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RANDOM f-EXPANSIONS[1]

BY JON. AARONSON

University of Tel Aviv

We consider the asymptotic distribution properties of f-expansion digits. In particular, if $x = 1/\varphi_0(x) - 1/\varphi_1(x) - \cdots$ etc., then

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi_k \to 3 \quad \text{in measure.}$$

0. Introduction. In [12] Lévy showed that if $x = 1/\varphi_0(x) + 1/\varphi_1(x) + \cdots$ etc. ($\varphi_k \in \mathbb{N}$) is the continued fraction expansion of $x \in [0,1]$ and $\phi(n)\uparrow\infty$ is regularly varying with index $1/\gamma$ as $n\uparrow\infty$ $(0 < \gamma \le 1)$, then

$$\operatorname{dist}\frac{1}{b(n)}\sum_{k=1}^{n}\phi(\varphi_k) \to_{n\to\infty} \operatorname{dist} Y_\gamma,$$

where Y_γ is the nonnegative random variable satisfying

$$E(e^{-pY_\gamma}) = e^{-\Gamma(2-\gamma)p^\gamma} \quad\text{and}\quad b(n) \sim n\int_0^1(\phi([1/x]) \wedge b(n))\frac{dx}{1+x}\Big/\log 2.$$

For Z a random variable defined on the probability space $(\Omega, \mathscr{A}, P)$, P-dist Z denotes the measure on $\mathbb{R}$ defined by $(P\text{-dist}\, Z)\,(A) = P(\{\omega \in \Omega\colon Z(\omega) \in A\})$. When there is no danger of confusion, we write P-dist = dist. The convergence is the usual weak convergence of measures.

Indeed the random variables $\{\varphi_k\}$ are considered as defined on the probability space $([0,1], \mathscr{B}, \lambda)$ where $d\lambda(x) = dx$ and $\mathscr{B}$ denotes the Lebesgue measurable sets, and $\operatorname{dist}(1/b(n))\Sigma_{k=1}^n\phi(\varphi_k)$ may be considered to mean λ-$\operatorname{dist}(1/b(n))\Sigma_{k=1}^n\phi(\varphi_k)$. In fact the λ is omitted in the statement of Lévy's theorem because (see below) it may be replaced by any λ-absolutely continuous probability.

If λ is replaced by the equivalent Gauss measure μ $(d\mu(x) = (\log_e 2(1+x))^{-1}\,dx)$ then $(\varphi_0, \varphi_1, \dots)$ becomes an ergodic stationary process with $\{\varphi_k\}_{k=0}^\infty$ "almost" independent random variables. This is the "reason" for Lévy's result.

The purpose of this paper is to study the asymptotic distribution properties of other f-expansion digits (see [13] for an introduction to f-expansions). For example, if $x = 1/\varphi_0(x) - 1/\varphi_1(x) - \cdots$ etc. ($\varphi_k \in \mathbb{N}$), then there is no λ-absolutely continuous probability with respect to which $(\varphi_0, \varphi_1, \dots)$ is

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stationary; moreover, $\int_0^1 \varphi_0 \, d\lambda = \infty$, but nevertheless (see the corollary in Section 3):

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi_k \to 3 \quad \text{in measure.}$$

The continued fraction process $(\varphi_0, \varphi_1, \dots)$, where $x = 1/\varphi_0(x) + 1/\varphi_1(x) + \cdots$, considered with respect to the Gauss measure μ satisfies a very strong mixing condition [12]: There exist $L < \infty$, $0 < \theta < 1$ so that for every $m, n \in \mathbb{N}$, $A \in \sigma(\{\varphi_k\colon 0 \le k \le m\})$, $B \in \sigma(\{\varphi_k\colon k \ge m+n\})$,

$$|\mu(A \cap B) - \mu(A)\mu(B)| \le L\theta^n \mu(A)\mu(B).$$

We shall need a generalisation of Lévy's result (which follows from a more general result of Davis—see [7], Theorem 2) to stationary processes with this mixing property, which we call *continued fraction-mixing* (and which is stronger than the $*$-mixing of [4]).

A stationary process $\{\varphi_n\}_{n=0}^{\infty}$ defined on the probability space $(\Omega, \mathscr{A}, P)$ is called *continued fraction-* (c.f.-) *mixing* if there exist $0 \le \varepsilon_n < \infty$ $(n \in \mathbb{N})$, $\varepsilon_n \to 0$ as $n \to \infty$ such that for every $m, n \in \mathbb{N}$, $A \in \sigma(\{\varphi_k\colon 0 \le k \le m\})$, $B \in \sigma(\{\varphi_k\colon k \ge m+n\})$,

$$|P(A \cap B) - P(A)P(B)| \le \varepsilon_n P(A)P(B).$$

THEOREM 1 (Davis [7]). *Suppose that $\{\phi_n\}_{n=0}^{\infty}$ is a nonnegative* c.f.*-mixing stationary process defined on $(\Omega, \mathscr{A}, P)$, that $E(\phi_0) = \infty$, and that $L(x) = E(\phi_0 \wedge x)$ is regularly varying with index $1 - \gamma$ as $x \uparrow \infty$ $(0 < \gamma \le 1)$. Then*

$$\operatorname{dist} \frac{1}{b(n)} \sum_{k=1}^{n} \phi_k \to_{n \to \infty} \operatorname{dist} Y_\gamma,$$

where Y_γ is as before and $b(n) \sim nL(b(n))$.

Here dist $(1/b(n))\sum_{k=1}^{n}\phi_k$ can be considered with respect to any P-absolutely continuous probability. This is because of a well-known fact concerning the type of distributional convergence considered in this paper: If $(\varphi_0, \varphi_1, \dots)$ is a stochastic process defined on the (minimal) probability space $(\Omega, \mathscr{A}, P)$ whose shift map $(\varphi_0, \varphi_1, \varphi_2, \dots) \to (\varphi_1, \varphi_2, \dots)$ is P-nonsingular, conservative, and ergodic, $n_k \to \infty$, $d_k \to \infty$, and

$$P_0\text{-dist} \frac{1}{d_k} \sum_{j=0}^{n_k} \varphi_j \to_{k \to \infty} \operatorname{dist} Y,$$

where Y is some random variable and P_0 is some P-absolutely continuous probability, then

$$P_1\text{-dist} \frac{1}{d_k} \sum_{j=0}^{n_k} \varphi_j \to \operatorname{dist} Y$$

for every P-absolutely continuous probability P_1. (A proof of this fact can be abstracted from the proof of proposition 0 in [1].)

It is possible that Theorem 1 can be proved by the methods of [12]. As mentioned above, it follows from Davis' result which is proved by studying the order statistics of $(\phi_0, \phi_1, \dots)$. We prove Theorem 1 by studying the tower T over the shift of $(\phi_0, \phi_1, \dots)$ with height function $[\phi_0]$ (see Section 1), showing that the base of the tower has a very special property (and is what we call a *Darling–Kac set* for T) by means of the main lemma. We then prove Theorem 1 using the Darling–Kac distributional limit theorem [6] and an asymptotic renewal equation. All of this is done in Section 1.

In Section 2 we apply Theorem 1 and obtain a result (Theorem 2) on the distributional limit properties of some asymptotically stationary f-expansion digits. Here a theorem of Adler establishes c.f.-mixing.

In Section 3 we study nonstationary f-expansions whose shifts have infinite invariant measures (see Theorem 4, whence our advertised corollary). To do this we need to prove also that transformations satisfying Thaler's conditions [16] are pointwise dual ergodic [1]. Theorem 3 (using Thaler's theorem) identifies the asymptotic types of these transformations.

In fact, our methods yield functional distributional convergence. Functional versions of Theorems 1, 2, and 4 are stated in Section 3.

1. Darling–Kac sets and the main lemma. Let $(X, \mathscr{B}, m, T)$ be a conservative ergodic measure preserving transformation of a σ-finite nonatomic measure space and let $\hat{T}$ be the operator of $L^1(X, \mathscr{B}, m)$ dual to T,

$$\int_X \hat{T}fg\,dm = \int_X fg\circ T\,dm, \qquad f \in L^1(X, \mathscr{B}, m), \quad g \in L^\infty(X, \mathscr{B}, m).$$

A set $A \in \mathscr{B}$, $0 < m(A) < \infty$, will be called a *Darling–Kac (DK) set for T* if there are constants $a_n > 0$ such that

$$\left\| \frac{1}{a_n} \sum_{k=1}^{n} \hat{T}^k 1_A - 1 \right\|_{L^\infty(A)} \to_{n\to\infty} 0 \cdot$$

Clearly, in this case, $a_n \sim \sum_{k=1}^{n} m(A \cap T^{-k}A)/m(A)$ as $n \to \infty$. Any transformation T having DK sets is pointwise dual ergodic (see [1]); that is, there are constants a_n such that

$$\frac{1}{a_n} \sum_{k=1}^{n} \hat{T}^k f \to_{n\to\infty} \int_X f\,dm \quad \text{a.e. } \forall f \in L^1(X, \mathscr{B}, m).$$

(We prove this at the end of Section 1.)

It is not known whether every pointwise dual ergodic transformation has a DK set.

We are now in a position to state and prove the

MAIN LEMMA. *Let $(X, \mathscr{B}, m, T)$ be a conservative ergodic measure preserving transformation of a σ-finite nonatomic measure space.*

Suppose that $A \in \mathscr{B}$, $0 < m(A) < \infty$, and let φ be the return time function of T on A: $\varphi(x) = \min\{n \geq 1: T^n x \in A\}$ and $T_A x = T^{\varphi(x)}x$ the induced transformation on A.

If there is a measurable function $\phi: A \to \mathbb{R}$ *such that*

(i) $\varphi = f(\phi)$ *some* $f: \mathbb{R} \to \mathbb{N}$,

(ii) $\sigma(\{\phi \circ T_A^k: k \geq 0\}) = \mathscr{B} \cap A$,

(iii) *the process* $\{\phi \circ T_A^k\}_{k=0}^{\infty}$ *defined on* $(A, \mathscr{B} \cap A, m_A)$ $(m_A(B) = m(A \cap B)/m(A))$ *is c.f.-mixing, then*

A is a DK set for T.

PROOF. Let $0 \leq \varepsilon_n < \infty$ $(n \in \mathbb{N})$, $\varepsilon_n \to 0$ be such that for $m, n \in \mathbb{N}$, $B \in \sigma(\{\phi \circ T_A^j: 0 \leq j \leq m\})(= \sigma_0^m)$, $C \in \sigma(\{\phi \circ T_A^k: k \geq m + n\})(= \sigma_{m+n}^{\infty}\}$

$$|m_A(B \cap C) - m_A(B)m_A(C)| \leq \varepsilon_n m_A(B)m_A(C).$$

Setting $\varphi_n = \sum_{k=0}^{n-1}\varphi \circ T_A^k$, and denoting by $\hat{T}_A$ the operator on $L^1(A)$ dual to $T_A(\int_A \hat{T}_A fg\,dm = \int_A fg \circ T_A\,dm)$, we have that

$$\int_B \hat{T}^n 1_A\,dm = m(A \cap T^{-n}B) = \sum_{k=1}^{n} m([\varphi_k = n] \cap T_A^{-k}B)$$

$$= \int_B \sum_{k=1}^{n} \hat{T}_A^k 1_{[\varphi_k = n]}\,dm.$$

Hence

$$\hat{T}^n 1_A = \sum_{k=1}^{n} \hat{T}_A^k 1_{[\varphi_k = n]}$$

and

$$\sum_{k=1}^{n} \hat{T}^k 1_A = \sum_{k=1}^{n} \hat{T}_A^k 1_{[\varphi_k \leq n]}.$$

Suppose now that $m \in \mathbb{N}$ and $B \in \sigma_0^m$. Then for $n \geq 1$ and $C \in \mathscr{B} \cap A$ we have

$$\int_C \hat{T}_A^{n+m} 1_B\,dm_A = m_A(B \cap T_A^{-(n+m)}C) = (1 \pm \varepsilon_n)m_A(B)m_A(C)$$

since $T_A^{-(n+m)}C \in \sigma_{n+m}^{\infty}$. (Here, and throughout, $A = (1 \pm \varepsilon)B$ means $(1 - \varepsilon)B \leq A \leq (1 + \varepsilon)B$ when $A, B > 0$.) Thus,

$$T_A^{n+m} 1_B = m_A(B)(1 \pm \varepsilon_n) \text{ a.e. } \forall n \geq 1.$$

The proof of the lemma is based on this fact and goes via two inequalities.

INEQUALITY 1. *For every* $1 \leq p \leq n$

$$\sum_{k=1}^{n} \hat{T}^k 1_A \leq p + (1 + \varepsilon_{p+1}) \sum_{k=1}^{n} m_A(T^{-k}A) \quad a.e.\ on\ A.$$

INEQUALITY 2. *For every* $1 \leq p \leq q \leq n - 1$

$$\sum_{k=1}^{n} \hat{T}^k 1_A \geq \left(1 - \varepsilon_{p+1} - (1 + \varepsilon_1)^2 m_A([\varphi_p \geq q])\right)$$

$$\times \sum_{k=1}^{n} m_A(T^{-k}A) - (1 + \varepsilon_1)^2 q - p \quad a.e.\ on\ A.$$

PROOF OF INEQUALITY 1.

$$\begin{aligned}\sum_{k=1}^{n} \hat{T}^k 1_A &= \sum_{k=1}^{n} \hat{T}_A^k 1_{[\varphi_k \le n]} \le \sum_{k=1}^{n+p} \hat{T}_A^k 1_{[\varphi_k \le n]}\\ &\le p + \sum_{k=1}^{n} \hat{T}_A^{k+p} 1_{[\varphi_{k+p} \le n]}\\ &\le p + \sum_{k=1}^{n} \hat{T}_A^{k+p} 1_{[\varphi_k \le n]}\\ &\le p + (1+\varepsilon_{p+1}) \sum_{k=1}^{n} m_A([\varphi_k \le n]) \quad \text{since } [\varphi_k \le n] \in \sigma_0^{k-1}\\ &= p + (1+\varepsilon_{p+1}) \sum_{k=1}^{n} m_A(T^{-k}A). \square\end{aligned}$$

PROOF OF INEQUALITY 2.

$$\begin{aligned}\sum_{k=1}^{n} \hat{T}^k 1_A = \sum_{k=1}^{n} \hat{T}_A^k 1_{[\varphi_k \le n]} &\ge \sum_{k=1}^{n} \hat{T}_A^{k+p} 1_{[\varphi_{k+p} \le n]} - p\\ &= \sum_{k=1}^{n} \hat{T}_A^{k+p} 1_{[\varphi_k \le n]} - \sum_{k=1}^{n} \hat{T}_A^{k+p} 1_{[\varphi_k \le n < \varphi_{k+p}]}.\end{aligned}$$

As above

$$\sum_{k=1}^{n} \hat{T}_A^{k+p} 1_{[\varphi_k \le n]} \ge (1-\varepsilon_{p+1}) \sum_{k=1}^{n} m_A(T^{-k}A).$$

Also

$$\begin{aligned}\sum_{k=1}^{n} \hat{T}_A^{k+p} 1_{[\varphi_k \le n < \varphi_{k+p}]} &\le (1+\varepsilon_1) \sum_{k=1}^{n} m_A([\varphi_k \le n < \varphi_{k+p}])\\ &\qquad \text{since } [\varphi_k \le n < \varphi_{k+p}] \in \sigma_0^{k+p-1}\\ &= (1+\varepsilon_1) \sum_{k=1}^{n} \sum_{l=1}^{n} m_A(\varphi_k = l, \varphi_p \circ T_A^k > n-l)\\ &\le (1+\varepsilon_1)^2 \sum_{k=1}^{n} \sum_{l=1}^{n} m_A(\varphi_k = l) m_A(\varphi_p \ge n-l)\\ &\qquad \text{since } [\varphi_k = l] \in \sigma_0^{k-1}\\ &= (1+\varepsilon_1)^2 \sum_{k=1}^{n} \sum_{l=1}^{n-q} + (1+\varepsilon_1)^2 \sum_{k=1}^{n} \sum_{l=n-q+1}^{n}\\ &= I + II.\end{aligned}$$

Now

$$\begin{aligned}I &\le (1+\varepsilon_1)^2 m_A(\varphi_p \ge q) \sum_{k=1}^{n} \sum_{l=1}^{n} m_A(\varphi_k = l)\\ &= (1+\varepsilon_1)^2 m_A(\varphi_p \ge q) \sum_{k=1}^{n} m_A(T^{-k}A)\end{aligned}$$

and

$$
\begin{aligned}
II &\le (1+\varepsilon_1)^2 \sum_{k=1}^{n} m_A(n-q+1 \le \varphi_k \le n) \\
&\le (1+\varepsilon_1)^2 \left[\sum_{k=1}^{n} m_A(\varphi_k \le n) - \sum_{k=1}^{n-q} m_A(\varphi_k \le n-q) \right] \\
&= (1+\varepsilon_1)^2 \sum_{k=n-q+1}^{n} m_A(T^{-k}A) \le q(1+\varepsilon_1)^2.
\end{aligned}
$$

Putting all this together proves inequality 2. □

To finish the proof of the main lemma, let $\varepsilon > 0$. Choose $p \ge 1$ so that $\varepsilon_{p+1} < \varepsilon/4$. Then choose $q > p$ so that $(1+\varepsilon_1)^2 m_A([\varphi_p \ge q]) < \varepsilon/4$. Lastly, choose $n_0 > q$ so large that

$$
(1+\varepsilon_1)^2 q \Big/ \sum_{k=1}^{n} m_A(T^{-k}A) < \varepsilon/4 \quad \forall\, n \ge n_0.
$$

It follows from the inequalities that for $n \ge n_0$

$$
\left\| \sum_{k=1}^{n} \hat{T}^k 1_A \Big/ \sum_{k=1}^{n} m_A(T^{-k}A) - 1 \right\|_{L^\infty(A)} < \varepsilon. \square
$$

Next, some properties of DK sets. For $A \in \mathscr{B}$, $0 < m(A) < \infty$, and $p > 0$, let

$$
u_A(p) = \sum_{n=0}^{\infty} m(A \cap T^{-n}A)e^{-pn}, \qquad c_A(p) = \sum_{n=0}^{\infty} m\left(T^{-n}A - \bigcup_{k=0}^{n-1} T^{-k}A\right) e^{-pn}.
$$

The first property we shall need is

THE ASYMPTOTIC RENEWAL EQUATION. *If A is a DK set for T, then* $p u_A(p) c_A(p) \to_{p\downarrow 0} m(A)^2$.

PROOF. (taken from [1], page 229).
Let $A \in \mathscr{B}$ be a DK set for T, and let $\varphi: A \to \mathbb{N}$ be the first return time function of T on A. For $n \ge 1$

$$
\begin{aligned}
m(A \cap T^{-n}A) &= \sum_{k=0}^{n-1} m(A \cap T^{-k}(A \cap [\varphi = n-k])) \\
&= \int_A \sum_{k=0}^{n-1} 1_{[\varphi = n-k]} \hat{T}^k 1_A \, dm.
\end{aligned}
$$

Hence, for $p > 0$

$$
\sum_{n=1}^{\infty} m(A \cap T^{-n}A)e^{-pn} = \int_A \sum_{k=0}^{\infty} e^{-pk} \hat{T}^k 1_A e^{-p\varphi} \, dm
$$

and

$$\int_A \sum_{k=0}^{\infty} e^{-pk}\hat{T}^k 1_A(1 - e^{-p\varphi})\, dm = m(A).$$

Now, since A is a DK set

$$\sum_{k=0}^{\infty} e^{-pk}\hat{T}^k 1_A / u_A(p) \to_{p\downarrow 0} \frac{1}{m(A)} \quad \text{uniformly on } A \pmod{m},$$

whence

$$u_A(p)\int_A (1 - e^{-p\varphi})\, dm \to_{p\to 0} m(A)^2.$$

The proof of the asymptotic renewal equation is completed by the easy $\int_A(1 - e^{-p\varphi})\, dm \sim pc_A(p)$ as $p \downarrow 0$. □

The other property is

THE DARLING–KAC DISTRIBUTIONAL LIMIT THEOREM. *Suppose that A is a DK set for T and that $a(n) = \sum_{k=1}^{n} m_A(T^{-k}A)$ is regularly varying with index $\gamma \in [0,1]$. Then*

$$m_A\text{-dist}\frac{1}{a(n)}\sum_{k=1}^{n} 1_A \circ T^k \to \text{dist}\, Z_\gamma,$$

where

$$E(e^{zZ_\gamma}) = \sum_{n=0}^{\infty} \Gamma(1+\gamma)^n z^n / \Gamma(1+\gamma n),$$

and, for $0 < \gamma \le 1$, $Y_\gamma = [\Gamma(2-\gamma)\Gamma(1+\gamma)Z_\gamma]^{-1/\gamma}$.

This theorem (in honour of which DK sets were introduced) is proven in [6]. It is proved for pointwise dual ergodic transformations, without recourse to the existence of DK sets in [1]. A functional version of the theorem was proven by Bingham [3]. (This is applied in Section 3.) We take the opportunity to remark here that Bingham's functional theorem remains true under the assumptions 1 − 1(b) of [1], Section 1.

Using these results, we can now give a proof of Theorem 1. There is no loss of generality in assuming that

$$\varphi_n \in \mathbb{N} \text{ a.e. } \forall n \ge 0,$$

since

$$\left|\sum_{k=1}^{n}([\varphi_k] + 1) - \sum_{k=1}^{n}\varphi_k\right| \le n = o(b(n)) \quad \text{as } n \to \infty,$$

$$\Omega = \mathbb{N}^\infty = \{\omega = (\omega_0, \omega_1, \dots) : \omega_n \in \mathbb{N}, n \ge 0\}$$

and $\varphi_n(\omega) = \omega_n$. If not, define $\pi\colon \Omega \to \mathbb{N}^\infty$ by

$$\pi(\omega) = (\varphi_0(\omega), \varphi_1(\omega), \dots), \qquad \mathscr{A}_1 = \{A \subseteq \mathbb{N}^\infty, \pi^{-1}A \in \mathscr{A}\},$$

and

$$P_1\colon \mathscr{A}_1 \to [0,1] \quad \text{by } P_1(A) = P(\pi^{-1}A).$$

The map $S\colon \Omega \to \Omega$ defined by $S(\omega_0, \omega_1, \dots) = (\omega_1, \omega_2, \dots)$ is a measure preserving transformation and is known as the shift of $(\varphi_0, \varphi_1, \dots)$. It is not hard to show, using the c.f.-mixing of $(\varphi_0, \varphi_1, \dots)$, that S is ergodic.

The tower transformation over S with height function $\varphi = \varphi_0$ is defined on $X = \{(\omega, n)\colon \varphi(\omega) \geq n\}$ by

$$T(\omega, n) = \begin{cases} (\omega, n+1) & \text{if } \varphi(\omega) \geq n+1, \\ (S\omega, 1) & \text{if } \varphi(\omega) = n. \end{cases}$$

The transformation T preserves the measure m, defined on the σ-algebra $\mathscr{B}$, generated by sets of form $(A, n) = \{(\omega, n)\colon \omega \in A\}$ where $A \in \mathscr{A}$, $A \subseteq [\varphi \geq n]$ by $m(A, n) = P(A)$. The transformation T is also ergodic [11] and $m(X) = E(\varphi) = \infty$ [10]. Clearly, $T_{\overline{\Omega}}(\omega, 1) = (S\omega, 1)$ and the conditions of the main lemma are satisfied with $\phi = \varphi$, $f = id$.

Thus, as advertised in the introduction, $\overline{\Omega}$ *is a DK set for* T. Since $L(n) = E(\varphi \wedge n) = m(\bigcup_{k=1}^n T^{-k}\overline{\Omega})$ is regularly varying with index $1 - \gamma$, we have by Karamata's Tauberian theorem [8] that

$$c_{\overline{\Omega}}(p) \sim \Gamma(2-\gamma) L\left(\frac{1}{p}\right) \quad \text{as } p \downarrow 0.$$

The asymptotic renewal equation gives that

$$u_{\overline{\Omega}}(p) \sim \frac{1}{\Gamma(2-\gamma)} \left(\frac{1}{p}\right) \Big/ L\left(\frac{1}{p}\right) \quad \text{as } p \downarrow 0$$

which is regularly varying with index γ as $p \downarrow 0$; whence, again by Karamata's Tauberian theorem

$$a(n) = \sum_{k=1}^n m_{\overline{\Omega}}(T^{-k}\overline{\Omega}) \sim \frac{1}{\Gamma(1+\gamma)\Gamma(2-\gamma)} \frac{n}{L(n)} \quad \text{as } n \uparrow \infty.$$

This latter is regularly varying with index γ as $n \uparrow \infty$ and so by the Darling–Kac limit theorem

$$P\text{-dist} \frac{1}{a(n)} \sum_{k=1}^n 1_{\overline{\Omega}} \circ T^k \to \text{dist } Z_\gamma.$$

From this follows Theorem 1 by Proposition 1 of [1]. □

To conclude this section, we show that any conservative, ergodic, measure preserving transformation which has DK sets is pointwise dual ergodic. Let $(X, \mathscr{B}, m, T)$ be the transformation and let $A \in \mathscr{B}$, $0 < m(A) < \infty$.

Set $A_0 = A$ and $A_m = A \setminus \bigcup_{k=1}^{m} T^{-k}A$ for $m \geq 1$. It can be verified that for every $B \in \mathscr{B}$

(a) $$\sum_{n=0}^{\infty} m(A_n \cap T^{-n}B) = m(B),$$

(b) $$\sum_{n=0}^{N} m(A \cap T^{-n}B) = \sum_{k=0}^{N} \sum_{n=0}^{N-k} m\big(A \cap T^{-n}A_k \cap T^{-(n+k)}B\big),$$

which imply

(a′) $$\sum_{n=0}^{\infty} \hat{T}^n 1_{A_n} = 1 \quad \text{a.e. on } X,$$

(b′) $$\sum_{n=0}^{N} \hat{T}^n 1_A = \sum_{k=0}^{N} \hat{T}^k \left(1_{A_k} \sum_{n=0}^{N-k} \hat{T}^n 1_A \right) \quad \text{a.e. on } X.$$

If A is a DK set for T, then there are constants $a(n) \uparrow \infty$, $a(n) \sim a(n+1)$ such that

$$\sum_{n=0}^{N} \hat{T}^n 1_A \sim a(N) \quad \text{uniformly on } A \text{ as } N \to \infty.$$

From (a′) and (b′) it now follows that

$$\sum_{n=0}^{N} \hat{T}^n 1_A \sim \sum_{k=0}^{N} a(N-k) \hat{T}^k 1_{A_k} \sim a(N) \quad \text{a.e. on } X,$$

and the pointwise dual ergodicity of T follows from the Chacon–Ornstein theorem.

It can also be shown that a conservative ergodic measure preserving transformation $(X, \mathscr{B}, m, T)$ is pointwise dual ergodic iff there are sets $A, B \in \mathscr{B}$ of positive finite measure, and constants $a(n)$ so that

$$\frac{1}{a(n)} \sum_{k=0}^{n} \hat{T}^k 1_A \to 1 \quad \text{a.e. on } B.$$

2. f-expansions with finite invariant measures. Let f: $(1, \infty) \to (0,1)$ be onto and strictly monotone of class C^2. Given $x \in [0,1]$ one can find $\varphi_0, \varphi_1, \ldots \in \mathbb{N}$ so that

$$x = f\big(\varphi_0 + f\big(\varphi_1 + f\big(\varphi_2 + \cdots \text{ etc.}\big)\big)\big).$$

In this case we say that $(\varphi_0, \varphi_1, \ldots)$ is the f-$(+)$-expansion of x.

Alternatively, we may find $\varphi_0, \varphi_1 \ldots \in \mathbb{N}$ so that

$$x = f\big(\varphi_0 - f\big(\varphi_1 - f\big(\varphi_2 - \cdots \text{ etc.}\big)\big)\big)$$

in which case we say that $(\varphi_0, \varphi_1, \ldots)$ is the f-$(-)$-expansion of x.

Either way, we shall regard $(\varphi_0, \varphi_1, \ldots)$ as a stochastic process on the probability space $([0,1], \mathscr{B}, \lambda)$ and attempt to study the asymptotic distributional behaviour of sums $\sum_{k=0}^{n} \phi(\varphi_k)$ for ϕ: $\mathbb{N} \to \mathbb{R}_+$.

To do this, we use the shift map T on $(\varphi_0, \varphi_1, \dots)$ defined by $T(\varphi_0, \varphi_1, \dots) = (\varphi_1, \varphi_2, \dots)$. This shift map can be written explicitly as a map of $[0,1]$:

In the case of f-$(+)$-expansions, set

$$Tx = \left(\left(f^{-1}(x)\right)\right) \quad \text{and} \quad \varphi(x) = \left[f^{-1}(x)\right].$$

Then

$$x = f(\varphi(x) + Tx) = f\big(\varphi(x) + f(\varphi(Tx) + T^2x)\big) = \cdots \text{ etc.},$$

so

$$\varphi_k(x) = \varphi(T^k x).$$

In the case of f-$(-)$-expansions, set

$$Tx = 1 - \left(\left(f^{-1}(x)\right)\right) \quad \text{and} \quad \varphi(x) = \left[f^{-1}(x)\right] + 1,$$

then

$$x = f(\varphi(x) - Tx) = f\big(\varphi(x) - f(\varphi(Tx) - T^2x)\big) = \cdots \text{ etc.}$$

so again

$$\varphi_k(x) = \varphi(T^k x).$$

For example, if $f(x) = 1/x$, then for f-$(+)$-expansions

$$Tx = \left(\left(\frac{1}{x}\right)\right), \qquad \varphi(x) = \left[\frac{1}{x}\right],$$

and for f-$(-)$-expansions

$$Tx = 1 - \left(\left(\frac{1}{x}\right)\right), \qquad \varphi(x) = \left[\frac{1}{x}\right] + 1.$$

The maps T have the property that for every $n \in \mathbb{N}$

$$T\colon (f(n+1), f(n)) \to (0,1) \quad \text{resp. } T\colon (f(n), f(n+1)) \to (0,1)$$

is a C^2 diffeomorphism.

We shall need the following theorem which is due to Adler [2].

ADLER'S THEOREM. *Suppose that Λ is a countable set, $\alpha = \{I_a\colon a \in \Lambda\}$ is a collection of disjoint open subintervals of $[0,1]$ so that $U = \bigcup_{a\in\Lambda} I_a$ has full λ-measure in $[0,1]$, and suppose that $T\colon U \to [0,1]$ is such that*

(i) *For every $a \in \Lambda$, T is a C^2 diffeomorphism of I_a onto $(0,1)$.*

(ii) *For some $m \geq 1$ there is a $\lambda > 1$ such that $|T^{m\prime}(x)| \geq \lambda$ for every $x \in \bigcap_{k=0}^{m-1} T^{-k}U$.*

(iii) *There exists $M < \infty$ so that $|T''(x)/T'(x)^2| \leq M$ for every $x \in U$.*

Then there is a T-invariant probability measure $\mu \sim \lambda$ such that ess sup$|\log(d\mu/d\lambda)| < \infty$ *and there exist $L < \infty$ and $0 \leq \theta < 1$ such that for every $m, n \geq 1$, $A \in \sigma(\{T^{-k}\alpha\colon 0 \leq k \leq m\})$, $B \in \sigma(\{T^{-k}\alpha\colon k \geq m+n\})$*

$$|\mu(A \cap B) - \mu(A)\mu(B)| \leq L\theta^{\sqrt{n}}\mu(A)\mu(B).$$

It has also been proved that under these conditions, $d\mu/d\lambda$ has a continuous version. This was proved by Halfant [9] in case $|\Lambda| < \infty$, and in general by Thaler [16]. We remark here that under the same conditions, $d\mu/d\lambda$ in fact has an absolutely continuous version, with bounded derivative. To see this, examine Thaler's proof of the existence of a continuous version ([16], page 82).

Define

$$h_n = \sum_{k_1 \cdots k_n} \left| f'_{k_1 \cdots k_n} \right|.$$

Thaler shows that a subsequence of any subsequence of $g_n = (1/n)\sum_{k=0}^{n-1} h_k$ converges uniformly to h, the continuous version. But for every n, h_n is differentiable and

$$h'_n = \sum_{k_1 \cdots k_n} \operatorname{sgn} f'_{k_1 \cdots k_n} f''_{k_1 \cdots k_n} \quad \left(\operatorname{sgn} f'_{k_1 \cdots k_n} \text{ is constant}\right),$$

whence

$$\begin{aligned} |h'_n| &\le \sum_{k_1 \cdots k_n} |f''_{k_1 \cdots k_n}| \le K \sum_{k_1 \cdots k_n} |f'_{k_1 \cdots k_n}| \\ &\le Ke^K \sum_{k_1 \cdots k_n} \inf |f'_{k_1 \cdots k_n}| \le Ke^K \int_{[0,1]} \sum_{k_1 \cdots k_n} |f'_{k_1 \cdots k_n}| \, d\lambda \\ &= Ke^K. \end{aligned}$$

There is a subsequence $g'_{n_k} \to g$ weak $*$ in $L^\infty([0,1])$. Clearly, for this g: $|g| \le Ke^K$ and

$$h(x) = \text{const} + \int_0^x g(t)\, dt.$$

Using these results, we now extend Lévy's result to other f-expansion digits with finite invariant measures.

THEOREM 2. *Suppose that* f: $(1, \infty) \to (0,1)$ *is strictly monotone, onto, and of class* C^2.

Consider the f*-expansion* $(\varphi_0(x), \varphi_1(x), \dots)$ *of* $x \in [0,1]$ *(plus or minus), and suppose that the associated transformation* T *of* $[0,1]$ $[Tx = ((f^{-1}(x)))$ *or* $1 - ((f^{-1}(x)))]$ *satisfies the conditions of Adler's theorem.*

If ϕ: $\mathbb{N} \to \mathbb{R}_+$ *is such that* $\int_0^1 \phi \circ \varphi_0 \, d\lambda = \infty$ *and* $L(x) = \int_0^1 (\phi(\varphi_0(t)) \wedge x)\, dt$ *is regularly varying with index* $1 - \gamma$ $(0 < \gamma \le 1)$ *as* $x \uparrow \infty$, *and* $b(n) = nL(b(n))$, *then*

$$\operatorname{dist} \frac{1}{b(n)} \sum_{k=1}^{n} \phi(\varphi_k) \to_{n \to \infty} \operatorname{dist} cY_\gamma,$$

where

$$0 < c < \infty.$$

PROOF. It follows from Adler's theorem that there is a T-invariant measure $\mu \sim \lambda$ under which the stochastic process $(\varphi_0, \varphi_1, \dots)$ is c.f.-mixing. In order to apply Theorem 1, we must examine $L_1(x) = \int_0^1(\phi(\varphi_0(t)) \wedge x)\,d\mu(t)$. Let h be the continuous version of $d\mu/d\lambda$.

If f is monotone increasing, then $\varphi_0(t) = [f^{-1}(t)]$ or $[f^{-1}(t)] + 1$ tends to infinity as $t \uparrow 1^-$ whence $L_1(x) = \int_0^1(\phi(\varphi_0(t)) \wedge x)h(t)\,dt \sim h(1)L(x)$ as $x \uparrow \infty$.

Similarly if f is monotone decreasing, then $\varphi_0(t) \uparrow \infty$ as $t \downarrow 0$ and $L_1(x) \sim h(0)L(x)$ as $x \uparrow \infty$.

Either way, for $h_0 = h(0)$ or $h(1)$, $L_1(x) \sim h_0 L(x)$ is regularly varying with index $1 - \gamma$ as $x \to \infty$, and if $b_1(n) = h_0^{1/\gamma} b(n)$, then $b_1(n) \sim nL_1(b_1(n))$, whence by Theorem 1

$$\operatorname{dist} \frac{1}{b_1(n)} \sum_{k=1}^{n} \phi(\varphi_k) \to \operatorname{dist} Y_\gamma.$$

Thus

$$\operatorname{dist} \frac{1}{b(n)} \sum_{k=1}^{n} \phi(\varphi_k) \to \operatorname{dist} h_0^{1/\gamma} Y_\gamma. \square$$

Note that the calculation of the constant c in this theorem involves the calculation of certain values of the continuous T-invariant density.

EXAMPLES. $f(x) = 1/x^{1/\alpha}$, $\alpha \geq 1$. For f-$(+)$-expansions, $Tx = ((1/x^\alpha))$, and the conditions of Adler's theorem are satisfied with $m = 1$, $\lambda = \alpha$, and $M = \alpha + 1$ for $\alpha > 1$ and, by [2] $m = 2$, $\lambda = 4$, and $M = 2$ for $\alpha = 1$.

For f-$(-)$-expansions, $Tx = 1 - ((1/x^\alpha))$ and the conditions of Adler's theorem are satisfied for $\alpha > 1$ with $m = 1$, $\lambda = \alpha$, and $M = \alpha + 1$. For $\alpha = 1$ the conditions of Adler's theorem are not satisfied as $T(1) = 1$, $T'(1) = 1$, and indeed, a different treatment is needed.

3. f-expansions with infinite invariant measures. Here we study some f-expansions whose associated maps do not satisfy the conditions of Adler's theorem, and have infinite invariant measures. Maps of this type have been studied by Thaler in [15] and [16].

Suppose that Λ is a countable set, $\alpha = \{I_a\colon a \in \Lambda\}$ a collection of disjoint open intervals in $[0,1]$ whose union U has full measure in $[0,1]$. We consider maps $T\colon U \to [0,1]$ which map each I_a, C^2-diffeomorphically onto $(0,1)$ in such a way that for each a, T extends to a C^1 diffeomorphism of I_a onto $[0,1]$. We denote the inverse of $T\colon \bar{I}_a \to [0,1]$ by f_a and the unique point of $\bar{I}_a$ fixed by T, by x_a.

We also assume that

(i) For every a, $x \in \bar{I}_a$; $|(T)|_{\bar{I}_a})'(x)| \geq 1$ with equality only when $x = x_a$ and $T'(x_a) = 1$.

(ii) The set $\Lambda_1 = \{a \in \Lambda\colon T'(x_a) = 1\}$ is finite.

(iii) There is an $M < \infty$ so that $|T''(x)/T'(x)^2| \leq M$ for all $x \in U$.

(iv) For every $\varepsilon > 0$ there is a $\rho(\varepsilon) > 1$ such that $|T'(x)| \geq \rho(\varepsilon)$ for all $x \in U - \bigcup_{a \in \Lambda_1}(x_a - \varepsilon, x_a + \varepsilon) \cap I_a$.

(v) There is an $\varepsilon > 0$ such that for every $a \in \Lambda_1$, $f_a'(x)$ increases on $I_a \cap (x_a - \varepsilon, x_a)$ and decreases on $I_a \cap (x_a, x_a + \varepsilon)$.

THALER'S THEOREM. ([16]) *If T satisfies conditions* (i), (ii), (iii), (iv), *and* (v), *then*

(a) *There is an infinite σ-finite measure $m \sim \lambda$ with $m \cdot T^{-1} = m$, and*

$$\frac{dm}{d\lambda}(x) = h(x) \prod_{a \in \Lambda_1} (x - x_a)(x - f_a(x))^{-1} \quad \text{for } \lambda \text{ a.e. } x,$$

where h: $[0,1] \to \mathbb{R}_+$ is continuous (in fact, h is absolutely continuous).

(b) *T is a conservative rationally ergodic, exact endomorphism,*

(c) *T has minimal wandering rates, in the sense that there is an $L(n) \uparrow \infty$ such that*

$$m\left(\bigcup_{k=0}^{n-1} T^{-k}A\right) \sim L(n) \quad \text{as } n \to \infty$$

for any measurable set A bounded away from $\{x_a\colon a \in \Lambda_1\}$ [that is, $A \subseteq [0,1] \setminus \bigcup_{a \in \Lambda_1}(x_a - \varepsilon, x_a + \varepsilon)$ for some $\varepsilon > 0$].

(d) $$L(n) \sim \sum_{a \in \Lambda_1} c_a \sum_{k=1}^{n-1} \left(f_a^k(1) - f_a^k(0)\right),$$

where

$$c_a = h(x_a) \prod_{\substack{b \in \Lambda_1 \\ b \neq a}} (x_a - x_b)(x_a - f_b(x_a))^{-1}.$$

We shall need

THEOREM 3. *Let T satisfy the conditions of Thaler's theorem. Then T has DK sets, and indeed, the following sets are DK sets for T:*

(i) $I_a a \notin \Lambda_1$,

(ii) $I_a \cap T^{-1}I_b$, $a, b \in \Lambda_1$, $a \neq b$.

Hence, T is pointwise dual ergodic and if the asymptotic type of T is $a_n(T) = \sum_{k=1}^n u_k$, then

$$\sum_{n=0}^{\infty} u_n e^{-pn} \sim \left(\frac{1}{p}\right) \Big/ \sum_{n=0}^{\infty} \left(\sum_{a \in \Lambda_1} c_a \left| f_a^n(1) - f_a^n(0)\right| \right) e^{-pn} \quad \text{as } p \downarrow 0.$$

PROOF. We consider a transformation like Schweiger's [14] jump transformation:

Define a: $U \to \Lambda$ by $x \in I_{a(x)}$,

$$\phi\colon \bigcap_{n=0}^{\infty} T^{-n}U \to \mathbb{N}$$

by

$$\phi(x) = \begin{cases} 1 & \text{if } a(x) \in \Lambda_0 \text{ and/or } a(x) \neq a(Tx) \quad (\Lambda_0 = \Lambda \setminus \Lambda_1), \\ \min\{n \geq 1\colon a(T^{n+1}x) \neq a(x)\}, & a(x) \in \Lambda_1(T), \quad a(x) = a(Tx), \end{cases}$$

and $Sx = T^{\phi(x)}x$.

Clearly if $n \geq 1$ and $a(T^n x) \in \Lambda_0$ [or $a(T^n x) \neq a(T^{n+1}x)$] then $\phi(x) \leq n$. Moreover, in this case there is an $m \geq 1$ so that $S^m x = T^n x$. To see this, write $S^m x = T^{k(m)}x$ where $k(m) = \sum_{j=0}^{m-1}\phi(S^j x)$. If $S^m x \neq T^n x$ for every m there exists m_0 so that $k(m_0) < n < k(m_0 + 1)$, and setting $y = T^{k(m_0)}x$ we have that $a(T^{n-k(m_0)}y) \in \Lambda_0$ [or $a(T^{n-k(m_0)}y) \neq a(T^{n-k(m_0)+1}y)$] but $\phi(y) = k(m_0 + 1) - k(m_0) > n - k(m_0)$—a contradiction.

This means that for $a \in \Lambda_0$ (or $a \neq b$), the transformation induced by T on I_a (or $I_a \cap T^{-1}I_b$) is identical to that induced by S.

We show that $J = I_a (a \in \Lambda_0)$ or $I_a \cap T^{-1}I_b$ $(a \neq b)$, is a DK set for T by showing that $T_J = S_J$ satisfies the conditions of Adler's theorem. This is done by studying S, which, although not piecewise onto, does satisfy conditions (ii) and (iii) of Adler's theorem. (The idea of finding intervals J for which T_J satisfies the conditions of Adler's theorem seems to originate in [5].)

For $a \in \Lambda_0(T)$, S is a diffeomorphism of I_a onto (0,1). For $a \in \Lambda_1(T)$, $a \neq b \in \Lambda(T)$ and $n \geq 1$, setting $([a]_n, b) = (a, \ldots, a, b)$ where a appears n times and $I_{([a]_n, b)} = \bigcap_{k=0}^{n-1} T^{-k}I_a \cap T^{-n}I_b$, we have that S is a diffeomorphism of $I_{([a]_n, b)}$ onto $I_{(a, b)} = I_a \cap T^{-1}I_b$ when $n \geq 2$, and onto I_b when $n = 1$.

Under Thaler's assumptions, there exists $\lambda > 1$ so that $|T'(f_a^2(y))| \geq \lambda$ for $y \notin I_a$, $a \in \Lambda_1(T)$ and $|T'(y)| \geq \lambda$ for $y \in I_a$, $a \in \Lambda_0(T)$.

If $y \in I_a$, $a \in \Lambda_0$ or $y \in I_{(a,b)}$ $(a \neq b)$, then $|S'(y)| = |T'(y)| \geq \lambda$. If $y \in I_{([a]_{n+1}, b)}$ $(n \geq 1)$ $b \neq a \in \Lambda_1(T)$ then writing $x = TSy = T^{n+1}y \in I_b$ we have that $S'(y) = T^{n\prime}(y) = \prod_{k=0}^{n-1} T'(T^k y) = \prod_{k=1}^{n} T'(f_a^{k+1}(x))$, whence $|S'(y)| \geq |T'(f_a^2(x))| \geq \lambda$ since $x \notin I_a$. Thus S satisfies condition (ii) of Adler's theorem.

Clearly $|S''(x)/S'(x)^2| \leq M$ for $x \in I_a$, $a \in \Lambda_0$, or $x \in I_{(a,b)}$, $a \neq b$, $(Sx = Tx)$. Suppose $x \in I_{([a]_{n+1}, b)}$ some n, a, b, then $Sx = T^n x$ and

$$S''(x)/S'(x)^2 = \sum_{k=0}^{n-1} \left(T^{k\prime}(x)/T^{n\prime}(x)\right)\left(T''(T^k x)/T'(T^k x)\right),$$

whence

$$\begin{aligned} \left|S''(x)/S'(x)^2\right| &\leq M \sum_{k=0}^{n-1} \left|T^{k\prime}(x) T'(T^k x)\right| \big/ \left|T^{n\prime}(x)\right| \\ &= M \sum_{k=1}^{n} 1\big/\left|T^{n-k\prime}(T^k x)\right| \\ &= M \sum_{k=0}^{n-1} 1\big/\left|T^{k\prime}(T^{n-k}x)\right|. \end{aligned}$$

Setting $y = T^{n+1}x \in I_b$ we have $T^{n-k}(x) = f_a^{k+1}(y)$ and

$$\left|S''(x)/S'(x)^2\right| \leq M \sum_{k=0}^{n} \left|f_a^{k\prime}(y)\right|.$$

It is not hard to show, using the lemma on page 305 of [15], that $\sup_{a \in \Lambda_1} \sup_{x \in I_a} \sum_{n=0}^{\infty} |f_a^{n\prime}(x)| = A < \infty$, whence

$$\left|S''(x)/S'(x)^2\right| \leq MA = M_1 < \infty.$$

Next, suppose that $a \in \Lambda_0(T)$ (or $a \neq b$, $a \in \Lambda_1$) and set $J = I_a$ (or $I_a \cap T^{-1}I_b$). Then $T_J = S_J$: $J \to J$ is piecewise onto. Clearly $\inf_x|S_J'(x)| \geq \lambda > 1$ and an argument proposed by Adler in his afterword to [5] shows that $|S_J''(x)/S_J'(x)^2| \leq M_1\lambda/(\lambda - 1)$. Thus S_J satisfies the conditions of Adler's theorem and hence, if a^*: $I \to \Lambda(T_J)$ is defined by $x \in I_{a^*(x)}$, then the process $\{a^* \circ T_J^k\}_{k=0}^{\infty}$ is c.f.-mixing. The first return time function of T on J depends only on a^* and so by the main lemma, J is a DK set for T.

We have shown that T always has DK sets. Now suppose that A is a DK set for T and that A is bounded away from $\{x_a$: $a \in \Lambda_1\}$, as are I_a $(a \in \Lambda_0)$ and $I_a \cap T^{-1}I_b$ $(a \neq b)$.

We have that T is pointwise dual ergodic (hence rationally ergodic). Suppose the asymptotic type of T is given by $a_n(T) = \sum_{k=1}^n u_k$, $u_k \geq 0$. Then $\sum_{n=1}^{\infty} u_n e^{-pn} \sim u_A(p)/m(A)^2$ and so by the asymptotic renewal equation:

$$\sum_{n=0}^{\infty} u_n e^{-pn} \sim \left(\frac{1}{p}\right)\Big/ c_A(p)$$

$$\sim \left(\frac{1}{p}\right)\Big/ \sum_{n=0}^{\infty}\left(\sum_{a\in\Lambda_1} c_a|f_a^n(1) - f_a^n(0)|\right)e^{-pn}$$

by (c) of Thaler's theorem. □

REMARK. The maps T for which Thaler's theorem and Theorem 3 have been stated have been assumed to increase on intervals with critical fixed points, (that is, T increases on I_a for $a \in \Lambda_1$). This assumption can be dropped provided $T'(x)$, and $f_a'(x)$ are replaced by their absolute values in assumptions (i), (ii), and (v). Indeed the proof of the existence of DK sets for such transformations (the first part of Theorem 3) proceeds as written. Also, as mentioned in [16], if T is such a transformation, then T^2 satisfies the assumptions for Thaler's theorem (unmodified), so T^2 (and hence T) is conservative, exact and rationally ergodic. The asymptotic type of T can be calculated from that of T^2 by the relationship $a_n(T) \sim 2a_{[n/2]}(T^2)$.

We can now state and prove our results on the distributional convergence of f-expansions with infinite invariant measures.

THEOREM 4. *Suppose that* f: $(1,\infty) \to (0,1)$ *is strictly increasing* (*decreasing*), *onto*, C^2 *and* $|f'(x)| < 1$ *for* $x > 1$, $|f'(1)| = 1$, $f''(1) < 0$ (> 0), *and there is an* $M < \infty$ *so that* $|f''(x)/f'(x)| \leq M$ *for* $x \in [1,\infty)$.

If $(\varphi_0(x), \varphi_1(x), \dots)$ *is the* f-$(+)$-*expansion* (f-$(-)$-*expansion*) *of* $x \in (0,1)$, ϕ: $\mathbb{N} \to \mathbb{R}_+$ *is such that*

$$\phi(1) = 0\ (\phi(2) = 0) \quad \textit{and} \quad L(x) = \int_0^1 (\phi(\varphi_0(t)) \wedge x)\,dt$$

is regularly varying with index $1 - \gamma$ $(0 < \gamma \leq 1)$ *as* $x \uparrow \infty$ *and* $b(t) = tL(b(t))$, *then*

$$\operatorname{dist}\frac{1}{b(n/\log n)}\sum_{k=1}^{n}\phi(\varphi_k)\to\operatorname{dist} cY_\gamma\qquad(0<c<\infty).$$

PROOF. We shall prove the theorem for the case of f-$(+)$-expansions with f increasing. The other case [of f-$(-)$-expansions with f decreasing] is analogous.

The associated transformation for the f-$(+)$-expansion is $Tx=((f^{-1}(x)))$ and this satisfies the conditions of Thaler's theorem:

$$\Lambda=\mathbb{N},\qquad I_n=(f(n),f(n+1)),\qquad \Lambda_1=\{1\},\qquad f_1(x)=f(x+1),$$

and $x_1=0$.

There is a T-invariant σ-finite measure $\mu\sim\lambda$ whose density satisfies

$$\frac{d\mu}{d\lambda}(x)=h(x)\frac{x}{x-f_1(x)}=\frac{h_1(x)}{x},$$

where $h,h_1\colon[0,1]\to\mathbb{R}_+$ are continuous.

By Theorem 3 T is pointwise dual ergodic and $a_n(T)=\sum_{k=1}^{n}u_k$, where

$$\sum_{n=0}^{\infty}u_ne^{-pn}\sim\left(\frac{1}{p}\right)\Big/\sum_{n=0}^{\infty}h(0)f_1^n(1)e^{-pt}.$$

Now $v_n=f_1^n(1)\downarrow 0$ and $v_{n+1}=f_1(v_n)=v_n+(f_1''(0)/2)v_n^2+o(v_n^2)$ as $n\to\infty$, whence $v_n\sim\text{const}/n$.

Thus, by Karamata's Tauberian theorem (see [8]) $a_n(T)\sim cn/\log n$, and by Theorem 1 of [1]

$$\sum_{k=0}^{n-1}g\circ T^k/a_n(T)\to\int_X g\,d\mu\quad\text{in measure}\quad\forall g\in L^1.$$

Next let $I=(f(2),1)$ and $S_n(x)=\sum_{k=1}^{n}1_I(T^kx)$. Then $S_n/a_n(T)\to\mu(I)$ in measure $[0<\mu(I)<\infty]$ and, since $\phi(1)=0$

$$\sum_{k=1}^{n}\phi\circ\varphi\circ T^k=\sum_{k=1}^{S_n}\phi\circ\varphi\circ T_I^k\quad\text{on } I.$$

The induced transformation T_I is readily seen to be piecewise onto. Moreover, if S is the jump transformation considered in the proof of Theorem 3, then $T_I=S_I$. An argument identical to the first part of the proof of Theorem 3 shows that T_I satisfies the conditions of Adler's theorem. So, if $a\colon I\to\Lambda(T_I)$ is defined by $x\in I_{a(x)}$, then the process $\{a\circ T_I^k\colon k\geq 0\}$ is c.f.-mixing. Moreover, on I, φ depends only on a, hence the process $\{\varphi\circ T_I^k\colon k\geq 0\}$ is c.f.-mixing and by Theorem 1 if $b(n)\sim nL(b(n))$ then

$$\operatorname{dist}\frac{1}{b(n)}\sum_{k=1}^{n}\phi\circ\varphi\circ T_I^k\to\operatorname{dist} c_0Y_\gamma,\quad\text{where } 0<c_0<\infty.$$

To finish recall that

$$\sum_{k=1}^{n}\phi\circ\varphi\circ T^k=\sum_{k=1}^{S_n}\phi\circ\varphi\circ T_I^k$$

and

$$S_n/(n/\log n) \to c_1 \quad \text{in measure} \quad (0 < c_1 < \infty).$$

Let $p \ll \lambda|_I$, $p([0,1]) = 1$, and $\varepsilon > 0$.

Writing $\Sigma(n) = \sum_{k=1}^{n} \phi \circ \varphi \circ T_I^k$ and $m(n) = (1+\varepsilon)c_1(n/\log n)$, we have for $0 < x < \infty$

$$\begin{aligned} p\left(\sum_{k=1}^{n} \phi \circ \varphi \circ T^k \le xb(n/\log n)\right) &= p\left(\sum(S_n) \le xb(n/\log n)\right) \\ &\ge p\left(\sum(S_n) \le xb(n/\log n),\, S_n \le m(n)\right) \\ &\ge p\left(\sum(m(n)) \le xb(m(n)/(1+\varepsilon)c_1),\, S_n \le m(n)\right) \\ &\ge p\left(\sum(m(n)) \le xb(m(n)/(1+\varepsilon)c_1\right) - p(S_n > m(n)) \end{aligned}$$

Now, $p(S_n > m(n)) \to_{n\to\infty} 0$ and $b(m(n)/(1+\varepsilon)c_1) \sim b(m(n))/(1+\varepsilon)^{1/\gamma}c_1^{1/\gamma}$ as $n \to \infty$. Thus

$$\begin{aligned} \liminf_{n\to\infty} p\left(\sum_{k=1}^{n} \phi \circ \varphi \circ T^k \le xb(n/\log n)\right) &\ge \liminf_{n\to\infty} p\left(\sum(m(n)) \le xb(m(n))/(1+\varepsilon)^{1/\gamma}c_1^{1/\gamma}\right) \\ &= \operatorname{Prob}\left(c_0 Y_\gamma \le x/c_1^{1/\gamma}(1+\varepsilon)^{1/\gamma}\right), \end{aligned}$$

since dist Y_γ is continuous.

This is true for every $\varepsilon > 0$, so

$$\liminf p\left(\sum_{k=1}^{n} \phi \circ \varphi \circ T^k \le xb(n/\log n)\right) \ge \operatorname{Prob}\left(\left(c_0 c_1^{1/\gamma}\right) Y_\gamma \le x\right).$$

Similarly,

$$\begin{aligned} \limsup p\left(\sum_{k=1}^{n} \phi \circ \varphi \circ T^k \le xb(n/\log n)\right) &\le \operatorname{Prob}\left(c_0 c_1^{1/\gamma} Y_\gamma \le x\right) \end{aligned}$$

and

$$\operatorname{dist} \frac{1}{b(n/\log n)} \sum_{k=1}^{n} \phi \circ \varphi \circ T^k \to \operatorname{dist} c_0 c_1^{1/\gamma} Y_\gamma. \square$$

More precise information is available for our advertised example, the $1/x(-)$-expansion.

Corollary. *If $x = 1/\varphi_0(x) - 1/\varphi_1(x) - \cdots$ etc., then*

$$\frac{1}{n}\sum_{k=1}^{n} \varphi_k \to 3 \quad \textit{in measure}.$$

PROOF. Here $Tx = 1 - ((1/x))$, $\Lambda = \mathbb{N}$,

$$I_n = \left(\frac{1}{n+1}, \frac{1}{n}\right), \qquad \Lambda_1 = \{1\}, \qquad \varphi(x) = \left[\frac{1}{x}\right] + 1, \qquad I = (0, \tfrac{1}{2}),$$

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{1-x} \quad \text{and} \quad a_n(T) \sim n/\log n.$$

Hence $S_n/(n/\log n) \to_\lambda \mu(0, \frac{1}{2}) = \log 2$.
Set $\phi(k) = k - 2$, then $\phi \circ \varphi = 0$ on I_1. We have that

$$L(x) = \int_0^{1/2} (\phi \circ \varphi \wedge x)\, d\mu/\log 2 \sim \log x/\log 2.$$

Thus

$$b(t) \sim tL(b(t)) = t \log b(t)/\log 2$$
$$\Rightarrow b(t) \sim t \log t/\log 2.$$

From Theorem 1

$$\sum_{k=1}^{n} \phi \circ \varphi \circ T_I^k / b(n) \to 1 = Y_1 \quad \text{in measure,}$$

whence

$$\sum_{k=1}^{n} \phi \circ \varphi \circ T^k / b(n/\log n) \to (\log 2)^{-1} \quad \text{in measure.}$$

Now $b(n/\log n) \sim n/\log 2$. So

$$\frac{1}{n} \sum_{k=1}^{n} \phi \circ \varphi \circ T^k \to 1 \quad \text{in measure,}$$

since

$$\varphi = 2 + \phi \circ \varphi\colon \frac{1}{n} \sum_{k=1}^{n} \varphi \circ T^k \to 3 \quad \text{in measure.} \ \square$$

Now suppose that

$$f\colon [1, \infty) \to [0, 1) \text{ is increasing, } C^2,$$
$$f'(1) = 1, \qquad f'(x) < 1$$

for $x > 1$, $f'(x)$ is decreasing near 1, $f(1 + x) = x - cx^{p+1} + o(x^{p+1})$ when $x \downarrow 0$, $(c > 0,\ p > 1)$, and $|f''(x)/f'(x)| \le M$.

The associated transformation T for f-(+)-expansions is $Tx = ((f^{-1}(x)))$ which satisfies the conditions of Thaler's theorem with $\Lambda = \mathbb{N}$, $I_n = (f(n), f(n+1))$, $\Lambda_1 = \{1\}$. Here $(d\mu/d\lambda)(x) = h(x)/x^p$ where h is continuous. The transformation T is pointwise dual ergodic and $a_n(T) \sim \text{const}\, n^{1/p}$ whence

$$\operatorname{dist} \frac{1}{n^{1/p}} \sum_{k=1}^{n} g \circ T^k \to \operatorname{dist} \text{const} \int_X g\, d\mu\, Y_{1/p}^{-1/p} \quad \text{for } g \in L^1(\mu).$$

Given $\phi: \mathbb{N} \to \mathbb{R}_+$, $\phi(1) = 0$ one may ask about the distributional behaviour of

$$\sum_{k=1}^{n} \phi \circ \varphi \circ T^k.$$

THEOREM 5. *In this situation, if* $\int_0^1 (\phi \circ \varphi \wedge x)\, d\lambda = L(x)$ *is slowly varying as* $x \uparrow \infty$ *and if* $b(n) = nL(b(n))$ *then*

$$\operatorname{dist} \frac{1}{b(n^{1/p})} \sum_{k=1}^{n} \phi \circ \varphi \circ T^k \to \operatorname{dist} \operatorname{const}(Y_{1/p})^{-1/p}.$$

PROOF. As in the proof of Theorem 4, setting

$$I = (f(2), 1), \qquad \sum(n) = \sum_{k=1}^{n} \phi \circ \varphi \circ T_I^k \quad \text{and} \quad S_n = \sum_{k=1}^{n} 1_I \cdot T^k,$$

we have that

$$\{\phi \circ \varphi \circ T_I^k\} \text{ is a c.f. mixing process, and so by Theorem 1:}$$

$$\sum(n)/b(n) \to c \quad \text{in measure on } I$$

where $0 < c < \infty$.

Choose $p \ll \lambda$, a probability measure on I. On I, $\sum_{k=1}^{n} \phi \circ \varphi \circ T^k = \Sigma(S_n)$. Choose $x \in \mathbb{R}_+$ and $x'' < x < x'$. Then

$$p\left(\sum_{k=1}^{n} \phi \circ \varphi \circ T^k \le cxb(n^{1/p}) \right) = p\left(\sum(S_n) \le cxb(n^{1/p}) \right)$$
$$= p\left(\sum(S_n) \le cxb(n^{1/p}), S_n \le x'n^{1/p} \right)$$
$$+ p\left(\sum(S_n) \le cxb(n^{1/p}), S_n > x'n^{1/p} \right).$$

Now

$$p\left(\sum(S_n) \le cxb(n^{1/p}), S_n > x'n^{1/p} \right) \le p\left(\sum(x'n^{1/p}) \le cxb(n^{1/p}) \right)$$
$$\approx p\left(\sum(x'n^{1/p}) \le c\frac{x}{x'} b(x'n^{1/p}) \right) \to 0$$

because $\Sigma(n)/b(n) \to c$ in measure and $b(n)$ is regularly varying with index 1. Thus

$$p\left(\sum_{k=1}^{n} \phi \circ \varphi \circ T^k \le cxb(n^{1/p}) \right) \approx p\left(\sum(S_n) \le cxb(n^{1/p}), S_n \le x'n^{1/p} \right)$$
$$\le p\left(S_n \le x'n^{1/p} \right) \to \operatorname{Prob}\left((y_{1/p})^{-1/p} \le kx' \right),$$

where $0 < k < \infty$.

Thus

$$\limsup_{n \to \infty} p\left(\sum_{k=1}^{n} \phi \circ \varphi \circ T^k \le xb(n^{1/p}) \right) \le \operatorname{Prob}\left((Y_{1/p})^{-1/p} \le kx' \right).$$

Next,

$$\begin{aligned} p\left(\sum(S_n) \le cxb(n^{1/p})\right) &\ge p\left(\sum(S_n) \le cxb(n^{1/p}),\ S_n \le x''n^{1/p}\right) \\ &\ge p\left(\sum(x''n^{1/p}) \le cxb(n^{1/p}),\ S_n \le x''n^{1/p}\right) \\ &\ge p\left(S_n \le x''n^{1/p}\right) - p\left(\sum(x''n^{1/p}) > cxb(n^{1/p})\right) \\ &\to \mathrm{Prob}\left((Y_{1/p})^{-1/p} \le kx''\right), \end{aligned}$$

as

$$p\left(\sum(x''n^{1/p}) > cxb(n^{1/p})\right) \approx p\left(\sum(x''n^{1/p}) > c\frac{x}{x''}b(x'n^{1/p})\right) \to 0.$$

[This again because $\Sigma(n)/b(n) \to c$ in measure and $b(n)$ is regularly varying with index 1.]

Thus

$$\liminf p\left(\sum(S_n) \le cxb(n^{1/p})\right) \ge \mathrm{Prob}\left((Y_{1/p})^{-1/p} \le kx'\right).$$

The continuity of $\mathrm{Prob}((Y_{1/p})^{-1/p} \le x)$ now shows that $p(\Sigma_{k=1}^{n}\phi \circ \varphi \circ T^k \le cxb(n^{1/p})) \to \mathrm{Prob}((Y_{1/p})^{-1/p} \le kx)$. □

CONJECTURE. If $L(x) = \int_0^1(\phi \circ \varphi \wedge x)\,d\lambda$ is regularly varying with index $1 - \gamma$ $(0 < \gamma \le 1)$ and $b(t) = tL(b(t))$ then

$$\mathrm{dist}\frac{1}{b(n^{1/p})}\sum_{k=1}^{n}\phi \circ \varphi \circ T^k \to \mathrm{dist\,const}\, Y_\gamma(Y_{1/p})^{-1/\gamma p},$$

where Y_γ and $Y_{1/p}$ are independent.

Lastly, we note the following functional versions of Theorems 1, 2, 4, and 5, which follow from the main lemma and Bingham's theorem (mentioned above):

THEOREM 1′. *Under the conditions of Theorem* 1,

$$\mathrm{dist}\frac{1}{b(n)}\sum_{k=0}^{[nt]}\phi(\varphi_k) \to \mathrm{dist}\, Y_\gamma(t),$$

where $Y_\gamma(t)$ is the stable subordinator of index γ, and the convergence is that of finite dimensional distributions.

THEOREM 2′. *Under the conditions of Theorem* 2,

$$\mathrm{dist}\frac{1}{b(n)}\sum_{k=0}^{[nt]}\phi(\varphi_k) \to \mathrm{dist}\, cY_\gamma(t), \quad \textit{where } 0 < c < \infty.$$

THEOREM 4′. *Under the conditions of Theorem* 4,

$$\mathrm{dist}\frac{1}{b(n/\log n)}\sum_{k=0}^{[nt]}\phi(\varphi_k) \to \mathrm{dist}\, cY_\gamma(t), \quad \textit{where } 0 < c < \infty.$$

THEOREM 5′. *Under the conditions of Theorem* 5,

$$\operatorname{dist}\frac{1}{b(n^{1/p})}\sum_{k=1}^{[nt]}\phi\circ\varphi\circ T^k \to \operatorname{dist} cY_{1/p}^{-1}(t),$$

where $0 < c < \infty$ *and* $Y_{1/p}^{-1}$ *is the inverse of the stable subordinator of index* $1/p$.

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SCHOOL OF MATHEMATICAL SCIENCES
TEL AVIV UNIVERSITY
RAMAT AVIV
69978 TEL AVIV
ISRAEL