qquad \psi'(x) \text{ is bounded on } x \neq 0 \text{ (with the bound depending on } v > 0), \quad \text{and} \quad \psi'(0+) - \psi'(0-) = -2.$$

This result can be applied in two directions. One can choose some V and use the equation (44) to solve for the Laplace transform of (42) (or at least find properties of the distribution of (42)); or one can try to use the representation (43) to derive properties of the solutions to (44). Kac in [K47], [K55], and [K177] gives examples of both kinds of applications. For instance if $V(x) = (1 + \operatorname{sgn}(x))/2$ then one recovers the arcsine law for the amount of time spent in the positive half-line. More generally, $V(x) = 1 - I_{[a,b]}(x)$ can be used to treat the amount of time spent outside the interval $[a, b]$. By taking the limit $u \to \infty$ in (43) one obtains an expression for the Laplace transform of

$$P_0\{B_s \text{ stays in } [a, b] \text{ up till } t\},$$

i.e., one obtains a handle on the two-sided absorption problem. Kac [K55] even started on this for stable processes (the Feynman–Kac formula generalizes to any other Markov process; see below) but the resulting replacement for (44) seemed too nasty to obtain explicit results. Nevertheless Widom and Getoor [76] soon afterwards obtained explicit expressions for the distribution of the exit *place* for stable processes (and less explicit expressions for the joint distribution of the exit time and place). For an application in the other direction Kac derives the asymptotic expressions for

$$\sum_{\lambda_j < \lambda} u_j^2(x) \quad \text{and} \quad \sum_{\lambda_j < \lambda} 1$$

where λ_j and u_j are the eigenvalues and corresponding eigenfunctions in L^2 of

$$\tag{45} \tfrac{1}{2}u''(x) - V(x)u(x) = -\lambda u(x)$$

for a V which tends to ∞ as $|x| \to \infty$. These expressions are closely related to the Weyl and Carleman results discussed in Section 2. In this application Kac also gives the expression

$$\lambda_1 = -\lim_{t \to \infty} \frac{1}{t} \log E_0\left\{\exp\left[-\int_0^t V(E_s)\, ds\right]\right\}$$

for the smallest eigenvalue of (45).

Kac's original example was $V(x) = |x|$, which he treated in [K39] in order to answer a question of Martin. For that example he approximated the Brownian motion by a Gaussian random walk and treated the random walk problem by an eigenfunction expansion of an integral equation (the latter is Kac's favorite tool, which is all pervasive in his work). Actually he could not solve the integral equation for the Gaussian case but instead dealt with the case where the variables have a density $\frac{1}{2}\exp(-|x|)$ and used Uhlenbeck's remark discussed in Section 4 that the limit (as the discretization interval goes to 0) should be the same for this as for the Gaussian case, i.e., a form of the invariance principle should hold. All these calculations masked the general Feynman–Kac formula and he did not discover this until [K47]. There he still used discretization, but this unnecessary device was no longer used in [K55] where the result was obtained by deriving an integral equation for the density at x of

$$E_0\left\{\exp\left[-u\int_0^t V(B_s)\, ds\right];\ B_t \in dx\right\}.$$

The way Kac arrived at his proof has been well described in several places (see [2], [K100], and [K182]). The proof has since been considerably streamlined and generalized; see [77], [78], Chapter 13.4, and [79]. Let L be the generator of a probability semigroup and $V(t, x)$, $f(t, x)$, and $g(x)$ measurable functions. These functions also have to satisfy some growth conditions to justify the argument in footnote 7. For simplicity we take V, f, and g bounded in x for t in compact time sets. These conditions can be relaxed considerably afterwards, but we shall not discuss this here (see [79]). Let $\{X_t\}_{t \geq 0}$ be a standard Markov process with generator L. Assume that $u(t, x)$ is a solution of

$$\tag{46} \frac{\partial u(t, x)}{\partial t} = Lu(t, x) + V(t, x)u(t, x) + f(t, x), \qquad u(0, x) = g(x),$$

which is sufficiently nice so that

$$\tag{47} \theta(t) := u(T - t, X_t) - \int_0^t \left\{ -\left[\frac{\partial u(r, X_s)}{\partial s}\right]_{r = T - s} + Lu(T - s, X_s)\right\} ds$$

is a martingale on $[0, T]$ with

$$E_x\Big\{\sup_{t\le T}|\theta(t)|\Big\} < \infty \quad \text{for all } x.$$

The Feynman–Kac result in its present version says that the only function u which can satisfy these conditions is given by[7]

$$(48)\qquad \begin{aligned} u(t,x) &= E_x\Big\{g(X_t)\exp\Big[\int_0^t V(t-s,X_s)\,ds\Big]\Big\} \\ &\quad + \int_0^t E_x\Big\{f(t-s,X_s)\exp\Big[\int_0^s V(t-r,X_r)\,dr\Big]\Big\}\,ds, \qquad t\le T. \end{aligned}$$

Note that (47) is indeed a martingale under very mild conditions by [66], Theorem 4.2.1, and the fact that

$$v(X_t) - \int_0^t Lv(X_s)\,ds$$

is a martingale if v is in the domain of L and $s \to E_x|Lv(X_s)|$ is locally integrable, by Dynkin's formula (see [78], page 133).

Applications of the Feynman–Kac formula are to be found throughout the literature (see [K55], [K100], [K177], [79], [80], and [81] for instance). Note that if $f \equiv 0$, then solving (46) amounts to constructing a semigroup with generator

[7] Here is a very general method to prove (48) which I learned from R. Holley; it basically follows Exercise 4.6.7 in [66]. Define

$$\eta(t) = \exp\Big[\int_0^t V(T-s,X_s)\,ds\Big]$$

and use Theorem 1.2.8 of [66] as well as

$$\theta(t) = u(T-t,X_t) + \int_0^t [V(T-s,X_s)u(T-s,X_s) + f(T-s,X_s)]\,ds$$

(by (46)) to obtain that

$$(*)\qquad \begin{aligned} &\theta(t)\eta(t) - \int_0^t \theta(s)\eta(ds) \\ &\quad = u(T-t,X_t)\exp\Big[\int_0^t V(T-s,X_s)\,ds\Big] \\ &\qquad + \exp\Big[\int_0^t V(T-s,X_s)\,ds\Big]\int_0^t [V(T-r,X_r)u(T-r,X_r) + f(T-r,X_r)]\,dr \\ &\qquad - \int_0^t u(T-s,X_s)V(T-s,X_s)\exp\Big[\int_0^s V(T-r,X_r)\,dr\Big]\,ds \\ &\qquad - \int_0^t ds\, V(T-s,X_s)\int_0^s \{V(T-r,X_r)u(T-r,X_r) \\ &\qquad + f(T-r,X_r)\}\,dr\exp\Big[\int_0^s V(T-q,X_q)\,dq\Big] \end{aligned}$$

is a martingale on $[0, T]$. Integration by parts of the last term and equating the expectations of $(*)$ with respect to P_x at times T and 0 gives (48) at $t = T$.

$L + V$, a perturbation of L. In quantum field theory (see [79], [80]) the Feynman–Kac formula (48) is apparently an important tool for deriving properties of this perturbed semigroup (and sometimes even for its construction).

We arbitrarily pick one more application to illustrate the usefulness of the Feynman–Kac formula. The Kolmogorov–Petrovsky–Piscounov equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + f(u(t,x)), \qquad x \in \mathbb{R}, \tag{49}$$

for a function f in $C^1[0,1]$ which satisfies

$$f(0) = f(1) = 0, \qquad f(u) > 0 \text{ on } (0,1) \quad \text{and}$$
$$f'(0) = 1, \qquad f'(u) \le 1 \text{ on } (0,1]$$

was introduced as a model for the spread of an advantageous gene in a one-dimensional medium. One is interested in the asymptotic behavior as $t \to \infty$ of the solutions of this nonlinear partial differential equation for different initial conditions. It turns out that in many cases (49) has asymptotically travelling wave solutions, i.e.,

$$u(t, x + m(t)) \to w(x), \qquad t \to \infty, \tag{50}$$

for some wave w centered at $m(t)$. In order to prove (50) and to find an asymptotic expansion for $m(t)$, Bramson [82] writes the solution by means of the Feynman–Kac formula as

$$u(t,x) = E_x\left\{u(0, B_t)\exp\left[\int_0^t V(t-s, B_s)\, ds\right]\right\} \tag{51}$$

with

$$V(t,x) = f(u(s,x))/u(s,x).$$

This V depends on the unknown function u, but nevertheless Bramson obtains sufficient information to find out which Brownian paths give the main contributions to (51) and to evaluate these asymptotically. This yields (50) with sharp information about m. Even though Uchiyama [83] obtained a good part of these results by purely analytic methods, the probability attack seems to give the sharpest results at present.

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