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ON THE AVERAGE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION

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There are many known asymptotic estimates of the expected number of zeros of a polynomial of degree n with independent random coefficients, for $n \to \infty$. The present paper provides an estimate of the expected number of times that such a polynomial assumes the real value K, where K is not necessarily zero. The coefficients are assumed to be normally distributed. It is shown that the results are valid even for $K \to \infty$, as long as $K = O(\sqrt{n})$.

1. Introduction. Let

$$P(x) = \sum_{i=0}^{n-1} a_i x^i, \tag{1.1}$$

where $a_0, a_1, a_2, \dots, a_{n-1}$ is a sequence of independent, normally distributed random variables with mathematical expectation zero and variance unity; let $N(a, b)$ be the number of real roots of the algebraic equation $P(x) = K$ in the interval (a, b), where K is a constant independent of x, and multiple roots are counted only once. Some years ago Kac ([4] and [5]) found that in the case of $K = 0$, the mathematical expectation of the number of real roots, $EN(-\infty, \infty)$, is asymptotic to $(2/\pi)\log(n)$. We know from the work of [2] that if the coefficients a_j ($j = 0, 1, 2, \dots, n-1$) are independent identically distributed random variables, belong to the domain of attraction of the normal law, and have zero means and $\text{Prob}(a_j = 0) > 0$, still we are able to get the same asymptotic relation. Further in case of $E(a_j) \neq 0$, they [3] proved that the asymptotic formula is exactly half of the previous case.

In this work it is proved:

THEOREM. *If the coefficients of* (1.1) *are independent, standard normal random variables, then for any constant K such that (K^2/n) tends to zero the mathematical expectation of the number of real roots of the equation $P(x) = K$ satisfies,*

$$EN(-1, 1) \sim (1/\pi)\log(n/K^2),$$

$$EN(-\infty, -1) = EN(1, \infty) \sim (2\pi)^{-1}\log(n).$$

2. Proof of the theorem. First we use the expected number of level crossings ([1], page 285) for our special equation $P(x) - K = 0$. The covariance and

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correlation coefficient of $P(x)$ and $P'(x)$ are

$$\gamma = \sum_{i=1}^{n-1} ix^{2i-1} \quad \text{and} \quad \rho = \gamma/(\alpha\beta)^{1/2}, \text{ respectively,}$$

where

$$\alpha = \sum_{i=0}^{n-1} x^{2i} \quad \text{and} \quad \beta = \sum_{i=0}^{n-1} i^2 x^{2i}.$$

Then we have

$$\begin{aligned} EN(a,b) = \int_a^b &(\Delta^{1/2}/\alpha)\phi(K\alpha^{-1/2})\big[2\phi(K\gamma\alpha^{-1/2}\Delta^{-1/2}) \\ &+K\gamma\alpha^{-1/2}\Delta^{-1/2}\{2\Phi(K\gamma\alpha^{-1/2}\Delta^{-1/2}) - 1\}\big]\,dx, \end{aligned} \tag{2.1}$$

where

$$\Delta = \alpha\beta - \gamma^2.$$

Then since $\Phi(x) = \frac{1}{2} + (2\pi)^{-1/2}\text{erf}(x)$ from (2.1) we can get the extended Kac–Rice formula [6],

$$\begin{aligned} EN(a,b) &= \int_a^b \big[\Delta^{1/2}/(\pi\alpha)\exp(-\beta K^2/2\Delta) \\ &\qquad + (|K|\gamma\sqrt{2}\,\alpha^{-3/2}/\pi)\exp(-K^2/2\alpha)\text{erf}\{|K|\gamma/\sqrt{2\alpha\Delta}\}\big]\,dx \\ &= \int_a^b I(x)\,dx. \end{aligned} \tag{2.2}$$

Since a_j and $-a_j$ $(j = 0,1,2,\ldots,n-1)$ both have the standard normal distribution, $EN(0,1) = EN(-1,0)$ and $EN(1,\infty) = EN(-\infty,-1)$.

Now we find the asymptotic relation for $EN(0,1)$ as $n \to \infty$. Since

$$\begin{aligned} \gamma &= \{(n-1)x^{2n+1} - nx^{2n-1} + x\}(1-x^2)^{-2} \\ &= x(1-x^{2n})(1-x^2)^{-2} - nx^{2n-1}(1-x^2)^{-1} \end{aligned}$$

for $0 \le x \le 1 - 1/n$ we have

$$\gamma \le x(1-x^{2n})(1-x^2)^{-2},$$

so

$$\gamma/(\alpha^{3/2}) \le x(1-x^{2n})^{-1/2}(1-x^2)^{-1/2} \le x(1-e^{-2})^{-1/2}(1-x^2)^{-1/2}.$$

On the other hand, for $1 - 1/n \le x \le 1$ we have

$$\gamma = \sum_{i=0}^{n-1} ix^{2i-1} \le (n/x)\sum_{i=0}^{n-1} x^{2i},$$

so in this range of x and for all sufficiently large n,

$$\begin{aligned}\gamma/(\alpha^{3/2}) &\le (n/x)(1-x^2)^{1/2}(1-x^{2n})^{-1/2}\\ &\le (n/x)\{1-(1-1/n)^2\}^{1/2}\{1-(1-1/n)^{2n}\}^{-1/2}\\ &\le (2n^{1/2}/x)(1-e^{-2})^{-1/2}.\end{aligned}$$

Hence, since $\operatorname{erf}(x) \le 1$,

$$\begin{aligned}&\int_0^1 \gamma|K|\sqrt{2}/(\pi\alpha^{3/2})\exp(-K^2/2\alpha)\operatorname{erf}\{|K|\gamma/\sqrt{2\alpha\Delta}\}\,dx\\ &\le |K|(2\pi)^{-1/2}(1-e^{-2})^{-1/2}\int_0^{1-1/n} x(1-x^2)^{-1/2}\\ &\qquad\times\exp\{-K^2(1-x^2)/(1-x^{2n})\}\,dx\\ &\qquad+2(2\pi)^{-1/2}|K|\sqrt{n}(1-e^{-2})^{-1/2}\exp\{-K^2/(2n)\}\int_{1-1/n}^1 (1/x)\,dx \qquad (2.3)\\ &\le |K|(2\pi)^{-1/2}(1-e^{-2})^{-1/2}\int_0^{1-1/n} x(1-x^2)^{-1/2}\exp\{-K^2(1-x^2)\}\,dx\\ &\qquad-2(2\pi)^{-1/2}|K|\sqrt{n}(1-e^{-2})^{-1/2}\exp\{-K^2/(2n)\}\log(1-1/n)\\ &\le (2\sqrt{2})^{-1}(1-e^{-2})^{-1/2}+4|K|(2\pi n)^{-1/2}(1-e^{-2})^{-1/2}\exp\{-K^2/(2n)\}.\end{aligned}$$

Also Kac ([4], page 318) obtained

$$\begin{aligned}\Delta &= \{x^{4n}-n^2x^{2(n+1)}+2(n^2-1)x^{2n}-n^2x^{2(n-1)}+1\}/(x^2-1)^4\\ &= \left[(1-x^{2n})^2\{1-n^2x^{2(n-1)}(1-x^2)^2(1-x^{2n})^{-2}\}\right]/(1-x^2)^4 \qquad (2.4)\\ &= \{1-h(x)^2\}(1-x^{2n})^2/(1-x^2)^4,\end{aligned}$$

where

$$h(x) = nx^{n-1}(1-x^2)/(1-x^{2n}), \qquad (2.5)$$

and since

$$\begin{aligned}\beta = \sum_{i=0}^{n-1} i^2x^{2i} &= \{-n^2x^{2n-2}(1-x^2)^2-2nx^{2n}(1-x^2)\\ &\qquad+(1+x^2)(1-x^{2n})\}(1-x^2)^{-3}, \qquad (2.6)\end{aligned}$$

it follows that

$$\begin{aligned}\beta/\Delta &= (1-x^2)\{(1-x^{2n})^2-n^2x^{2n-2}(1-x^2)^2\}^{-1} \qquad (2.7)\\ &\qquad\times\{-n^2x^{2n-2}(1-x^2)^2-2nx^{2n}(1-x^2)+(1+x^2)(1-x^{2n})\}.\end{aligned}$$

But for $0 \le x \le 1 - 1/n$ and all sufficiently large n we have

$$\operatorname{Max}\left\{x^{2n-2}(1-x^2)^2\right\} = 4/(n^2e^2)\{1 + O(1/n)\},$$

(2.8)

$$\operatorname{Max}\{x^{2n-2}(1-x^2)\} = 2/(ne^2)\{1 + O(1/n)\}.$$

Then from (2.7) in this range of x and for all sufficiently large n we have

$$\beta/\Delta > (1 - 5e^{-2})(1 - x^2). \tag{2.9}$$

On the other hand, by [4], page 319, $\Delta^{1/2}/\alpha = \{1 - h(x)^2\}^{1/2}/(1 - x^2)$ and for $0 \le x < 1$ satisfies the following inequalities,

$$\Delta^{1/2}/\alpha < (2n-1)^{1/2}/(1-x)^{1/2} \quad \text{and} \quad \Delta^{1/2}/\alpha \le (1-x^2)^{-1}. \tag{2.10}$$

Let $\lambda = (1 - 5e^{-2})K^2$. Then from (2.9) and (2.10) we have

$$\begin{aligned}
&\int_0^{1-1/n}(\Delta^{1/2}/\alpha)\exp\{-K^2\beta/(2\Delta)\}\,dx \\
&\quad\le \int_0^{1-1/n}(1-x^2)^{-1}\exp\{-\lambda(1-x^2)\}\,dx \\
&\quad\le \int_0^{1-1/n}(1-x^2)^{-1}\{1+\lambda(1-x^2)\}^{-1}\,dx \\
&\quad\le \int_0^{1-1/n}\left[(1-x^2)^{-1} - \lambda\{1+(1-x^2)\}^{-1}\right]dx \\
&\quad= \tfrac{1}{2}\log\left(\frac{2-1/n}{1/n}\right) - \tfrac{1}{2}(1-1/\lambda)^{-1/2}\log\left\{\frac{(1+1/\lambda)^{1/2}+1-1/n}{(1+1/\lambda)^{1/2}-1+1/n}\right\} \\
&\quad= \tfrac{1}{2}\log(n) + \tfrac{1}{2}\log(2-1/n) - \tfrac{1}{2}(1+1/\lambda)^{-1/2}\log(\lambda) \\
&\qquad - \tfrac{1}{2}(1+1/\lambda)^{-1/2}\log(4-1/n) \\
&\quad\le \tfrac{1}{2}\log(n/K^2) + 0.27.
\end{aligned} \tag{2.11}$$

Also from (2.10) we have

$$\begin{aligned}
\int_{1-1/n}^{1}(\Delta^{1/2}/\alpha)\exp\{-K^2\beta/(2\Delta)\}\,dx &\le \int_{1-1/n}^{1}(\Delta^{1/2}/\alpha)\,dx \\
&\le \int_{1-1/n}^{1}(2n-1)^{1/2}(1-x)^{-1/2}\,dx \\
&\le 2(2-1/n)^{1/2}.
\end{aligned} \tag{2.12}$$

Finally from (2.2), (2.3), (2.11), and (2.12) we have

$$EN(0,1) < (2\pi)^{-1}\log(n/K^2) + 1.1. \tag{2.13}$$

In order to obtain a lower estimate for $EN(0,1)$ from (2.7) and (2.8), and for $0 \le x \le 1 - 1/n$ we have,

$$\begin{aligned}\beta/\Delta &= (1-x^2)(1-x^{2n})\left\{(1-x^{2n}) - n^2x^{2n-2}(1-x^2)^2\right\}^{-1}\\ &\quad \times\left\{-n^2x^{2n-2}(1-x^2)^2/(1-x^{2n})\right.\\ &\quad \left. -2nx^{2n}(1-x^2)/(1-x^{2n}) + (1+x^2)\right\}\\ &\le 2(1-x^2)\left\{(1-e^{-2})^2 - 4e^{-2}\right\}^{-1} < 9.7(1-x^2)\end{aligned} \tag{2.14}$$

for all sufficiently large n.

Now let $\lambda' = 9.7K^2$ and $t = 1 - x$. Then from (2.14) we have

$$\begin{aligned}EN(0,1) &\ge \int_0^{1-1/n} \Delta^{1/2}/(\pi\alpha)\exp\{-K^2\beta/(2\Delta)\}\,dx\\ &\ge (2\pi)^{-1}\int_{1/n}^1 t^{-1}\exp(-\lambda' t)\,dt\\ &= (2\pi)^{-1}\log(n) - (2\pi)^{-1}\int_0^{\lambda'}(1-e^{-t})/t\,dt\\ &\quad +(2\pi)^{-1}\int_0^{\lambda'/n}(1-e^{-t})/t\,dt.\end{aligned} \tag{2.15}$$

Since, by hypothesis, $(\lambda'/n) \to 0$ it follows that the last integral is $(\lambda'/n) + O(\lambda'^2/n^2)$ and also,

$$\begin{aligned}\int_0^{\lambda'}(1-e^{-t})/t\,dt &= \int_0^1(1-e^{-t})/t\,dt + \int_1^{\lambda'}(1-e^{-t})/t\,dt\\ &\le \int_0^1(1-e^{-t})/t\,dt + \log(\lambda') + O(1/\lambda') + O(\lambda'/n)\\ &< \log(\lambda') + 1\end{aligned} \tag{2.16}$$

for all sufficiently large n. Then from (2.15) and (2.16) we have

$$EN(0,1) \ge (2\pi)^{-1}\log(n/K^2) - 0.53 \tag{2.17}$$

for all sufficiently large n. So from (2.13) and (2.17) we have the asymptotic formula

$$EN(0,1) \sim (2\pi)^{-1}\log(n).$$

Now we shall find the asymptotic relation for $EN(1,\infty)$. By putting $y = 1/x$ we have

$$\int_1^\infty I(x)\,dx = \int_1^\infty I(1/y)y^{-2}\,dy.$$

In this case we have

$$\gamma(x) = \sum_{i=1}^{n-1} ix^{2i-1} < (n/x)\sum_{i=1}^{n-1} x^{2i} = (n/x)(x^{2n}-1)/(x^2-1)$$

and so for $x \in (1, \infty]$ we have

$$\gamma(x)\{\alpha(x)\}^{-3/2} < (n/x)(x^2-1)^{1/2}(x^{2n}-1)^{-1/2}$$
$$< ny^n(1-y^2)^{1/2}(1-y^{2n})^{-1/2}$$
$$= ny^n\{\alpha(y)\}^{-1/2}.$$

Hence

$$(2.18)\quad \begin{aligned} &\left(|K|\sqrt{2}/\pi\right)\int_1^{\infty}\gamma\alpha^{-3/2}\exp\{-K^2/(2\alpha)\}\operatorname{erf}\left(\gamma|K|/\sqrt{2\alpha\Delta}\right)dx \\ &\le \left(|K|/\sqrt{2\pi}\right)\int_0^1\gamma(y)\alpha(y)^{-3/2}y^{-2}\,dy \\ &\le \left(|K|/\sqrt{2\pi}\right)\int_0^{1-1/\sqrt{n}}ny^{n-3}\,dy \\ &\quad+\left(|K|/\sqrt{2\pi}\right)\int_{1-1/\sqrt{n}}^1 ny^{n-3}(1-y^2)^{1/2}(1-y^{2n})^{-1/2}\,dy \\ &\le n\left(|K|/\sqrt{2\pi}\right)/(n-2)\exp(-\sqrt{n}) \\ &\quad+n\left(|K|/\sqrt{2\pi}\right)/(n-2)\left\{n(n-1/\sqrt{n})\right\}^{-1/2} \end{aligned}$$

and also

$$(2.19)\quad \begin{aligned} \beta &= \sum_{i=1}^{n-1} i^2(1/y)^{2i-2} \\ &= y^{-(2n-4)}\left\{(1+y^2)(1-y^{2n})(1-y^2)^{-3}\right. \\ &\qquad\left.+n^2(1-y^2)^{-1}-2n(1-y^2)^{-2}\right\}. \end{aligned}$$

Now from (2.4), (2.5) and since $h(y) = h(1/y)$ we have

$$(2.20)\quad \{\Delta(1/y)\}^{1/2} = y^{-(2n-4)}\left\{1-h(y)^2\right\}^{1/2}(y^{2n}-1)(y^2-1)^{-2}.$$

Hence, from (2.10), (2.20), and the relation

$$\alpha(1/y) = y^{-(2n-2)}(y^{2n}-1)(y^2-1)^{-1},$$

we have

$$(2.21)\quad \begin{aligned} &\int_1^{\infty}(\Delta^{1/2}/\alpha)\exp\{-\beta K^2/(2\Delta)\}\,dx \\ &< \int_0^1(\Delta^{1/2}/\alpha)y^{-2}\,dy \\ &< \int_0^{1-1/n}(1-y^2)^{-1}\,dy+\int_{1-1/n}^1(2n-1)^{1/2}(1-y)^{-1/2}\,dy \\ &< \tfrac{1}{2}\log(n)+1.36. \end{aligned}$$

Finally from (2.18) and (2.21) we have

$$EN(1,\infty) \le (2\pi)^{-1}\log(n)+1.36.$$

For getting the lower estimate of $EN(1,\infty)$ from (2.8) and (2.19), and for $0 \le y \le 1 - 1/n$ we have

$$\begin{aligned}
\beta/\Delta &= y^{2n-4}(1-y^2)(1-y^{2n})^{-2}\left\{1-h(y)^2\right\}^{-1}\\
&\quad\times\left\{(1+y^2)(1+y^{2n})+n^2(1-y^2)^2-2n(1-y^2)\right\}\\
&< y^{2n-4}(1-y^2)\left\{(1-y^{2n})^2-n^2y^{2n-2}(1-y^2)^2\right\}^{-1}\\
&\quad\times\left\{n^2(1-y^2)^2+1+y^2-2n(1-y^2)\right\}\\
&< y^{2n-4}(1-y^2)\left\{(1-e^{-2})^2-4e^{-2}+O(1/n)\right\}^{-1}\\
&\quad\times\left\{n^2(1-y^2)^2-2+1/n\right\}\\
&< 5n^2y^{2n-4}(1-y^2)^3
\end{aligned} \tag{2.22}$$

for all sufficiently large n.

Now let $\lambda'' = 9K^2/e^2$. Since in this range of y, $\mathrm{Max}\{y^{n-4}(1-y^2)^2\} \le 9/(n^2e^2)$ we have

$$\begin{aligned}
&\int_1^\infty (\Delta^{1/2}/\alpha)\exp\{-\beta K^2/(2\Delta)\}\,dx\\
&\quad\ge \int_0^{1-1/n}(1-y^2)^{-1}\exp\left\{-\lambda'' e^2n^2y^{2n-4}(1-y^2)^3/18\right\}dy\\
&\quad\ge \int_0^{1-1/n}(1-y^2)^{-1}\exp\left\{-\lambda'' y^n(1-y^2)/2\right\}dy\\
&\quad\ge \tfrac{1}{2}\int_0^{1-1/n}(1-y)^{-1}\exp\{-\lambda'' y^n(1-y)\}\,dy.
\end{aligned}$$

Now for large n, $\mathrm{Max}\{y^n(1-y)\} < 1/(en)$. Then for this range of y we have

$$\exp\{-y^n\lambda''(1-y)\} = 1 - \lambda'' y^n(1-y) + O\{\lambda''^2/(e^2n^2)\};$$

and finally by (2.22) we have

$$\begin{aligned}
&\int_1^\infty (\Delta^{1/2}/\alpha)\exp\{-\beta K^2/(2\Delta)\}\,dy\\
&\quad\ge \tfrac{1}{2}\int_0^{1-1/n}(1-y)^{-1}\,dy - \tfrac{1}{2}\int_0^{1-1/n}\lambda'' y^n\,dy + \tfrac{1}{2}\int_0^{1-1/n}O(\lambda''^2/n^2)\,dy\\
&\quad= \tfrac{1}{2}\log(n) + O(\lambda''/n).
\end{aligned}$$

Hence

$$EN(1,\infty) \ge (2\pi)^{-1}\log(n) + O(K^2/n).$$

So

$$EN(1,\infty) \sim (2\pi)^{-1}\log(n).$$

We could use the same method to obtain the asymptotic formula when (K^2/n) tends to a nonzero positive constant, and it is interesting to know that in this case

$$EN(-\infty, \infty) \sim (1/\pi)\log(n).$$

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