The Annals of Probability
1986, Vol. 14, No. 1, 119–135

A PROCESS IN A RANDOMLY FLUCTUATING ENVIRONMENT

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For every integer x, construct a stationary continuous-time Markov process $\gamma(x; t)$, with state space $\{-1, +1\}$ (all processes independent, and having the same distributions). Consider a particle moving at unit speed along the real line, with its direction completely determined by the γ's, as follows: if S_t is its position at time t, then $S_0 = 0$ and $S_{t+1} = S_t + \gamma(S_t; i)$ for $i = 0, 1, 2, \dots$. The increments are not stationary, nor is S_n Markov, yet this process has much in common with the classical random walk, including zero-one laws, a strong law of large numbers, and an invariance principle. The main result of the paper is the proof of the natural conjecture that the process is recurrent if and only if $P\{\gamma(0;0) = +1\} = \frac{1}{2}$. We also show how the FKG inequality can be used to investigate this process.

1. Introduction and summary. Let $\{\gamma(t),\ t \geq 0\}$ be a stationary two-state continuous-time Markov process on the state space $\{-1, +1\}$, with transition probability matrix

$$\begin{bmatrix} p_{-1,-1}(t) & p_{-1,+1}(t) \\ p_{+1,-1}(t) & p_{+1,+1}(t) \end{bmatrix} = \begin{bmatrix} q + pr^t & p - pr^t \\ q - qr^t & p + qr^t \end{bmatrix}, \tag{1.1}$$

where

$$p = \beta/(\beta + \delta), \qquad q = \delta/(\beta + \delta), \quad \text{and} \quad r = e^{-(\beta+\delta)}. \tag{1.2}$$

(β and δ are positive parameters in this model.) By stationarity, $\{\gamma(0) = +1\}$ has probability p. On a probability space $(\Omega, \mathscr{F}, P)$, construct a family of independent copies of $\gamma(t)$, indexed by $x \in \mathbb{Z}$; call these processes $\gamma(x; t)$. Now, think of a particle moving along the real line at unit speed, starting at time $t = 0$, from position $x = 0$. Its speed remains constant, but its direction is determined by the $\gamma(x; t)$ processes, as follows: If $\gamma(0;0) = +1$, then the particle begins by moving to the right; if $\gamma(0;0) = -1$, then it moves to the left. The direction does not change at nonintegral times. In general, if the particle is at position x' ($\in \mathbb{Z}$) at time t (where t is an integer), then it moves in the $\gamma(x'; t)$ direction.

Explicitly, let S_t be the position of the particle at time $t \geq 0$; let $X_n = S_n - S_{n-1}$. We define

$$\begin{aligned} &S_0 = 0, \\ &X_{i+1} = \gamma(S_i; i), \qquad && i = 0, 1, 2, \dots, \\ &S_t = S_i + (t - i)X_{i+1}, && i < t \leq i + 1. \end{aligned} \tag{1.3}$$

We remark here that the process may be easily generalized so that S_n takes

Received December 1983; revised August 1984.
AMS 1980 *subject classifications.* Primary 60K99; secondary 60J27.
Key words and phrases. Randomly fluctuating environment, Poisson point process, coupling, FKG inequality.

values in $\mathbb{Z}^d$; in this case, the state space of γ is a subset of $\mathbb{Z}^d$. This general model will be discussed only in Sections 2 and 3. We will refer to the process in which $X_n \in \{-1, +1\}$ as the "simple model."

The process $\{S_n\}$ is not Markovian, nor are the increments $\{X_n\}$ stationary. However, it turns out that $\{X_n\}$ is asymptotically stationary, and $\{S_n\}$ has many of the properties of classical random walks. For example, in Section 2 it will be shown that the following are equivalent: $P\{S_n = 0 \text{ for some } n > 0\} = 1$; S_n visits every point infinitely often, a.s.; and $E(\text{card}\{n\colon S_n = 0\}) = \infty$. We say that S_n is recurrent if these properties hold; otherwise, we say that S_n is transient. In Section 3, we use a convergence theorem of Norman (1968) to show that $\{X_n\}$ is exponentially ϕ mixing, which helps us to prove a zero-one law, a strong law of large numbers, and an invariance principle.

Section 4 presents the main result of the paper: the criterion for recurrence in the simple model. The result is exactly what one would naively guess: The process is recurrent if and only if $\beta = \delta$. However, no simple proof of this is known. The proof given here uses coupling and relies heavily upon the one-dimensional nearest-neighbor feature.

Finally, Section 5 presents another tool for investigating the simple model: the correlation inequality of Fortuin, Kasteleyn, and Ginibre (FKG) (1971). We use it to prove the following result, which is well known for classical simple random walk: $\lim_{n\to\infty} S_n = +\infty$ if and only if $E(\tau_1) < +\infty$, where $\tau_1 = \inf\{n\colon S_n = 1\}$ is the first hitting time of the point 1. We also use it in the case $\beta = \delta$ to derive a bound for probabilities associated with the Gambler's Ruin Problem in terms of the corresponding classical probabilities.

2. Some basic properties. Let E be a finite subset of $\mathbb{Z}^d$ such that the additive semigroup generated by E is all of $\mathbb{Z}^d$. Let $\gamma(t)$ be an irreducible stationary continuous-time Markov process with state space E. For $a, b \in E$ and $t \geq 0$, let $p_{ab}(t) = P\{\gamma(t) = b | \gamma(0) = a\}$ and $p_b = P\{\gamma(0) = b\}$. It is well known that there exists a constant $r < 1$ for this process such that

$$|p_{ab}(t) - p_b| \leq r^t \quad \text{for all } a,\ b \in E,\ t \geq 0. \tag{2.1}$$

Let $\{\gamma(x; t)\colon x \in \mathbb{Z}^d\}$ be i.i.d. copies of $\gamma(t)$ on a probability space $(\Omega, \mathscr{F}, P)$. We define $\{S_t^{(z,T)}\colon t \geq T\}$, the path of a particle starting from position $z \in \mathbb{Z}^d$ at time $T \in \mathbb{N}_0$ (let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$):

$$\begin{aligned} &S_T^{(z,T)} = z \\ &S_{i+1}^{(z,T)} = S_i^{(z,T)} + \gamma\big(S_i^{(z,T)}; i\big), \qquad i = T, T+1, \dots. \end{aligned} \tag{2.2}$$

(We write $S_t \equiv S_t^{(0,0)}$ and $X_i = S_i - S_{i-1}$.) Observe that the sequences $\{(S_{T+n}^{(z,T)} - z)\colon n \geq 0\}$ and $\{S_n\colon n \geq 0\}$ have the same distributions. By construction, these processes *coalesce*: i.e., if $S_u^{(z,T)} = S_u^{(y,T)}$ for some $u \geq T$, then $S_t^{(z,T)} = S_t^{(y,T)}$ for all $t \geq u$.

We define a *walk* W to be a finite sequence of points in $\mathbb{Z}^d$, $W = (w_0, w_1, \dots, w_n)$, such that $w_0 = 0$ and $w_i - w_{i-1} \in E$ for $i = 1, 2, \dots, n$. W will also denote the event $\{S_0 = w_0, \dots, S_n = w_n\}$. If $V = (v_0, \dots, v_m)$ is a second walk, we define a third walk $V * W = (v_0, \dots, v_m, v_m + w_1, \dots, v_m + w_n)$.

Let $\Delta W_i = w_i - w_{i-1}$, and $T(k, W) = \sup\{j\colon\ j < k,\ w_j = w_k\}$ (as usual, the sup of the empty set is $-\infty$). The probability of $W = (w_0, \dots, w_n)$ is computed as follows:

$$\begin{aligned} P(W) &= \prod_{i=1}^{n} P\{X_i = \Delta W_i | X_j = \Delta W_j,\ j = 1, \dots, i-1\} \\ &= \prod_{i=1}^{n} P\{\gamma(w_{i-1}; i-1) = \Delta W_i | \gamma(w_{i-1}; T(i-1, W)) = \Delta W_{T(i-1, W)+1}\} \qquad (2.3) \\ &= \prod_{i=1}^{n} p_{\Delta W_{T(i-1,W)+1}, \Delta W_i}(i - 1 - T(i-1, W)). \end{aligned}$$

[To understand (2.3), look at this example for the simple model: $P((0,1,0,1,2)) = pq(p + qr^2)(p - pr^2)$.]

Although the process $\{S_n\}$ is obviously not Markovian, we can use the following lemma to show that it shares many properties with classical random walks and Markov chains.

Lemma 2.1. *There exist constants L and U ($0 < L < 1 < U < \infty$), depending only on the transition probabilities of γ, such that for any walks V and W we have*

$$LP(V)P(W) \le P(V * W) \le UP(V)P(W).$$

(*Note that in the classical random walk,* $L = U = 1$.)

Proof of Lemma 2.1. Let $V = (v_0, \dots, v_m)$ and $W = (w_0, \dots, w_n)$ be two walks, and let $J = V * W$. Notice that

$$\begin{aligned} &\text{(i) } T(k, J) = T(k, V) \quad \text{if } k < m; \quad \text{and} \\ (2.4) \qquad &\text{(ii) } T(k, J) - m = T(k - m, W) \quad \text{if } k \ge m \quad \text{and} \\ &\qquad \text{either} \quad T(k, J) = -\infty \quad \text{or} \quad T(k - m, W) \ne -\infty. \end{aligned}$$

Expand $P(V)$, $P(W)$, and $P(J)$ as products of conditional probabilities, as in (2.3). By (2.4)(i),

$$\frac{P(J)}{P(V)P(W)} = \prod_{i=1}^{n} Q_i, \qquad (2.5)$$

where Q_i equals

$$\frac{p_{\Delta J_{T(m+i-1, J)+1}, \Delta J_{m+i}}(m + i - 1 - T(m + i - 1, J))}{p_{\Delta W_{T(i-1, W)+1}, \Delta W_i}(i - 1 - T(i - 1, W))}.$$

By (2.4)(ii), $Q_i = 1$ unless $T(m + i - 1, J) \ne -\infty$ and $T(i - 1, W) = -\infty$. [This is the case where $i - 1$ is the first time that W hits x, but J hits $x + v_m$ at some time $t < m$ (i.e., $J_t = J_{m+i-1} = x + v_m$).]

So if $Q_i \neq 1$, then Q_i is of the form $p_{ab}(t(i))/p_b$, where $t(i) \geq i$ and $a, b \in E$. Let $M = \max\{p_a^{-1}: a \in E\}$. Then by (2.2), $|Q_i - 1| \leq Mr^{t(i)} \leq Mr^i$. So by (2.5), the right-hand inequality of the lemma holds with $U = \prod_{i=1}^{\infty}(1 + Mr^i)$. The left-hand inequality is similar: We use the facts that $Q_i \geq 1 - Mr^i$ and $\inf\{p_{ab}(t)/p_b: a, b \in E, t \geq 1\} > 0$. □

We use the notation $\sigma(C)$ to denote the sigma algebra generated by C, where C is either a collection of sets or a collection of random variables. We define the following sub-sigma algebras of $\mathscr{F}$:

$$\mathscr{F}_n = \sigma(\{X_1, \ldots, X_n\}), \qquad \mathscr{F}_\infty = \sigma\left(\bigcup_{n=1}^{\infty} \mathscr{F}_n\right),$$

$$\mathscr{F}_n' = \sigma(\{X_k: k > n\}), \qquad \mathscr{F}' = \bigcap_{n=1}^{\infty} \mathscr{F}_n'.$$

We now define the shift operator θ to be the set transformation $\theta: \mathscr{F}_\infty \to \mathscr{F}_\infty$ satisfying

$$(2.6) \qquad \theta A = \{\omega \in \Omega: \exists\, \omega' \in A \text{ such that, for all } k \in \mathbb{N},\ X_{k+1}(\omega) = X_k(\omega')\}.$$

It is straightforward to extend Lemma 2.1 to

LEMMA 2.2. *Let τ be a "stopping time" for S_n; i.e., τ is a random variable taking values in $\{0, 1, 2, \ldots, \infty\}$ such that $\{\tau \leq n\} \in \mathscr{F}_n$. Let $\mathscr{F}_\tau = \{B \in \mathscr{F}_\infty: B \cap \{\tau = n\} \in \mathscr{F}_n$, for every $n\}$. If $P(B \cap \{\tau = +\infty\}) = 0$, then for $A \in \mathscr{F}_\infty$ and $B \in \mathscr{F}_\tau$,*

$$LP(B)P(A) \leq P(B \cap \theta^\tau A) \leq UP(B)P(A).$$

For $x \in \mathbb{Z}^d$, define $\Omega_x = \{\omega \in \Omega: S_n(\omega) = x$ for infinitely many $n\}$. Let $\tau(x, i)$ be the ith smallest integer in $\{j > 0: S_j = x\}$ (or $+\infty$, if this set has fewer than i elements). For a set A, let $|A|$ denote its cardinality.

PROPOSITION 2.3. *The following are equivalent*:

(a) $P\{\tau(0,1) < \infty\} = 1$;
(b) $P(\Omega_0) > 0$;
(c) $P(\bigcup_{y \in \mathbb{Z}^d} \Omega_y) = 1$;
(d) $P(\bigcap_{y \in \mathbb{Z}^d} \Omega_y) = 1$;
(e) $E(|\{n: S_n = 0\}|) = \infty$.

PROOF. (i) (a) implies (b): If (a) holds, then

$$P\{\tau(0, n) < \infty, \tau(0, n+1) = \infty\} = P(\{\tau(0, n) < \infty\} \cap \theta^{\tau(0, n)}\{\tau(0, 1) = \infty\})$$

equals 0 by Lemma 2.2. Summing over all n gives $P(\Omega_0^c) = 0$.

(ii) (b) implies (c) by the zero-one law for $\mathscr{F}'$ which will be proven in the next section, independently of this proposition.

(iii) (c) implies (d): For any $e \in E$, $\inf\{p_{ae}(t): a \in E, t \geq 1\} > 0$; it is then easy to see that $P(\Omega_x \setminus \Omega_{x+e}) = 0$, and consequently $P(\Omega_x \Delta \Omega_y) = 0$ for every x, $y \in \mathbb{Z}^d$.

(iv) (d) implies (e) is obvious.

(v) (e) implies (a): Assume $P\{\tau(0,1) < \infty\} < 1$. Then (d) can not hold, so the above shows that $P(\Omega_0) = 0$. Hence for some m, $P\{\tau(0, m) < \infty\} < (2U)^{-1}$. As in (i), we can show

$$P\{\tau(0, i+j) < \infty\} \leq UP\{\tau(0, i) < \infty)\}P\{\tau(0, j) < \infty\}. \tag{2.7}$$

Now, $E(|\{n: S_n = 0\}|) = 1 + \sum_{i=1}^{\infty} P\{\tau(0, i) < \infty\}$. This is a sum of nonincreasing terms, so it is finite if and only if $\sum_{k=1}^{\infty} P\{\tau(0, km) < \infty\}$ is finite. Using induction on (2.7), we find

$$\begin{aligned}\sum_{k=1}^{\infty} P\{\tau(0, km) < \infty\} &\leq \sum_{k=1}^{\infty} U^{k-1} P\{\tau(0, m) < \infty\}^k \\ &\leq \sum_{k=1}^{\infty} U^{-1} 2^{-k} \\ &< \infty.\end{aligned}$$

□

3. Ergodicity. In this section we show that $P(\theta^n \cdot)$ converges to a probability measure $P^\infty(\cdot)$, with respect to which $\{X_n\}$ is stationary and exponentially ϕ mixing. To this end, the following two paragraphs show how to apply a convergence theorem of Norman (1968) to our model; the reader is advised to look at the first two sections of Norman's paper before continuing. (Alternatively, the reader may skip directly to Proposition 3.1, since the intervening material is not used in the sequel.)

Consider a process $\zeta = \{\zeta_n: n \in \mathbb{N}_0\}$ defined by $\zeta_n = (X_n, X_{n-1}, \dots, X_1)$. ζ takes values in $\Sigma = \bigcup_{n=0}^{\infty} E^n$, where $E^n = \{(e_1, \dots, e_n): e_i \in E, i = 1, \dots, n\}$ for $n \geq 1$, and E^0 consists of the "empty string" (in our setting, it is the value of ζ at time 0). We will use this notation: If $u \in E^n$, then $u_i = e_i$ for $i = 1, \dots, n$ and $u_i = \Delta$ for $i > n$. We will view ζ as a (time-homogeneous) Markov process on $S \equiv \Sigma \cup E^{\mathbb{N}}$, with "event operators" $f_e(u) \equiv (e, u_1, u_2, \dots)$ for each $e \in E$, and transition probabilities

$$\phi_e(u) \equiv P\{\zeta_{i+1} = f_e(u) | \zeta_i = u\} = p_{u_T, e}(T), \tag{3.1}$$

where $T \equiv T(u) = \min\{k \in \mathbb{N}: u_1 + \cdots + u_k = 0\}$ ($T = \infty$ if no such k exists, or if $u \in E^0$). Thus we can define the process ζ starting from any $\zeta_0 \in S$.

Define the following metric on S:

$$d(u, u') = \sum_{i=1}^{\infty} r^i \delta(u_i, u_i');$$

where $\delta(a, b) = 0$ if $a = b$, and 1 otherwise. Then (S, d) is a compact metric space. We will now check the rest of Norman's conditions. Since $\phi_e(u) > 0$ for every $u \in S$ and $e \in E$, conditions H8 and H9 are trivial; H7 is just as easy, since $d(f_e(u), f_e(u')) = rd(u, u')$ for every $u, u' \in S$, $e \in E$. The only remaining condition to verify is H6:

$$\sup_{u \neq u'} \frac{|\phi_e(u) - \phi_e(u')|}{d(u, u')} < \infty. \tag{3.2}$$

Let u and u' be distinct elements of S; let $n = \min\{k\colon\ u_k \neq u'_k\}$. By (3.1), if $T(u) < n$, then $\phi_e(u) = \phi_e(u')$; if not, then $|\phi_e(u) - \phi_e(u')| \leq 2r^n$. But $d(u, u') \geq r^n$, so the left-hand side of (3.2) is dominated by 2; hence H6 holds. We can now apply Norman's Theorem 2.4, obtaining:

Proposition 3.1. *There exists a probability measure P^∞ on $(\Omega, \mathscr{F}_\infty)$, and constants $K < \infty$, $\alpha < 1$ such that: for all $j \geq 1$, $A \in \mathscr{F}_j$, and $n \geq 1$:*

$$\sup_{u \in S} |P\{\theta^n A | \xi_0 = u\} - P^\infty(A)| \leq K\alpha^n. \tag{3.3}$$

In particular, convergence of $P(\theta^n \cdot)$ to $P^\infty(\cdot)$ follows.

From (3.3) we easily conclude:

(i) $P^\infty(\theta \cdot) = P^\infty(\cdot)$; i.e., $\{X_n\}$ is strictly stationary with respect to P^∞.

(ii) P and P^∞ are equivalent measures on $(\Omega, \mathscr{F}_\infty)$; in fact, $LP^\infty \leq P \leq UP^\infty$ (let $B = \Omega$ and $\tau = n$ in Lemma 2.2, and let $n \to \infty$).

(iii) P^∞ is exponentially ϕ mixing; i.e., let

$$\phi^\infty(m) = \sup\left\{\frac{|P^\infty(B \cap C) - P^\infty(B)P^\infty(C)|}{P^\infty(B)}:\right.$$
$$\left. n \geq 0,\ P^\infty(B) > 0,\ B \in \mathscr{F}_n,\ C \in \mathscr{F}'_{n+m}\right\}.$$

Then $\phi^\infty(m) \leq K\alpha^m$ for all m.

(iv) P (the law of the chain started from ζ_0 = empty string) is exponentially ϕ mixing: If $\phi(m)$ is defined as above with P^∞ replaced by P, then $\phi(m) \leq 2K\alpha^m$.

(v) In particular: $|\mathrm{cov}(X_k, X_{m+k})| \leq 2K\alpha^m$ for all m and k, when cov is the covariance with respect to either P or P^∞.

(vi) Zero-one law: If $C \in \mathscr{F}'$, then $P(C)$ is either 0 or 1.

(vii) S_n is recurrent if $d = 1$ and γ is symmetric [i.e., $p_{a,b}(t) \equiv p_{-a,-b}(t)$]. (Proof: $P\{\lim S_n = +\infty\} = P\{\lim S_n = -\infty\}$; hence both must be 0 by (vi). Since E is finite, (c) of Proposition 2.3 must hold.)

(viii) Strong law of large numbers: Let $\hat{\mu} = \int_\Omega X_1(\omega) P^\infty(d\omega)$. Then by Birkhoff's ergodic theorem (using (i), (ii), and (vi)), $\lim_{n\to\infty} S_n/n = \hat{\mu}$ a.s. $[P^\infty]$, hence a.s. $[P]$.

Remarks on (viii). (a) If S_n is recurrent, then $\hat{\mu} = 0$.

(b) For $\lambda > 0$, let $\hat{\mu}^{[\lambda]}$ be the constant for $\{S_n^{[\lambda]}\}$, which is determined as in (1.3) by processes $\gamma^{[\lambda]}(\cdot\,;\cdot)$ with rescaled transition probabilities $p_{ab}^{[\lambda]}(t) \equiv p_{ab}(\lambda t)$ (e.g., in the simple model the parameters are $\beta^{[\lambda]} = \lambda\beta$ and $\delta^{[\lambda]} = \lambda\delta$).

Then it is not hard to prove that $\lim_{\lambda \to \infty} \hat{\mu}^{[\lambda]} = \sum_{a \in E} a p_a$ ("classical limit") and $\lim_{\lambda \to 0} \hat{\mu}^{[\lambda]} = 0$ ("frozen environment limit"). In particular, in the simple model, $\hat{\mu} \neq p - q$ in general.

(ix) Invariance principle ($d = 1$): For each n, define the process $W_n(t) = (S_{nt} - nt\hat{\mu})/\sigma n^{1/2}$ for $0 \le t \le 1$, where σ is a positive constant defined in the lemma below. Then as $n \to \infty$, W_n converges in distribution to the standard Wiener process in the space of continuous functions with the uniform topology. (This holds for the law of W_n induced by either P^∞ or P.) This follows immediately from Theorems 20.1 and 20.2 of Billingsley (1968), with the aid of the following lemma.

LEMMA 3.2. *Let V and V^∞ denote variance with respect to P and P^∞, respectively. Then there exists a constant $\sigma > 0$ such that* $\lim_{n \to \infty} V^\infty(S_n)/n = \sigma^2$.

PROOF. By (v) above, it is obvious that

$$\lim_{n \to \infty} \frac{V(S_n)}{n} = \lim_{n \to \infty} \frac{V^\infty(S_n)}{n} = c_0 + 2 \sum_{i=1}^{\infty} c_i,$$

where $c_i = \operatorname{cov}^\infty(X_1, X_{i+1}) = \lim_{n \to \infty} \operatorname{cov}(X_n, X_{i+n})$. Thus we only need to show that the right-hand side is not 0. Assume it is; then

$$V^\infty(S_n) = nc_0 + 2 \sum_{i=1}^{n-1} (n - i)c_i = -2n \sum_{i=n}^{\infty} c_i - 2 \sum_{i=1}^{n-1} ic_i$$

which is *bounded* by (v). Hence $E(|S_n - n\hat{\mu}|^2)$ is bounded (where E is expectation with respect to P) by (ii); and this in turn dominates $V(S_n)$.

Let $\mu_n = E(S_n)$. Since $V(S_n)$ is a bounded sequence, there exists an integer k such that

(3.4) $$P\{|S_n - \mu_n| \ge k/3\} \le 1/4 \quad \text{for all } n.$$

By Minkowski's inequality, since $E(S_n^{(k,0)}) = \mu_n + k$,

$$\left(E\left|S_n^{(k,0)} - S_n\right|^2\right)^{1/2} \le 2V(S_n)^{1/2} + k.$$

The right-hand side is bounded, so $\liminf_{n \to \infty} \left|S_n^{(k,0)} - S_n\right| < \infty$ a.s. by Fatou's lemma. By an argument similar to (iii) in the proof of Proposition 2.3, $\liminf_{n \to \infty} \left|S_n^{(k,0)} - S_n\right| = 0$ a.s. (Note: if S_n is periodic, we must exercise a bit of care in choosing k.) These paths coalesce when they meet; i.e., there exists a random $N < \infty$ such that $S_n^{(k,0)} = S_n$ for all $n \ge N$. Therefore, $\lim_{n \to \infty} P\{S_n^{(k,0)} = S_n\} = 1$; but this yields a contradiction, because (3.4) implies

$$P\left\{S_n^{(k,0)} = S_n\right\} \le P\left\{\left|S_n^{(k,0)} - (\mu_n + k)\right| \ge k/3 \text{ or } |S_n - \mu_n| \ge k/3\right\} \le \tfrac{1}{2}. \qquad \square$$

4. Proof of the main theorem. In this section we will restrict attention to the simple model [i.e., γ is described by (1.1) and (1.2)]. We will prove that

$\hat{\mu} \equiv \hat{\mu}(\beta, \delta)$ is a strictly increasing (respectively, decreasing) function of β (respectively, δ). In conjunction with (vii) and (viii) of Section 3, this will prove

THEOREM 4.1. *S_n is recurrent if and only if $\beta = \delta$.*

By symmetry, it suffices to prove the following: Fix $\beta > \beta' > 0$ and $\delta > 0$; then $\hat{\mu}(\equiv \hat{\mu}(\beta, \delta)) > \hat{\mu}'(\equiv \hat{\mu}(\beta', \delta))$.

The proof uses two kinds of coupling. First, let $L = \{(z, t) \in \mathbb{Z} \times \mathbb{N}_0: (z+t)/2 \in \mathbb{Z}\}$. If $(z, T) \in L$, then $S_i^{(z,T)} \in L$ for all $i \geq T$. The nearest-neighbor characteristic gives a useful coupling of these processes: If (y, T) is also in L, and $y < z$, then $S_t^{(y,T)} \leq S_t^{(z,T)}$ for all $t \geq T$. [If we regard these paths as continuous curves in the (z, t) plane, by linear interpolation for nonintegral t, then the coupling says that these paths do not cross one another; of course, paths can coalesce.]

Second, we will construct a coupled family of stationary Markov processes $\gamma'(x; t)$ with transition probabilities described by (1.1) and (1.2), with β replaced by β', and satisfying

$$\gamma'(x; t) \leq \gamma(x; t) \quad \text{for all } x \text{ and } t. \tag{4.1}$$

Now define X_t', S_n', and $S_n'^{(z,T)}$ as in (1.3) and (2.2), with γ replaced by γ'. It follows from (4.1) and the preceding paragraph that for any $(z, T) \in L$,

$$S_n'^{(z,T)} \leq S_n^{(z,T)} \quad \text{for every } n \geq T. \tag{4.2}$$

From (4.2), $\hat{\mu}' \leq \hat{\mu}$ is obvious; it remains to prove that the inequality is strict. This will be done in two steps, which we will now outline.

STEP 1. $\lim_{n\to\infty} E(S_n - S_n') = +\infty$ (Lemma 4.2). Idea: for a large integer V. Then for sufficiently large N', it is unlikely that $S_n - S_n' \leq V$ for all $n \leq N'$; and once these processes get far apart, it can be shown that they remain far apart (in expectation) at later times.

STEP 2. We define a process S^* which approximates S_n from below. Let $S_0^* = 0$, and let S_n^* "run naturally" for a long (fixed) time N [i.e., $S_{n+1}^* = S_n^* + \gamma(S_n^*; n)$ for $0 \leq n < N$]. Then let S^* take a step to the left at each unit of time, for a random time T_1, which is long enough to forget the past. Then let S^* continue to run naturally for another N time units; then it moves left for T_2 time units, and so on. These random times have two key properties:

(a) if $\phi[k] = (T_1 + N) + \cdots + (T_k + N)$ for $k \geq 0$, then $\{S_{\phi[k]}^* - S_{\phi[k-1]}^*: k \geq 1\}$ are i.i.d. random variables, with mean $E(S_N) - E(T_1)$; and

(b) as $N \to \infty$, $E(T_1)$ remains bounded. If $\beta' = \delta$, then $E(S_N') = 0$, so by Lemma 4.2, we can choose N so that $E(S_N) - E(T_1) > 0$; then Theorem 4.1 follows from the classical law of large numbers and the fact that $S_n^* \leq S_n$. The general proof is similar.

In both steps we will need to be able to say when we are sure that our processes have forgotten the past. We will use an event I or I', whose occurrence ensures that the past and future are (conditionally) independent.

We begin by constructing an explicit coupling of γ and γ'. For each $x \in \mathbb{Z}$, let $\rho^{(x)}$ be a Poisson point process on the real line with intensity $\beta + \delta$ (i.e., $E|\rho^{(x)} \cap [s, t]| = (\beta + \delta) \cdot (t - s)$ for any real interval $[s, t]$). Define the random sets $\rho_1^{(x)}$, $\rho_2^{(x)}$, and $\rho_3^{(x)}$ by assigning each point of $\rho^{(x)}$ independently to one of these sets with probabilities $\beta'/(\beta + \delta)$, $\delta/(\beta + \delta)$, and $(\beta - \beta')/(\beta + \delta)$, respectively; hence $\rho_1^{(x)}$, $\rho_2^{(x)}$, and $\rho_3^{(x)}$ are independent Poisson processes with intensities β', δ, and $(\beta - \beta')$, respectively. We construct all of these processes to be independent on a single probability space $(\Omega, \mathscr{F}, P)$. For $t \in \mathbb{R}$, $x \in \mathbb{Z}$, define

$$\gamma(x; t) = \begin{cases} +1 & \text{if } \sup\{s \le t\colon s \in \rho^{(x)}\} \text{ is a point of } \rho_1^{(x)} \cup \rho_3^{(x)}, \\ -1 & \text{otherwise,} \end{cases}$$

$$\gamma'(x; t) = \begin{cases} +1 & \text{if } \sup\{s \le t\colon s \in (\rho_1^{(x)} \cup \rho_2^{(x)})\} \text{ is a point of } \rho_1^{(x)}, \\ -1 & \text{otherwise.} \end{cases}$$

It is easy to check that γ and γ' are stationary Markov processes having the desired distributions and satisfying (4.1). Let $F = \{(-1, -1), (-1, +1), (+1, +1)\}$ and let $\eta(x; t) = (\gamma'(x; t), \gamma(x; t))$. Then for each fixed x, $\eta(x; \cdot)$ is a stationary F-valued Markov process. Let $\pi(\cdot) \equiv P\{\eta(0; 0) = \cdot\}$ be the stationary distribution of η:

$$\begin{aligned} \pi((+1, +1)) &= P\{\gamma'(0; 0) = +1\} = \beta'/(\beta' + \delta) \\ \pi((-1, -1)) &= P\{\gamma(0; 0) = -1\} = \delta/(\beta + \delta) \\ \pi((-1, +1)) &= 1 - \beta'/(\beta' + \delta) - \delta/(\beta + \delta). \end{aligned} \tag{4.3}$$

Let $a, b \in \mathbb{R}$ $(a < b)$ and $x \in \mathbb{Z}$. Let $A \in \sigma\{\gamma(x; t)\colon t \le a\}$. Then it is easy to see from the construction that

$$\begin{aligned} P\{\gamma(x; b) = +1 | A \cap [\rho^{(x)} \cap \{t\colon a < t < b\} \ne \phi]\} &= \beta/(\beta + \delta) \\ &= P\{\gamma(x; b) = +1\}. \end{aligned} \tag{4.4}$$

[Roughly speaking, any point of $\rho^{(x)}$ landing in (a, b) cuts off the effect of any earlier points of $\rho^{(x)}$, and restarts γ; thus, A is forgotten.] We need two extensions of (4.4). In the following, f, g, and h are functions from $\mathbb{Z}$ into $\mathbb{R}$ such that $f(x) < g(x) < h(x)$ for all $x \in \mathbb{Z}$. Define

$$\begin{aligned} \mathscr{H}_f &= \sigma\{\gamma(x; t)\colon x \in \mathbb{Z}, t \le f(x)\} \\ \mathscr{H}^g &= \sigma(\{\gamma(x; t)\colon x \in \mathbb{Z}, t \ge g(x)\} \\ \mathscr{G}_{g,f} &= \sigma\{(\rho^{(x)} \cap \{t\colon f(x) < t < g(x)\})\colon x \in \mathbb{Z}\} \\ I(f, g) &= \{\omega \in \Omega\colon \text{for each } x \in \mathbb{Z}, \rho^{(x)} \cap \{t\colon f(x) < t < g(x)\} \ne \phi\} \\ \bar{\mathscr{H}}_f &= \sigma\{\eta(x; t)\colon x \in \mathbb{Z}, t \le f(x)\} \\ \bar{\mathscr{H}}^h &= \sigma\{\eta(x; t)\colon x \in \mathbb{Z}, t \ge h(x)\} \\ I'(f, g, h) &= I(g, h) \cap \{\omega \in \Omega\colon \text{for each } x \in \mathbb{Z}, \\ &\qquad [(\rho_1^{(x)} \cup \rho_2^{(x)}) \cap \{t\colon f(x) < t < g(x)\}] \ne \phi\} \end{aligned}$$

First extension. If $C \in \mathscr{G}_{g,f}$ and $C \subset I(f, g)$, then we have, as in (4.4), that the conditional distribution of $\{\gamma(x; g(x)): x \in \mathbb{Z}\}$ given the event $A \cap C$ is simply the distribution of a collection of $\{-1, +1\}$-valued i.i.d. random variables with $P\{\gamma(x; g(x)) = +1\} = \beta/(\beta + \delta)$. But this is just the *unconditional* distribution of $\{\gamma(x; g(x)): x \in \mathbb{Z}\}$. So, using the Markov property of $\gamma(x; \cdot)$ we have

$$(4.5)\qquad P\{B|A \cap C\} = P(B), \quad \text{for } A \in \mathscr{H}_f,\ B \in \mathscr{H}^g, \text{ and } C \in \mathscr{G}_{f,g} \text{ with } C \subset I(f, g).$$

[Visualize $I(f, g)$ on the (x, t) plane: For each x, a point of $\rho^{(x)}$ (or rather, of $\{x\} \times \rho^{(x)}$) falls on the line segment $\{x\} \times (f(x), g(x))$. Thus the γs above the curve $t = g(x)$ have lost all memory of anything that happened below $t = f(x)$.]

Second extension. Let $A \in \overline{\mathscr{H}}_f$, and let $x \in \mathbb{Z}$. Then, using (4.4), we have

$$P\{\gamma(x; h(x)) = -1|I'(f, g, h) \cap A\} = \delta/(\beta + \delta)$$
$$P\{\gamma'(x; h(x)) = +1|I'(f, g, h) \cap A\} = \beta'/(\beta' + \delta);$$

so reasoning as before, we obtain from (4.3) that

$$(4.6)\qquad P\{B|I'(f, g, h) \cap A\} = P(B) \quad \text{for any } B \in \overline{\mathscr{H}}^h.$$

[Given that $I'(f, g, h)$ occurs, the γs and γ's above the curve $t = h(x)$ are stationary and independent of those below $t = f(x)$; i.e., the processes "restart" at the curve $t = h(x)$.]

The following type of observation will hereafter be used frequently: since $|S_t^{(z,T)} - z| \le |T - t|$ for $t \ge T$, we have $\sigma\{S_t^{(z,T)}: t \ge T\} \subset \mathscr{H}^f$, where $f(x) = T + |x - z|$. With this in mind, we proceed to the key lemma.

LEMMA 4.2. $\lim_{n\to\infty} E(S_n - S_n') = +\infty$.

PROOF. Choose $M > 0$; we will show that there exists an N' such that $E(S_n - S_n') > M$ for all $n \ge N'$. Let $J = \prod_{i=1}^{\infty}(1 - e^{-i\alpha})^8 > 0$, where $\alpha = (\beta' + \delta)/2$. Choose an integer $V > M/J$. Define the random time $\sigma \equiv \sigma_V = \min\{n: S_n - S_n' = 2V\}$. Since $\inf\{P\{\eta(x; 1) = a|\eta(x; 0) = b\}: a, b \in F\} > 0$, it is easy to see that $\sigma < \infty$ a.s. Choose N' such that $P\{\sigma \le N'\} > 1/2$. Since $S_n - S_n' \ge 0$, we have

$$(4.7)\qquad E(S_n - S_n') \ge \sum_{k=V}^{n} \sum_{j=-k}^{k} E\big((S_n - S_n')1_{\{\sigma=k, S_k'=j\}}1_{I_{k,j}}\big)$$

where 1_C is the indicator function of the event C, $I_{k,j} = I'(f_k, g_{kj}, h_{kj})$, and

$$f_k(z) = k - 1$$
$$h_{kj}(z) = k + \min\{|z - j|, |z - (j + 2V)|\}$$
$$g_{kj}(z) = \big(f_k(z) + h_{kj}(z)\big)/2.$$

[See Figure 1. Observe that if $\{\sigma = k, S_k' = j\}$ occurs, then the paths of $S'$and S after time k lie above the curve $t = h_{kj}(x)$; and if $I_{k,j}$ also occurs, then S' and S

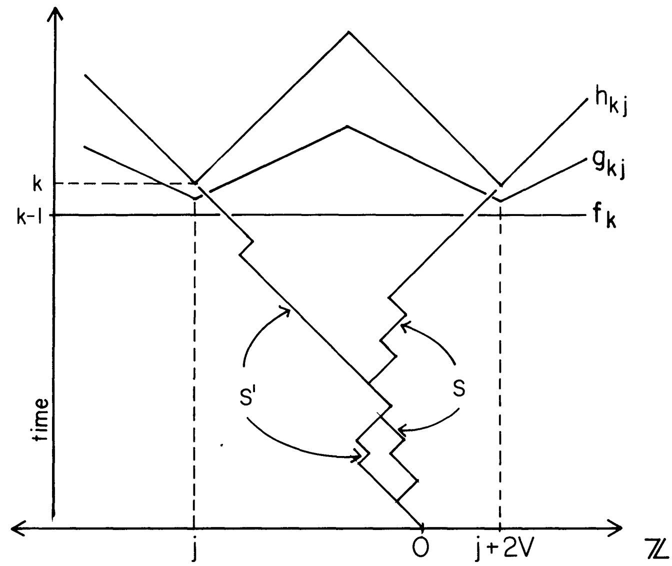

FIG. 1. *The functions f_k, g_{kj}, and h_{kj} and the event $\{\sigma = k, S'_k = j\}$.*

after time k will forget that $\{\sigma = k, S'_k = j\}$ has happened.] Now, for any j and k,

$$
\begin{aligned}
&P(I_{k,j}) \\
&= \prod_{x \in \mathbb{Z}} \left[1 - \exp\{-(\beta + \delta)(h_{kj}(x) - g_{kj}(x))\}\right] \\
&\qquad \cdot \left[1 - \exp\{-(\beta' + \delta)(g_{kj}(x) - f_k(x))\}\right] \\
&> \prod_{x \in \mathbb{Z}} \left[1 - \exp\{-\tfrac{1}{2}(\beta' + \delta)(\min\{|x - j|, |x - j - 2V|\} + 1)\}\right]^2 \\
&> \prod_{x \in \mathbb{Z}} \left[1 - \exp\{-\alpha(|x - j| + 1)\}\right]^2 \left[1 - \exp\{-\alpha(|x - j - 2V| + 1)\}\right]^2 \\
&> J.
\end{aligned} \tag{4.8}
$$

$S_n'^{(j,k)}$ and $S_n^{(j+2V,k)}$ are $\mathscr{H}^{h_{kj}}$-measurable, and $\{\sigma = k, S'_k = j\} \in \overline{\mathscr{H}}_{f_k}$, so from (4.6) it follows that the expectation term on the right side of (4.7) is

$$
\begin{aligned}
&E\left[\left(S_n^{(j+2V,k)} - S_n'^{(j,k)}\right)1_{\{\sigma = k, S'_k = j\}} 1_{I_{k,j}}\right] \\
&\qquad = E\left(S_n^{(j+2V,k)} - S_n'^{(j,k)}\right) P\{\sigma = k, S'_k = j\} P(I_{k,j}).
\end{aligned} \tag{4.9}
$$

Using the observation following (2.2), as well as (4.2), we obtain

$$
\begin{aligned}
E\left(S_n^{(j+2V,k)} - S_n'^{(j,k)}\right) &= E(2V + S_{n-k}) - E(S'_{n-k}) \\
&\ge 2V.
\end{aligned} \tag{4.10}
$$

Now, combine (4.7), (4.9), (4.8), and (4.10) to get, for $n \geq N'$,

$$E(S_n - S'_n) \geq \sum_{k=V}^{n} \sum_{j=-k}^{k} 2VP\{\sigma = k, S'_k = j\}J$$
$$= 2VJP\{\sigma \leq n\}$$
$$> M. \qquad \square$$

Now we proceed to Step 2. Fix a large integer N (to be specified later). Define the "independence indicator" function

$$K(z, t, t'; \omega) = \begin{cases} 1 & \text{if } \left(\rho^{(x)} \cap \{s\colon t - |z - x| < s < t'\}\right) \neq \phi \\ & \text{for each } x = z - N, \dots, z + N, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\phi[0] = 0$, $S_0^* = 0$, and recursively define

$$S_n^* = S_n^{(S^*_{\phi[k]}, \phi[k])}, \quad \text{for } \phi[k] < n \leq \phi[k] + N;$$
$$\phi[k+1] = \min\left\{t \in \mathbb{N}\colon t > \phi[k] + N, K\left(S^*_{\phi[k]+N}, \phi[k] + N, t; \omega\right) = 1\right\};$$
$$S_n^* = S^*_{\phi[k]+N} - (n - \phi[k] - N), \quad \text{for } \phi[k] + N < n \leq \phi[k+1].$$

We can think of $\phi[k+1]$ as the first time that S^* can be allowed to run naturally, with the assurance that the influence of $\{S_t^*\colon \phi[k] < t \leq \phi[k] + N\}$ has been forgotten, as in (4.5) (and by extension, $\{S_t^*\colon 0 < t \leq \phi[k] + N\}$ has been forgotten). With this intuitive picture, the proof of the following lemma is a straightforward but tedious exercise, which is omitted. (We will only check, in passing, that $P\{\phi[k] < \infty\} = 1$.)

LEMMA 4.3. *Each of the following is an i.i.d. sequence* $\{(\phi[k] - \phi[k-1])\colon k \in \mathbb{N}\}$, *and* $\{(S^*_{\phi[k]} - S^*_{\phi[k-1]})\colon k \in \mathbb{N}\}$.

For arbitrary $z \in \mathbb{Z}$, $t < t'$:

$$P\{K(z, t, t') = 0\} \leq \sum_{x=z-N}^{z+N} P\left\{\left(\rho^{(x)} \cap \{s \in \mathbb{R}\colon t - |z - x| < s < t'\}\right) = \phi\right\}$$
$$= \sum_{y=-N}^{N} \exp[-(\beta + \delta)(t' - t + |y|)]$$
$$< 2r^{|t'-t|}(1 - r)^{-1}.$$

Therefore

$$P\{\phi[1] - N > a\} < 2r^a(1 - r)^{-1} \qquad (a \in \mathbb{N}). \tag{4.11}$$

The left-hand side equals $P\{\phi[k+1] - (\phi[k] + N) > a | \phi[k] < \infty\}$, so we know that $\phi[k]$ is finite a.s. for every k. (4.11) also implies

$$E(\phi[1] - N) < \sum_{a=0}^{\infty} 2r^a(1 - r)^{-1} = 2(1 - r)^{-2}. \tag{4.12}$$

By Lemma 4.3 and the usual strong law of large numbers, $\lim_k S^*_{\phi[k]}/k = E(S_N - (\phi[1] - N))$ and $\lim_k \phi[k]/k = E(\phi[1])$ a.s. $[P]$, so $\lim_k S^*_{\phi[k]}/\phi[k] = E(S_N - (\phi[1] - N))/E(\phi[1])$ a.s. Since $S_n \geq S_n^*$, it follows that

$$\hat{\mu} \geq \frac{E(S_N - (\phi[1] - N))}{E(\phi[1])} \tag{4.13}$$

Now we can define an analogous process $S_n'^*$ which is always to the right of S_n', and a sequence $\phi'[k]$, exactly as above except that $S_n'^* = S_{n-1}'^* + 1$ for $\phi'[k] + N < n \leq \phi'[k+1]$; then we have, as in (4.12) and (4.13),

$$E(\phi'[1] - N) \leq 2(1 - r')^{-2}, \quad \text{where } r' = e^{-(\beta' + \delta)}, \quad \text{and} \tag{4.14}$$

$$\hat{\mu}' \leq \frac{E(S_N' + (\phi'[1] - N))}{E(\phi'[1])} \tag{4.15}$$

Combining (4.13) and (4.15) and rearranging, $\hat{\mu}' < \hat{\mu}$ will follow if

$$E(\phi'[1] - N) + E(\phi[1] - N)\frac{E\phi'[1]}{E\phi[1]} + E(S_N)\left[\frac{E\phi[1] - E\phi'[1]}{E\phi[1]}\right] < E(S_N - S_N'). \tag{4.16}$$

The left-hand side is a bounded function of N, by (4.12) and (4.14); so Lemma 4.2 implies that there exists an N satisfying (4.16), and we are done.

5. The FKG inequality, with applications. In this section we again restrict attention to the simple model. Let B be a finite subset of $L = \{(x, t) \in \mathbb{Z} \times \mathbb{N}_0: (x + t)/2 \in \mathbb{Z}\}$, and let $\Gamma_B = \sigma\{\gamma(x; t): (x, t) \in B\}$. For $C \subset B$, let C also denote the event $\{\gamma(x; t) = +1$ for each $(x, t) \in C$, and $\gamma(x; t) = -1$ for each $(x, t) \in B \setminus C\}$. Then P is a probability measure on the lattice L_B of subsets of B. As usual, we identify Γ_B-measurable random variables with functions on L_B. A real-valued function h on the lattice L_B is called increasing (respectively decreasing) if for every $C, D \in L_B$ such that $C \subset D$ we have $h(C) \leq h(D)$ [respectively, $h(C) \geq h(D)$]. (We will sometimes suppress the phrase "on L_B", or decline to specify the precise set B.) To show that a function is increasing, we would typically argue as in the proof of the following lemma.

LEMMA 5.1. *If B is a subset of L which contains $L \cap ([-n, n] \times [0, n])$, then $S_n^{(0,0)}$ is an increasing function on L_B.*

PROOF. Clearly $S_n^{(0,0)}$ is Γ_B-measurable. It suffices to show the following: If $(x, t) \in B \setminus C$, then $S_n^{(0,0)}(C) \leq S_n^{(0,0)}(C')$, where $C' = C \cup \{(x, t)\}$. If $S_t^{(0,0)}(C) \neq x$, or if $t \geq n$, then $S_n^{(0,0)}(C) = S_n^{(0,0)}(C')$. On the other hand, if $S_t^{(0,0)}(C) = x$ and $t < n$, then $S_{t+1}^{(0,0)}(C) = x - 1$ and $S_{t+1}^{(0,0)}(C') = x + 1$. Now, using the fact that S paths do not jump over one another (see beginning of Section 4), we have that $S_n^{(0,0)}(C') = S_n^{(x+1,t+1)}(C') = S_n^{(x+1,t+1)}(C) \geq S_n^{(x-1,t+1)}(C) = S_n^{(0,0)}(C)$, as desired. □

Proposition 5.2 (FKG inequality). *If f and g are both increasing functions on L_B (for some finite $B \subset L$), then $E(fg) \geq E(f)E(g)$.*

Proof. This follows from Fortuin, Kasteleyn, and Ginibre (1971), once we verify the "convexity condition": For any $C, D \in L_B$, $P(C \cap D)P(C \cup D) \geq P(C)P(D)$. Since the processes $\gamma(x; \cdot)$ are independent for different x's, it suffices to check the special case where $B \subset \{0\} \times 2\mathbb{N}_0$. In addition, for notational simplicity, we will assume $B = \{(0,0),(0,2),\dots,(0,T)\}$ for some even integer T.

For $C \subset B$, we can write

$$P(C) = f_C(0) \prod_{i=1}^{T-2} g_C(i, i+2), \tag{5.1}$$

where $f_C(i)$ equals p if $(0,i) \in C$ and equals q if $(0,i) \in B \setminus C$, and

$$g_C(i, i+2) = \begin{cases} p + qr^2 & \text{if } \{0\} \times \{i, i+2\} \subset C \\ q + pr^2 & \text{if } \{0\} \times \{i, i+2\} \subset B \setminus C \\ f_C(i+2)(1-r^2) & \text{otherwise.} \end{cases}$$

Now we proceed by inspection. First we see that $f_{C \cap D}(0) f_{C \cup D}(0) = f_C(0) f_D(0)$. Next,

$$g_{C\cap D}(i, i+2) g_{C \cup D}(i, i+2) > g_C(i, i+2) g_D(i, i+2) \tag{5.2}$$

if $(0,i) \in C \setminus D$ and $(0, i+2) \in D \setminus C$ or vice versa. In all other cases, the two sides of (5.2) are equal. Thus, using (5.1), the condition is verified. □

Corollary 5.3. *Let $W = (w_0, \dots, w_n)$ be a walk such that*

$$w_n > w_i \quad \text{for each } i = 0, 1, \dots, n-1. \tag{5.3}$$

Let Q be an increasing function on L_B, where B is a finite subset of $L \cap \{(x,t): t \geq n\}$. Then $E(Q|W) \geq E(Q)$.

Proof. Let $v = \min\{w_0, \dots, w_n\}$, $\Delta W_i = w_i - w_{i-1}$, and $t_W(x) = \max\{j: w_j = x\}$. Then (5.3) implies $\Delta W_{t_W(x)+1} = +1$ for each $x = v, v+1, \dots, w_{n-1}$. So $W \subset W'$, where $W' = \{\gamma(x; t_W(x)) = +1 \text{ for each } x = v, v+1, \dots, w_n - 1\}$. Moreover, the Markov property of the processes $\gamma(x; \cdot)$ implies that $E(Q|W) = E(Q|W')$. $1_{W'}$ (the indicator function of W') and Q are both increasing functions, so, by Proposition 5.2, $E(Q 1_{W'}) \geq E(Q)E(1_{W'})$. Equivalently, $E(Q|W') \geq E(Q)$. The lemma follows, since $E(Q|W) = E(Q|W')$. □

We are now in a position to reap some benefits from the FKG inequality. Our focus will be on first hitting times of points, which enables us to take advantage of Corollary 5.3. For $(x, T) \in L$ and $y \in \mathbb{Z}$, define $\tau_y^{(x,T)} = \min\{i \geq T: S_i^{(x,T)} = y\}$. Let $\tau_y = \tau_y^{(0,0)}$.

Proposition 5.4. *Suppose $\beta \geq \delta$. Then, for positive integers x and y, $E(\tau_{x+y}) \leq E(\tau_x) + E(\tau_y)$.*

Proof. We decompose the event $\{\tau_{x+y} \geq k\}$ according to the path of S_t up to its first visit to x (we know that τ_x is finite almost surely, by results of previous sections):

$$P\{\tau_{x+y} \geq k\} = \sum_n \sum_W P\big(\{\tau_{x+y} \geq k\} \cap W\big), \tag{5.4}$$

where the outer sum is over all integers n, and the inner sum is over all walks W of length n such that $w_n = x$ and $w_i < x$ for each $i < n$. Now

$$P\big(\{\tau_{x+y} \geq k\} \cap W\big) = P\Big(\big\{\tau_{x+y}^{(x,n)} \geq k\big\} \cap W\Big) \leq P\big\{\tau_{x+y}^{(x,n)} \geq k\big\} P(W)$$

by Corollary 5.3, since W satisfies (5.3), and since the indicator function of $\{\tau_{x+y}^{(x,n)} \geq k\}$ is a decreasing function (on L_B, where $B = L \cap ([-k, k] \times [n, k])$). We substitute this into (5.4):

$$P\{\tau_{x+y} \geq k\} \leq \sum_n P\big\{\tau_{x+y}^{(x,n)} \geq k\big\} \sum_W P(W) = \sum_n P\{\tau_y \geq k - n\} P\{\tau_x = n\}.$$

Thus, if τ_x' is a random variable with the same distribution as τ_x, but independent of τ_y, then we have

$$P\{\tau_{x+y} \geq k\} \leq P\{\tau_y + \tau_x' \geq k\}. \tag{5.5}$$

Summing over all $k = 1, 2, \ldots$ in (5.5) gives $E(\tau_{x+y}) \leq E(\tau_y + \tau_x')$. The proposition follows. □

Theorem 5.5. *$E(\tau_1) < \infty$ if and only if $\beta > \delta$.*

Proof. We divide the proof into three cases.

(i) $\beta < \delta$: $\lim_{n\to\infty} S_n = -\infty$ a.s., so $P\{\tau_1 = \infty\} > 0$ by Proposition 2.3.
(ii) $\beta = \delta$: By (viii) of section 3, $\lim_{n\to\infty}(S_{\tau_n}/\tau_n) = 0$; therefore $\lim_{n\to\infty}(\tau_n/n) = +\infty$ a.s. By Fatou's lemma, $\lim_{n\to\infty} E(\tau_n/n) = +\infty$. But $E(\tau_n) \leq nE(\tau_1)$ for all $n \geq 1$, by Proposition 5.4; hence $E(\tau_1) = +\infty$.
(iii) $\beta > \delta$: For each integer x, let $N(x) = |\{n \geq 0\colon S_n = x\}|$. $\tau_1 \leq \sum_{x \leq 0} N(x)$, so it suffices to show that $\sum_{x\leq 0} E(N(x))$ is finite. Lemma 2.2 tells us that $P\{N(x) = n\} \leq UP\{\tau_x < \infty\} P\{N(0) = n\}$ for any n; so it suffices to show that $\sum_{x \leq 0} P\{\tau_x < \infty\} < \infty$, by Proposition 2.3(e). Since $\lim_{n\to\infty} S_n = +\infty$ a.s., there exists an integer $z < 0$ such that $P\{\tau_z < \infty\} < (2U)^{-1}$; the proof now is analogous to (v) of Proposition 2.3. □

The last part of this section is devoted to a particular case of the Gambler's Ruin Problem for the process S_n (Theorem 5.7). We introduce the notation $P[a, b] = P\{\tau_a < \tau_b\}$ for $a, b \in \mathbb{Z}$.

Lemma 5.6. *Fix integers x, y, and z such that $x < 0 < y < z$. Then*

$$P\{\tau_y < \tau_x \text{ and } \tau_x < \tau_z\} \leq P[y, x] P[x - y, z - y].$$

REMARK. An application of Lemma 2.2 will prove that

$$LP[y,x]P[x-y,z-y] \le P\{\tau_y < \tau_x \text{ and } \tau_x < \tau_z\} \le UP[y,x]P[x-y,z-y].$$

PROOF OF LEMMA 5.6. Let Q be the indicator function of the event $R = \{\tau_x^{(y,n)} < \tau_z^{(y,n)}, \tau_x^{(y,n)} \le m\}$ (for some fixed integers $m > n > 0$). Then

$$P(R \cap \{\tau_y < \tau_x, \tau_y = n\}) = \sum_W E(Q|W)P(W),$$

where the sum is over all walks W of length n such that $w_n = y$ and $x < w_i < y$ for all $i < n$. Since Q is a decreasing function, we can apply Corollary 5.3, to get $E(Q|W) \le E(Q)$. Therefore

$$P(R \cap \{\tau_y < \tau_x, \tau_y = n\}) \le P(R)\sum_W P(W) = P(R)P\{\tau_y < \tau_x, \tau_y = n\}.$$

Let $m \to \infty$ in the above inequality; then $P(R)$ converges to $P[x-y, z-y]$, so summing on n proves the lemma. □

THEOREM 5.7. *Suppose* $\beta = \delta$. *For any nonzero integer c, and any positive integer k,*

$$\frac{1}{k+1} \le P[(kc), -c] \le \frac{1}{L(k+1)}.$$

(*Note that the left-hand side is the exact value for the classical simple symmetric random walk.*)

PROOF. By symmetry, we may assume that $c > 0$. We begin by proving the following: For any positive integers c and d, there exists a number $\Lambda \equiv \Lambda(c,d)$ such that

$$P[d+c, -c] = \frac{P[d,-c]}{1 + \Lambda P[d,-c]} \quad \text{and} \quad L \le \Lambda \le 1. \tag{5.6}$$

To do this, we write $P[d,-c] = P[d+c,-c] + P\{\tau_d < \tau_{-c} \text{ and } \tau_{-c} < \tau_{d+c}\}$. Now, let $\Lambda = P\{\tau_d < \tau_{-c} \text{ and } \tau_{-c} < \tau_{d+c}\}/(P[d,-c]P[-c-d,c])$. Then $L \le \Lambda \le 1$ by Lemma 5.6 and its accompanying remark. Now rearrange terms and use $P[-c-d,c] = P[d+c,-c]$ to obtain (5.6).

We will now prove the theorem by induction on k. Our inductive hypothesis will be the following:

$$P[kc, -c] = \frac{1}{(k+1)\Theta_k} \quad \text{for some } \Theta_k \in [L,1]. \tag{5.7}$$

For $k = 1$, (5.7) holds with $\Theta_1 = 1$, by symmetry. Assume (5.7) for k. Let d equal kc in (5.6); upon substituting (5.7) into (5.6), we obtain $P[(kc+c), -c] = 1/((k+1)\Theta_k + \Lambda)$. Then (5.7) holds for $k+1$, with $\Theta_{k+1} = ((k+1)\Theta_k + \Lambda)/(k+2)$. Thus (5.7) holds for all $k \ge 1$, by induction. The theorem follows immediately. □

6. Acknowledgments. I would like to thank my thesis advisor, Frank Spitzer, for his encouragement, and for many stimulating conversations. I am also grateful to Harry Kesten and Thomas M. Liggett for their help and advice.

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