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CENTRAL LIMIT THEOREM FOR THE CONTACT PROCESS

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If $(\xi^A(t), t \geq 0)$ is the contact process with initial configuration A, $f\colon \mathscr{P}(\mathbb{Z}) \to \mathbb{R}$ is any cylindrical function and $|A| = \infty$, we prove a central limit theorem for $(f(\xi^A(t)), t \geq 0)$ when the rate of infection is supercritical.

Consider the contact process with initial configuration $A \subset \mathbb{Z}$ and rate of infection λ, $(\xi^A(t), t \geq 0)$ [3], [4], [6]. It is known that if $\lambda > \lambda_* = \sup\{\lambda > 0\colon \xi^{\mathbb{Z}}(t) \to \delta_\varnothing \text{ weakly as } t \to \infty\}$, μ is the nontrivial extremal invariant measure and $|A| = \infty$, then for any cylindrical $f\colon \mathscr{P}(\mathbb{Z}) \to \mathbb{R}$

$$T^{-1}\int_0^T f\big(\xi^A(t)\big)\,dt \to \int f\,d\mu$$

almost surely as $T \to \infty$.

Here we prove a corresponding central limit theorem:

THEOREM 1. *If $\lambda > \lambda_*$ and $|A| = \infty$, for any cylindrical $f\colon \mathscr{P}(\mathbb{Z}) \to \mathbb{R}$,*

$$T^{1/2}\left[T^{-1}\int_0^T f\big(\xi^A(t)\big)\,dt - \int f\,d\mu\right] \to_L N\big(0, \sigma_f^2\big)$$

as $T \to \infty$, with $0 \leq \sigma_f^2 < \infty$. ($\to_L$ means convergence in law).

We use theorems stated in [7], [8] and an estimation of the decay of time correlations for the contact process based on a result in [2]. This approach was motivated by similar methods used in [1].

First, we construct the family of processes $\{(\xi^A(t), t \geq 0)\colon A \subset \mathbb{Z}\}$ and a stationary process, all on the same probability space. Consider the following percolation structure on $\mathbb{Z} \times \mathbb{R}$. For each $i \in \mathbb{Z}$ consider three independent Poisson processes on $\mathbb{R}$: $(\vec{\tau}_n^{\,i})_{n \in \mathbb{Z}}$, $(\overleftarrow{\tau}_n^{\,i})_{n \in \mathbb{Z}}$, and $(\tau_n^{+\,i})_{n \in \mathbb{Z}}$ with parameters λ, λ, and 1, respectively. We suppose that for i varying in $\mathbb{Z}$ the processes are all independent. Now for $i \in \mathbb{Z}$ we draw arrows in $\mathbb{Z} \times \mathbb{R}$ from $(i, \vec{\tau}_k^{\,i})$ to $(i+1, \vec{\tau}_k^{\,i})$, $k, i \in \mathbb{Z}$. Secondly we draw arrows from $(i, \overleftarrow{\tau}_k^{\,i})$ to $(i-1, \overleftarrow{\tau}_k^{\,i})$, $k, i \in \mathbb{Z}$. Finally we put down $+$ signs at each of the points $(i, \tau_k^{+\,i})$, $k, i \in \mathbb{Z}$.

We call a segment linking (x, t) to (x, s) a time segment. We give it the orientation from (x, t) to (x, s) if $s > t$. Given two points (i, s) and (j, t) in the space time $\mathbb{Z} \times \mathbb{R}$, with $s < t$, we say that there is a path from (i, s) to (j, t) if there is a connected chain of oriented time segments and arrows, leading from (i, s) to (j, t), following the direction of the time segments and the arrows and without passing through a $+$ sign.

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Now, given $A \subset \mathbb{Z}$ we define the process $(\xi^A(t), t \geq 0)$ in the following way: $\xi^A(0) = A$, and for $t > 0$, $\xi^A(t) = \{j \in \mathbb{Z}$: there is a path from $(i, 0)$ to (j, t), for some $i \in A\}$.

Using the same percolation structure we define $(\zeta(t), t \in \mathbb{R})$ by $\zeta(t) = \{j \in \mathbb{Z}$: for any $s < t$ there is a site $i(s) \in \mathbb{Z}$ such that there is a path from $(i(s), s)$ to $(j,t)\}$. So $(\zeta(t), t \in \mathbb{R})$ is a strictly stationary Markov process. Also

PROPOSITION 1. *If $\lambda > \lambda_*$, the distribution of $\zeta(0)$ is μ.*

PROOF. We must prove that for any $A \subset \mathbb{Z}$,

$$P(\zeta(0) \cap A \neq \varnothing) = \mu(\eta\colon \eta \cap A \neq \varnothing).$$

For fixed A consider the events

$$E_N = \{\exists j \in A \text{ s.t. } \exists \text{ a path from } (i, -N) \text{ to } (j, 0) \text{ for some } i \in \mathbb{Z}\}.$$

So $(E_N, N \geq 1)$ is a decreasing sequence of events converging to $[\zeta(0) \cap A \neq \varnothing]$. But by the homogeneity of the Poisson processes, $P(E_N) = P(\xi^{\mathbb{Z}}(N) \cap A \neq \varnothing)$ and this converges to $\mu(\eta\colon \eta \cap A \neq \varnothing)$ as $N \to \infty$. $\square$

So $(\zeta(t), t \geq 0)$ is the contact process with random initial condition taken with distribution μ.

Now we prove

LEMMA 1. *If $\lambda > \lambda_*$, for any cylindrical $f\colon \mathscr{P}(\mathbb{Z}) \to \mathbb{R}$,*

$$T^{1/2}\left[\frac{1}{T}\int_0^T f(\zeta(t))\,dt - \int f\,d\mu\right] \to_L N\left(0, \sigma_f^2\right)$$

as $T \to \infty$, where

$$\sigma_f^2 = \int_{-\infty}^{+\infty} \operatorname{cov}(f(\zeta(0)), f(\zeta(s)))\,ds.$$

REMARK. If f is increasing, then $\operatorname{cov}(f(\xi(0)), f(\xi(s))) \geq 0$, and if f is also not constant, then $\operatorname{var}(f(\xi(0))) > 0$, so that $\sigma_f^2 > 0$ by continuity. We do not know if it is true that $\sigma_f^2 > 0$ whenever f is nonconstant.

PROOF. In what follows f is fixed and Λ is its support. We identify $\mathscr{P}(\mathbb{Z})$ with $\{0,1\}^{\mathbb{Z}}$ in the usual way. So we write for $\eta \in \mathscr{P}(\mathbb{Z})$, $x \in \mathbb{Z}$: $\eta(x) = 1$ if $x \in \eta$, $\eta(x) = 0$ if $x \notin \eta$. We use the notation $\xi(t, x)$ instead of $(\xi(t))(x)$.

We employ Theorem 3 in [7], so we first define an associated (FKG) system of random variables $(Y_k, k \in \mathbb{Z})$. In [7] these random variables are supposed to be real but this is not necessary. In fact the Y_k may assume values in any partially ordered measurable set, and $(Y_k, k \in \mathbb{Z})$ must be associated with respect to this partial order. We define Γ as the set of functions from $[0,1]$ to $\{0,1\}^{|\Lambda|}$ which are right continuous and have left limits, with the usual partial order: if $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \dots, b_n) \in \mathbb{R}^n$, $a \geq b$ iff $a_i \geq b_i$, $i = 1, \dots, n$. If $\phi_1, \phi_2 \in \Gamma$, $\phi_2 \geq \phi_1$ iff $\phi_2(x) \geq \phi_1(x)$, $\forall x \in [0,1]$. Let Γ be endowed with the Skorokhod topology. Next define the random variables Y_k, $k \in \mathbb{Z}$ assuming values on Γ by

$$Y_k(x) = (\zeta(k+x, i), i \in \Lambda)$$

The system of random variables $(Y_k, k \in \mathbb{Z})$ is associated; this means that for any $m \le n$, $g_1(Y_m, \dots, Y_n)$ and $g_2(Y_m, \dots, Y_n)$ are positively correlated whenever g_1 and g_2 are bounded, increasing and continuous functions from $\Gamma^{n-m+1} \to \mathbb{R}$. This fact is a consequence of Harris' theorem in [5] (Theorem 2.14 of Chapter II of [6]). First μ has positive correlations by Harris' theorem and then $\{\xi(t), t \in \mathbb{R}\}$ has positive correlations by a corollary to Harris' theorem: Corollary 2.21 in [6].

In fact the definition of associativity given above is a little less restrictive than the definition in [7], but it is not difficult to see that modifying their definition of D to be the set

$$\{F(Y_m, \dots, Y_n): m \le n,\ F \text{ is real, coordinatewise nondecreasing, bounded and continuous}\},$$

their Theorem 2 still holds.

For each $j \in \mathbb{Z}$ and each cylindrical f the random variable

$$X_j = \int_j^{j+1} f(\zeta(t))\, dt$$

is a bounded and continuous function of Y_j, almost surely well defined, and

$$N^{1/2}\left[N^{-1}\int_0^N f(\zeta(t))\,dt - \int f\,d\mu\right] = N^{-1/2}\sum_{j=0}^{N-1}\left(X_j - E(X_j)\right) := X^N.$$

As f is cylindrical it can be represented by $f = f_+ - f_-$ with f_+ and f_- being increasing functions. We define $f' = f_+ + f_-$ and

$$X_j' = \int_j^{j+1} f'(\zeta(t))\, dt.$$

Then $X_j' \gg X_j$; this means $X_j' - \mathrm{Re}(e^{i\alpha}X_j) \in D$ for all $\alpha \in \mathbb{R}$ [7], and Lemma 1 will follow since we prove that

$$\sum_{j \in \mathbb{Z}} \mathrm{cov}(X_0', X_j') < \infty.$$

This is a consequence of Lemma 2 below. □

LEMMA 2. *If $\lambda > \lambda_*$, for any cylindrical f: $\mathscr{P}(\mathbb{Z}) \to \mathbb{R}$ there are constants $C = C(\lambda) > 0$, $\gamma = \gamma(\lambda) > 0$, such that*

$$|\mathrm{cov}(f(\zeta(r)), f(\zeta(s)))| \le Ce^{-\gamma|s-r|}.$$

PROOF. Without loss of generality we consider $r = 0$, $s > 0$. We use the notation: given $A \subset \mathbb{Z}$, $\bar{A} = \{\eta \subset \mathbb{Z}: \eta \cap A \ne \varnothing\}$, $I_{\bar{A}}(\cdot)$ = indicator of $\bar{A}$. As any cylindrical function is a finite linear combination of these indicators it is enough to prove for any pair $A, B \subset \mathbb{Z}$, $|A| < \infty$, $|B| < \infty$, that

$$|\mathrm{cov}(I_{\bar{A}}(\zeta(0)), I_{\bar{B}}(\zeta(s)))| \le Ce^{-\gamma s}.$$

We will construct some auxiliary processes. First we define a dual percolation structure. Consider the percolation structure where $(\zeta(t), t \in \mathbb{R})$ is constructed.

Take the inverse time scale $l = s - t$ and invert the direction of the time segments so that they are oriented according to increasing l. Also invert the direction of the arrows. Using l as time scale and given two points (i, l_1), $(j, l_2) \in \mathbb{Z} \times \mathbb{R}$, with $l_1 < l_2$, we say that there is an inverted path from (i, l_1) to (j, l_2) if there is a connected chain of time segments and arrows leading from (i, l_1) to (j, l_2), following the new orientations of the time segments and arrows.

Now consider the processes $(Z_l, l \geq 0)$ and $(W_l, l \geq s)$ defined by (we are using l as time scale):

$$Z_l = \{j \in \mathbb{Z}: \text{there is an inverted path from } (i,0) \text{ to } (j,l) \text{ for some } i \in B\},$$

$$W_l = \{j \in \mathbb{Z}: \text{there is an inverted path from } (i,s) \text{ to } (j,l) \text{ for some } i \in A\}.$$

The processes $(Z_l, l \geq 0)$ and $(W_l, l \geq s)$ have, respectively, the same laws as $(\xi^B(t), t \geq 0)$ and $(\xi^A(t), t \geq 0)$, the first under the correspondence $l \to t$ and the second under $l \to t + s$.

We define the events

$$A' = [I_{\bar{A}}(\zeta(0)) = 1] = [W_l \neq \varnothing, \forall l \geq s],$$

$$B' = [I_{\bar{B}}(\zeta(s)) = 1] = [Z_l \neq \varnothing, \forall l \geq 0],$$

$$E = [Z_s \neq \varnothing].$$

Then

$$\begin{aligned}|\text{cov}(I_{\bar{A}}(\zeta(0)), I_{\bar{B}}(\zeta(s)))| &= |P(A' \cap B') - P(A')P(B')| \\ &= |P(A' \cap B' \cap E) + P(A' \cap B' \cap E^c) \\ &\quad - P(A')P(B' \cap E) - P(A') \cdot P(B' \cap E^c)| \\ &= |P(A' \cap B' \cap E) - P(A') \cdot P(B' \cap E)|.\end{aligned}$$

The events E and A' are independent, since the former depend on the Poisson processes defining the dual percolation structure during the time interval $0 < l \leq s$ and the latter depend on these processes during the time interval $l > s$. Then

$$\begin{aligned}|\text{cov}(I_{\bar{A}}(\zeta(0)), I_{\bar{B}}(\zeta(s)))| &= |P(A' \cap B' \cap E) \\ &\quad - P(A' \cap E) + P(A')P(E) - P(A')P(B' \cap E)| \\ &= |P(A') \cdot P(E \cap (B')^c) - P(A' \cap E \cap (B')^c)|.\end{aligned}$$

But

$$0 \leq P(A') \cdot P(E \cap (B')^c) \leq P(E \cap (B')^c),$$

$$0 \leq P(A' \cap E \cap (B')^c) \leq P(E \cap (B')^c).$$

Then

$$\begin{aligned}|\text{cov}(I_{\bar{A}}(\zeta(0)), I_{\bar{B}}(\zeta(s)))| &\leq P(E \cap (B')^c) \\ &= P(Z_s \neq \varnothing, Z_l = \varnothing \text{ for some } l > s) \leq Ce^{-\gamma s},\end{aligned}$$

where the last inequality is Theorem 5 in [2]. □

PROOF OF THEOREM 1. Define the random variable

$$\Theta_{f,A} = \inf\{t > 0: f(\zeta(s)) = f(\xi^A(s)), \forall s \ge t\}.$$

It is known (see the proof of Theorem 6 in [4]) that $\forall A$ s.t. $|A| = \infty$, $\Theta_{f,A} < \infty$ a.s. So given $\varepsilon > 0$,

$$P\left(T^{-1/2}\left|\int_0^T f(\xi^A(t))\,dt - \int_0^T f(\zeta(t))\,dt\right| > \varepsilon\right)$$
$$\le P\left(T^{-1/2}\Theta_{f,A} \cdot 2|f| > \varepsilon\right) \to 0 \quad \text{as } T \to \infty,$$

where $|f| = \sup_{B \subset \mathbb{Z}} |f(B)|$.

This combined with Lemma 1 finishes the proof. □

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