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SCALING LIMITS FOR ASSOCIATED RANDOM MEASURES

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The problem of estimating $\text{Prob}(X(B) \leq x)$ for large regions $B \subseteq \mathbb{R}^d$ when X is a random measure is solved under a condition of positive dependence and summable correlations. Several applications are given and, in cases in which the applications have been examined previously, it is shown that the results are true under milder moment conditions than known before.

1. Introduction. An important problem in the theory of random measures is the problem of estimating $\text{Prob}(X(B) \leq x)$, for large regions $B \subseteq \mathbb{R}^d$, where $X(B)$ represents the mass of the set B for the random measure X. Of particular interest to us in the applications is the case when X is a point random field; see also Burton and Waymire (1984) for related results. Previous approaches to the problem in this latter case have relied on various types of mixing conditions; see Brillinger (1975), Daley and Vere-Jones (1972), and Ivanoff (1982), for example. While the results based on mixing enjoy applications to important classes of point random fields, for example the Poisson cluster fields, the moment requirements typically appear stronger than one would expect to be necessary.

In one of the truly important recent results of probability theory, Newman (1980) has established the central limit theorem for stationary families of random variables indexed by Z^d under a simple "summability decay rate" condition on the correlations by exploiting an often natural additional condition of positive dependence (association). Newman's central limit theorem has subsequently been refined in several directions; see Newman and Wright (1981), Cox and Grimmett (1984).

In the present paper Newman's ideas are investigated in the case of nonlattice random fields. For this the corresponding notions of positive dependence are introduced and then shown to hold for various important classes of random measures. As a consequence, it is shown that one can get estimates for probabilities of the form mentioned at the start under milder moment conditions than previously established by other techniques (e.g., mixing).

Preliminary definitions are given in Section 2 as well as a precise statement of the problem and a few references to previous results. In Section 3 the notion of associated random measures is introduced and some basic properties are given. The extension of Newman's central limit theorem to the present context is stated in Section 4 and applications are provided in Section 5. All proofs have been relegated to Section 6.

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2. Preliminaries. Let $\mathscr{B}^d$ denote the collection of Borel subsets of d-dimensional Euclidean space $\mathbb{R}^d$. The set M of all nonnegative measures μ defined on $(\mathbb{R}^d, \mathscr{B}^d)$ and finite on bounded sets (i.e., Radon measures) will be equipped with the smallest sigma field $\mathscr{M}$ containing basic sets of the form $\{\mu \in M\colon \mu(A) \le r\}$ for $A \in \mathscr{B}^d$, $0 \le r \le \infty$.

A *random measure* X is a measurable map from a probability space $(\Omega, \mathscr{F}, P)$ into $(M, \mathscr{M})$. The induced measure $P_X = P \circ X^{-1}$ on $(M, \mathscr{M})$ is the *distribution* of X. In the special case when the distribution of X is concentrated on the class N of nonnegative integer valued Radon measures we refer to X as a *point random field*.

M becomes a Polish space when equipped with the vague topology and the sigma field $\mathscr{M}$ coincides with the Borel sigma field for this topology (cf., Kallenberg, 1976). Moreover N is closed in the vague topology for M (and therefore measurable). We shall denote the restriction of the sigma field $\mathscr{M}$ to N by $\mathscr{N}$.

If X, X_n $(n = 1, 2, \dots)$ are random measures, then we say that X_n *converges in distribution* to X as $n \to \infty$ iff P_{X_n} converges weakly to P_X; i.e., for continuous bounded functions f on M, with the vague topology, $\lim_n \int_M f(\mu) P_{X_n}(d\mu) = \int_M f(\mu) P_X(d\mu)$. In particular, if $B_1, \dots, B_m \in \mathscr{B}^d$ are disjoint then the multivariate distribution of $(X_n(B_1), \dots, X_n(B_m))$ converges weakly to that of $(X(B_1), \dots, X(B_m))$ in $\mathbb{R}^m$ as $n \to \infty$. The converse is also true.

In the case when X is a point random field the realizations of X may be regarded as configurations of points in $\mathbb{R}^d$, together with their multiplicities, such that there are at most finitely many points in each compact subset of $\mathbb{R}^d$ by the Radon property. We shall require regularity in the distribution of X to the extent that for each bounded Borel set $D \subseteq \mathbb{R}^d$ and nonnegative integer n there are measurable functions $r_D = r_D^{(n)}\colon D^n \to [0, \infty)$ such that

$$P(X(A_1) = k_1, \dots, X(A_\ell) = k_\ell)$$
$$(2.1)\quad = \sum_{m=0}^{\infty} \binom{m+k}{k_1, \dots, k_\ell, m} \int \cdots \iint_{A_1^{k_1} \times \cdots \times A_\ell^{k_\ell} \times C^m} \frac{1}{(m+k)!} r_D(x_1, \dots, x_{m+k})\, d\mathbf{x}$$

for disjoint measurable sets $A_1, \dots, A_\ell$ in D and nonnegative integers $k_1 \cdots k_\ell$, $\ell \ge 1$, where $k = k_1 + \cdots + k_\ell$, $C = D \setminus \bigcup_{i=1}^{\ell} A_i$. The functions r_D shall be referred to as the *absolute product densities* of X. We are assuming, of course, that the probability of exactly n occurrences in D at $x_1, \dots, x_n$ is given by $(1/n!) r_D(x_1 \cdots x_n) \Delta x_1 \cdots \Delta x_n$ in the limit of sufficiently small neighborhoods Δx_i of x_i, $1 \le i \le n$, from which such a representation as (2.1) would follow.

Closely related are the ordinary product densities $p^{(n)}(x_1, \dots, x_n)$. If D is a bounded Borel set containing $x_1, \dots, x_n$ then define

$$(2.2)\qquad p^{(n)}(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{D^k} r_D(x_1, \dots, x_{n+k})\, dx_{n+1} \cdots dx_{n+k}$$

So $p^{(n)}(x_1, \dots, x_n)(\Delta x)^n$ is approximately the probability that there are point occurrences in fixed regions of volume Δx about $x_1, \dots, x_n$ for small enough Δx. This definition is independent of the set D containing $x_1, \dots, x_n$.

If X is a point random field with product densities $p^{(n)}$, $n = 1, 2, \dots,$ then the cumulant densities, or correlation functions, are indirectly defined by means of the *cluster expansion*

$$p^{(1)}(x_1) = q^{(1)}(x_1) \tag{2.3}$$

$$p^{(2)}(x_1, x_2) = q^{(2)}(x_1, x_2) + q^{(1)}(x_1)q^{(1)}(x_2) \tag{2.4}$$

$$\begin{aligned} p^{(3)}(x_1, x_2, x_3) = {} & q^{(3)}(x_1, x_2, x_3) + q^{(2)}(x_1, x_2)q^{(1)}(x_3) \\ & + q^{(2)}(x_1, x_3)q^{(1)}(x_2) + q^{(2)}(x_2, x_3)q^{(1)}(x_1) \\ & + q^{(1)}(x_1)q^{(1)}(x_2)q^{(1)}(x_3). \end{aligned} \tag{2.5}$$

In general, the nth equation in the hierarchy is obtained by decomposing $p^{(n)}(x_1, \dots, x_n)$ as a sum of factorizations with respect to subdivisions of the configuration $x_1, \dots, x_n$.

The product densities and cumulant densities, when they exist, are simply densities of the factorial moment and factorial cumulant measures, respectively, when these are absolutely continuous with respect to Lebesgue measure; see Daley and Vere-Jones (1972). In particular, it follows that

$$EX(B) = \int_B p^{(1)}(x)\,dx = \int_B q^{(1)}(x)\,dx, \qquad B \in \mathscr{B}^d \tag{2.6}$$

$$\begin{aligned} \operatorname{Cov}(X(A), X(B)) &= \int\!\!\int_{A \times B} q^{(2)}(x_1, x_2)\,dx_1\,dx_2, \\ &\text{if } A \cap B = \phi, \qquad A, B \in \mathscr{B}^d \end{aligned} \tag{2.7}$$

$$\begin{aligned} \operatorname{Var}(X(A)) &= \operatorname{Cov}(X(A), X(A)) \\ &= \int\!\!\int_{A \times A} q^{(2)}(x_1, x_2)\,dx_1\,dx_2 + \int_A q^{(1)}(x)\,dx, \qquad A \in \mathscr{B}^d \end{aligned} \tag{2.8}$$

A random measure X is *stationary* (or homogeneous) if for all bounded $B_1, \dots, B_n \in \mathscr{B}^d$ the distribution of $(X(B_1 + x), \dots, X(B_n + x))$ is independent of $x \in \mathbb{R}^d$. Most random measures to be discussed in this paper will be stationary. Let $I = [0,1]^d$ be the unit cube, then the *intensity* of a stationary random measure is $E[X(I)]$ (which may be infinite).

If X is a stationary random measure we say that X satisfies a *classical scaling limit* if X lies in the domain of attraction of Gaussian white noise for the scaling parameter $\lambda^{d/2}$, i.e., for all disjoint rectangles (products of finite intervals) $A_1, \dots, A_n$,

$$\left(\frac{X(\lambda A_1) - E[X(\lambda A_1)]}{\lambda^{d/2}}, \dots, \frac{X(\lambda A_n) - E[X(\lambda A_n)]}{\lambda^{d/2}} \right)$$

converges in distribution (as $\lambda \to \infty$) to a multivariate normal with mean vector 0 and diagonal covariance matrix whose diagonal terms are $\sigma^2|A_1|, \dots, \sigma^2|A_n|$ (where $|A_i|$ equals the Lebesgue measure of A_i) for some positive parameter σ^2.

A natural approach to finding classes of random measures with classical scaling limits is to investigate mixing properties. In the point random field case a well-known mixing condition is that of *B mixing* (B for Brillinger). A point random field is B mixing if cumulant densities of all orders exist and are integrable. Ivanoff (1982) has shown that every stationary, B-mixing point random field satisfies a classical scaling limit. In this paper we take another approach in which we investigate positive dependence properties, such as association, of point random fields. The underlying objective is to try to exploit ideas behind a recent central limit theorem for associated random variables due to C. M. Newman (1980).

3. Association of random measures. There is a partial ordering on $\mathbb{R}^n$ given by $\underline{x} = (x_1, \dots, x_n) \leq \underline{y} = (y_1, \dots, y_n)$ if $x_i \leq y_i$ for each coordinate, $1 \leq i \leq n$. Recall that an infinite family $\mathfrak{F}$ of random variables is *associated* if for any finite subfamily $Y_1, \dots, Y_n \in \mathfrak{F}$ and f, g: $\mathbb{R}^n \to [0,1]$, continuous and increasing with respect to the above ordering on $\mathbb{R}^n$, the $\mathrm{Cov}(f(Y_1, \dots, Y_n), g(Y_1, \dots, Y_n)) \geq 0$; see Esary, Proschan, and Walkup, (1967). We want to extend this notion to random measures.

DEFINITION 3.1. A random measure X is *associated* if and only if the family of random variables $\mathfrak{F} = \{X(B)\colon B \text{ a Borel set}\}$ is associated.

There is a natural partial ordering on M given by $\mu \leq \nu$ for $\mu, \nu \in M$ if for every Borel set B we have $\mu(B) \leq \nu(B)$. In the point random field case this amounts to stipulating that every occurrence in μ also be an occurrence in ν.

THEOREM 3.2. *A random measure X with distribution P_X is associated if and only if whenever $F, G\colon M \to [0,1]$ are increasing with respect to the ordering on M and are P_X-continuous (i.e., the set of discontinuities for the vague topology on M has P_X-measure 0) then $\mathrm{Cov}_X(F, G) \geq 0$, where*

$$\mathrm{Cov}_X(F, G) = \int_M F(\mu) G(\mu) P_X(d\mu) - \int_M \int_M F(\lambda) G(\mu) P_X(d\lambda) P_X(d\mu).$$

As will be seen, the following result provides a useful means for checking association in the case of point random fields.

THEOREM 3.3. *Suppose a point random field X has piecewise continuous absolute product densities that satisfy*

$$r_D(x_1, \dots, x_n) r_D(x_i, \dots, x_j) \geq r_D(x_1, \dots, x_j) r_D(x_i, \dots, x_n) \tag{3.4}$$

for all cubes $D \subseteq \mathbb{R}^d$, all $x_1, \dots, x_n \in \mathbb{R}^d$ and $1 \leq i \leq j \leq n$. Then X is associated.

One consequence of the above theorem is that a renewal process is associated (as a point process) if the lifetime density is log convex. Details will appear in a separate publication as an application of the results given here.

We list the properties of association for random measures.

(3.5) If X is Poisson then X is associated.

(3.6) If X, Y are independent and associated then $X + Y$ is associated.

(3.7) If X is a random measure and F: $M \to M$ is increasing then $F(X)$ is associated.

(3.8) If X_n converges to X in distribution and if each X_n is associated then X is associated.

To see (3.5) suppose X is Poisson and take $A_1, \dots, A_n \in \mathscr{B}^d$ bounded. We must show the random variables $X(A_1), \dots, X(A_n)$ are associated. Disjointify $A_1, \dots, A_n$; i.e., find disjoint sets $B_1, \dots, B_m$ so that each A_i is a union of B_j's. Then $(X(B_1), \dots, X(B_m))$ are independent, hence associated, and each $X(A_i)$ is a sum of some of the $X(B_j)$'s so the $X(A_i)$'s are associated because they are increasing functions of associated random variables. (3.6) and (3.8) follow directly from the corresponding properties for associated random variables. (3.7) follows directly from definition.

4. A classical scaling limit. In 1980 C. Newman proved that the renormalized block sums of stationary associated random variables indexed by the lattice Z^d converge in distribution to iid Gaussian random variables if the covariance function is summable. We extend this to random measures as follows.

THEOREM 4.1. *Suppose that X is a stationary associated point random measure such that $EX^2(B) < \infty$ for bounded $B \in \mathscr{B}^d$ and*

$$\sum_{\mathbf{k} \in Z^d} \operatorname{Cov}(X(I), X(I + \mathbf{k})) = \eta < \infty, \tag{4.2}$$

where I is the unit cube. Then X satisfies a classical scaling limit with parameter η. Moreover the assertion remains true if X is a stationary associated (finitely additive) random interval function satisfying (4.2).

The notion of random interval functions is discussed in Daley and Vere-Jones, Definition 2.12, page 317, (1972).

Notice that because X is associated we have $\eta > 0$ for nondegenerate X. We will say that a random measure X that satisfies (4.2) has a summable second-order correlation.

In the case when X is a stationary point random field with cumulant densities $q^{(1)}$ and $q^{(2)}$, then by (2.7)–(2.9), the condition (4.2) is satisfied if $\lambda = \int_I q^{(1)}(x)\,dx < \infty$ and

$$\gamma = \int_{\mathbb{R}^d} q^{(2)}(x_1, 0)\,dx_1 < \infty \quad \text{with } \eta = \lambda + \gamma \text{ in (4.2)}.$$

As pointed out to us by the referee, the result of Theorem (4.1) can actually be strengthened to get a functional scaling limit in dimensions one and two by an application of the results in Newman and Wright (1981, 1982). For this one defines the rescaled random field indexed by a multidimensional parameter $t = (t_1, \dots, t_d)$, $d = 1, 2$, by

$$X_\lambda(t) = \lambda^{-d/2}\left[X((0, \lambda t_1] \times \cdots \times (0, \lambda t_d]) - \lambda^d \rho t_1 \cdots t_d\right]. \tag{4.3}$$

Then X_λ converges for the Skorokhod topology on the appropriate function space to the d-parameter Brownian sheet as $\lambda \to \infty$ under the conditions of Theorem 4.1, for $d = 1, 2$.

5. Applications.

5.1 Poisson center cluster random measures. Let U be a Poisson random field with intensity ρ and let V be a random measure with $EV(\mathbb{R}^d) = \gamma$. The random measure X is defined by letting the occurrences of U act as centers or initiators and then superimposing iid random measures distributed as V but centered at the occurrences of U. More precisely, let U have occurrences $\{x_i\}$ and let $\{V_i\}$ be iid random measures independent of U and distributed as V. For bounded $B \in \mathscr{B}^d$ we set

$$X(B) = \sum_i V_i(B + x_i). \tag{5.1}$$

We denote X by $[U, V]$ and observe that X is a stationary random measure with intensity $\rho\gamma$.

The most studied class of such random measures is the case where V is required to be a point random field. A particularly tractable special case was used by Neymen and Scott (1958) to model clusters of galaxies. Cluster point random fields with Poisson centers have also been used to model earthquake occurrences by Vere-Jones (1970) as well as space-time rainfall by Waymire, Gupta and Rodriquez (1984). Foundational calculations for these models appear in Westcott (1971).

Also worthy of note is the case where the random measure V is assumed to be absolutely continuous. These models are referred to as smoothed Poisson random fields and arise in the theory of shot noise (see Vanmarcke, 1983) and in the description of gravitational fields as in Feller (1966) and Daley (1971).

We have the perhaps surprising theorem.

THEOREM 5.2. *X defined as above is a well defined stationary random measure that is associated.*

Combining theorems 4.1 and 5.2 we get

THEOREM 5.3. *If $X = [U, V]$ as above with V a point random field such that $E[V(\mathbb{R}^d)^2] = \xi < \infty$, then X satisfies a classical scaling limit with parameter $\rho\xi$.*

Using mixing techniques G. Ivanoff (1982) proved Theorem 5.3 in the case V is a point random field with $E[V(\mathbb{R}^d)^3] < \infty$ and whose first four cumulant densities exist.

5.2 Critical branching point random fields. This much studied example is an evolution of point random fields. X_o is a Poisson point random field on $\mathbb{R}^d$ where $d \geq 3$. The occurrences in X_o undergo independent Brownian motion. Each particle independently of the others is either erased or splits into two particles (each with probability $\frac{1}{2}$) after an exponentially distributed length of time. X_t is the evolved field at time t. Since X_t is a Poisson center cluster point random field it is associated and has a classical scaling limit. X_t converges to a steady state distribution X_∞ as $t \to \infty$ (Dawson, 1977) which must also be associated by (3.8). However, X_∞ has a nonclassical scaling limit (the so-called massless free field); see Dawson (1977).

5.3 Dependent thinning. Let X be a Poisson point random field with parameter λ. We define a derived point random field as follows. Let f: $\mathbb{R}^d \to [0, \infty)$ be Borel measurable and have compact support. Define F: $N \to N$ (recall N is identified with the set of countable subsets of $\mathbb{R}^d$ with the property that only a finite number of points may belong to bounded sets) by $F(\omega) = \bar{\omega}$ where $\omega = \{x_i\}$, $\bar{\omega} = \{\bar{x}_i\}$, and $\bar{x}_i \in \bar{\omega} \Leftrightarrow \bar{x}_i \in \omega$ and $\sum_j f(x_i - x_j) \geq 1$ [> 1 would also do]. Then set $X' = F(X)$. Roughly speaking, X' is obtained from X by deleting all occurrences that do not have "enough" nearby occurrences. For example, if f is the indicator function of the closed unit ball then X' is obtained by deleting all occurrences whose nearest neighbor is further away than one unit.

Since F is increasing on N it is immediate by (3.7) that X' is associated. Also since f has compact support and X has independent increments there is a finite subset $J \subseteq Z^d$ so $\mathbf{k} \notin J \Rightarrow \operatorname{Cov}(X'(I), X'(I + \mathbf{k})) = 0$. Then using Cauchy–Schwarz we have

$$\begin{aligned}
\sum_{\mathbf{k} \in Z^d} \operatorname{Cov}(X'(I), X'(I+k)) &= \sum_{\mathbf{k} \in J} \operatorname{Cov}(X'(I), X'(I+k)) \\
&\leq \sum_{\mathbf{k} \in J} E[X'(I) X'(I+\mathbf{k})] \\
&\leq \sum_{\mathbf{k} \in J} \left(E\left[X'(I)^2\right] E\left[X'(I+k)^2\right]\right)^{1/2} \\
&\leq \sum_{\mathbf{k} \in J} E\left[X'(I)^2\right]^{1/2} E\left[X'(I+\mathbf{k})^2\right]^{1/2} \\
&= |J|(\lambda + \lambda^2) < \infty.
\end{aligned}$$

Thus X' has summable correlations and we get the following theorem.

THEOREM 5.4. *X' defined as above is associated and satisfies a classical scaling limit.*

5.4 Doubly stochastic point random fields. Let Λ be a stationary, associated random measure and let Z be doubly stochastic with environment Λ; that is Z is conditionally Poisson with intensity measure Λ. Then we have the following theorem.

THEOREM 5.5. *Z is associated. Further if Λ has summable correlations then so does Z, so in this case, Z satisfies a classical scaling limit.*

A more general version of this theorem is given in Burton and Waymire (1984).

6. Proofs of preceding statements. In several of the proofs below the following approximation will be used. Suppose X is a random measure. Partition the half-open cube $[-n, n)^d$ into half-open cubes $A_1, \ldots, A_m$ of side length $(\frac{1}{2})^n$ (so $m = (2n2^n)^d$). Let $D_n = \{x_1, \ldots, x_m\}$ be the set of lower left hand corner points of $A_1, \ldots, A_m$. Define the random measure X_n by

$$X_n(B) = \sum_{i:\, x_i \in B} X(A_i)$$

for each bounded Borel $B \subseteq \mathbb{R}^d$. Intuitively, X_n is obtained from X by erasing all the mass outside of $[-n, n)^d$ and moving the mass in each A_i down to the point x_i. Clearly, since X is a.s. countably additive, X_n converges to X in distribution. Each X_n is an easier object to analyze because it only depends on the finite number of random variables $X(A_1), \ldots, X(A_m)$.

PROOF OF THEOREM 3.2. If X satisfies the conditions of the theorem then it is easy to see that $X(B)$, for bounded $B \in \mathscr{B}^d$, is an associated family of random variables because if f, g: $\mathbb{R}^n \to [0,1]$ are continuous and increasing then both $f(X(B_1), \ldots, X(B_n))$ and $g(X(B_1), \ldots, X(B_n))$ are increasing with respect to the ordering on M and are P_X-continuous so their P_X-convariance is nonnegative. Conversely suppose $X(B)$, bounded $B \in \mathscr{B}^d$, is an associated family of random variables and F, G: $M \to [0,1]$ are increasing and P_X-continuous. Then, if X_n is the random measure that approximates X as in the preceding discussion we have $\mathrm{Cov}_{X_n}(F, G) \geq 0$ because with respect to the measure induced by X_n, the distribution of F depends only on the random variables $X(A_1), \ldots, X(A_m)$ and is increasing in these values. Further, since F, G are bounded and P_X-continuous $\mathrm{Cov}_{X_n}(F, G)$ converges to $\mathrm{Cov}_X(F, G)$ which is thus also nonnegative. □

PROOF OF THEOREM 5.2. If $X = [U, V]$ with U Poisson with parameter ρ and $E[V(\mathbb{R}^d)] < \infty$ then X is associated: this statement is proven in stages. First suppose V is deterministic and atomic with a finite number of atoms, that is $V = \sum_{i=1}^n a_i \delta_{x_i}$ with $a_i > 0$ and where δ_{x_i} is the Dirac measure concentrated at $x_i \in \mathbb{R}^d$. Then $\{X(A)\}$ is associated because $X(A) = \sum_{i=1}^n a_i U(A - x_i)$ which is an increasing function of random variables from $U(B)$, bounded $B \in \mathscr{B}^d$, which are associated because U is Poisson. Next suppose V is discrete, that is $V = V_i$ with probability p_i $1 \leq i \leq k$ ($\sum_1^k p_i = 1$) and where V_i is deterministic and atomic with at most a finite number of atoms. Now for each i, $1 \leq i \leq k$, let U_i be an independent Poisson point random field with parameter ρp_i. Set $X_i = [U_i, V_i]$. Then $X_1, \ldots, X_k$ are independent and each is associated. Then $X_1 + \cdots + X_k$

has the same distribution as X which is associated by property (3.6). Now suppose V is any random measure satisfying $E[V(\mathbb{R}^d)] < \infty$. Let V be the random measure approximating V as in the discussion preceding the proofs. Set $X^{(n)} = [U, V_n]$ so $X^{(n)}$ is associated. We show $X^{(n)} \to X$ which will complete the proof because association is preserved under convergence of random measures (Property 3.9). Let λ_n be the intensity measure of V_n and λ be the intensity measure of V (i.e., $\lambda(B) = E[V(B)]$). Let B_s be the open ball of radius s and center 0 in $\mathbb{R}^d$ and B_s^c its complement. By Fleischman, (page 54, 1978) it is enough to show $\lim_{s\to\infty}\sup_n \int_{B_s^c}\lambda_n(B_r - x)\,dx = 0$. But

$$\begin{aligned}
&\int_{B_s^c}\lambda_n(B_r - x)\,dx\\
&= \int_{B_s^c}\int_{\mathbb{R}^d} 1_{\{B_r - x\}}(y)\lambda_n(dy)\,dx = \int_{\mathbb{R}^d}\int_{B_s^c} 1_{B_r}(y+x)\,dx\,\lambda_n(dy)\\
&= \int_{B_{s-r}^c}\int_{B_s^c} 1_{B_r}(y+x)\,dx\,\lambda_n(dy) \le \int_{B_{s-r}^c} |B_r|\lambda_n(dy)\\
&= |B_r|\lambda_n(B_{s-r}^c) \le |B_r|\lambda\left(B_{s-r-\sqrt{d}}^c\right) \to 0
\end{aligned}$$

as $s \to \infty$ since λ is a finite measure. □

Proof of Theorem 3.3. As before partition $K = [-n, n)^d$ into half-open cubes of side length $(\frac{1}{2})^n$ and volume $\Delta x = (\frac{1}{2})^{nd}$. Let D_n be the set of lower left hand corner points of the partition sets. We put a measure P_n on the power set $\mathscr{P}(D_n)$, the lattice of configurations consisting of points from D_n, by $P_n(R) = k_n r(x_1, \dots, x_m)(\Delta x)^m$ for $R = \{x_1, \dots, x_m\} \subseteq D_n$. k_n is a normalizing constant chosen to make P_n a probability measure, so $k_n \to 1$. Then (3.4) implies $P_n(R \cap S)P_n(R \cup S) \ge P_n(R)P_n(S)$ which is the log convexity condition in Proposition 3.1 of Fortuin, Kastelyn, and Ginibre (1971). This proposition then shows that the Bernoulli random variables $\{X_i\colon i \in D_n\}$,

$$X_i = \begin{cases} 1 & \text{if } i \in R \\ 0 & \text{if } i \notin R \end{cases}$$

are associated (where $R \subseteq D_n$ is chosen according to the distribution P_n). This extends to a point random field $\overline{X}_n$ defined by $\overline{X}_n(B) = \sum_{i\in B} X_i$, for bounded $B \in \mathscr{B}^d$, which is also associated by (3.7). The proof will be complete if we show $\overline{X}_n$ converges to X in distribution [because of (3.9)]. By Kallenberg (page 22, 1976) it is enough to show $(\overline{X}_n(A_1), \dots, \overline{X}_n(A_\ell)) \to (X(A_1), \dots, X(A_\ell))$ where $A_1, \dots, A_\ell$ are disjoint and for some N, each $A_i \subseteq [-N, N)^d$ and each A_i is made up of the partition elements of $[-N, N)^d$ (i.e., the half open cubes of side length $(\frac{1}{2})^n$). Set $C = K_N - \cup_i A_i$. We show $(X_n(A_1), \dots, X_n(A_\ell), X_n(C)) \to (X(A_1), \dots, X(A_\ell), X(C))$. Fix nonnegative integers $m_1, \dots, m_{\ell+1}$ and set $m = \sum_{i=1}^{\ell} m_i$. Then because the absolute product densities can be approximated by

Riemann sums and $k_n \to 1$ we have

$$\begin{aligned}
&\operatorname{Prob}[X_n(A_i)=m_i, 1 \le i \le \ell;\ X_n(C)=m_{\ell+1}] \\
&= \frac{1}{m_1! \cdots m_{\ell+1}!} \\
&\quad \times \sum_{x_i \in A_1 \cap D_n;\ 1 \le i \le m_1} \cdots \sum_{x_i \in C \cap D_N;\ m \le i \le m_{\ell+1}} k_n r(x_1, \dots, x_{m+m_{\ell+1}})(\Delta x)^{m+m_{\ell+1}} \\
&= \frac{1}{m_1! \cdots m_{\ell+1}!} \int_{A_1^{m_1}} \cdots \int_{C^{m_{\ell+1}}} r(x_1, \dots, x_{m+m_{\ell+1}})\, dx_1 \cdots dx_{m+m_{\ell+1}} \\
&= \operatorname{Prob}[X(A_i) = m_i, 1 \le i \le \ell;\ X(C) = m_{\ell+1}]. \quad \square
\end{aligned}$$

LEMMA 6.1. *Suppose that X is a stationary associated random interval function such that* $\operatorname{Var} X(I) = \sigma^2 < \infty$ *where I is the unit cube. If*

$$\sum_{\mathbf{k} \in Z^d} \operatorname{Cov}(X(I), X(I+\mathbf{k})) = \zeta < \infty$$

then for $B = \bigcup_{\mathbf{k} \in \Lambda}(I + \mathbf{k})$ where $\Lambda \subset Z^d$ is finite and $I + \mathbf{k}$ is a translate of I for $\mathbf{k} \in \Lambda$, we have

$$\operatorname{Var} X(B) \le \operatorname{card}(\Lambda) \cdot (\sigma^2 + \zeta).$$

PROOF. The proof is the simple computation

$$\begin{aligned}
\operatorname{Var} X(B) &= \sum_{\mathbf{k} \in \Lambda} \operatorname{Var} X(I + \mathbf{k}) + \sum_{\mathbf{l}, \mathbf{k} \in \Lambda;\ \ell \ne \mathbf{k}} \operatorname{Cov}(X(I+\mathbf{k}), X(I + \mathbf{l})) \\
&\le \sigma^2 \operatorname{card}(\Lambda) = \sum_{\mathbf{l} \in \Lambda} \sum_{\mathbf{k} \in Z^d} \operatorname{Cov}(X(I), X(I+\mathbf{k})) \\
&= (\sigma^2 + \zeta)\operatorname{card}(\Lambda). \quad \square
\end{aligned}$$

PROOF OF THEOREM 4.1. Let $I = [0,1)^d$. Let X denote a random interval function subject to the conditions of the theorem.

Consider first the distribution of

$$\frac{X(\lambda I) - EX(\lambda I)}{\lambda^{d/2}}$$

as $\lambda \to \infty$. Let $Z_{\mathbf{n}} = X(I + \mathbf{n})$, $\mathbf{n} \in Z^d$. Then $\{Z_{\mathbf{n}}\colon \mathbf{n} \in Z^d\}$ is an associated family of stationary random variables for which

$$\eta = \sum_{\mathbf{n} \in Z^d} \operatorname{Cov}(Z_{\mathrm{o}}, Z_{\mathbf{n}}) < \infty.$$

From Newman's Theorem [see Newman (1980)], it follows that

$$\left\{ \frac{S_{\mathbf{n}}^k - ES_{\mathbf{n}}^k}{k^{d/2}} : \mathbf{n} \in Z^d \right\} \to \{W_{\mathbf{n}} \colon \mathbf{n} \in Z^d\} \quad \text{as } k \to \infty,$$

in the sense of convergence of finite dimensional distributions, where

$\{W_{\mathbf{n}}: \mathbf{n} \in Z^d\}$ are iid mean zero, variance η, Gaussian random variables and

$$S_{\mathbf{n}}^k = \sum_{\nu \in [k(I+\mathbf{n})]} Z_\nu$$

where

$$[B] = \{\mathbf{n} = (n_1, \dots, n_d) \in Z^d: \mathbf{n} \in B\} = Z^d \cap B.$$

Let

$$D_\lambda = \{\mathbf{n} \in Z^d: I + \mathbf{n} \subseteq [\lambda] \cdot I = [([\lambda] - 1)I]\},$$

where $[\lambda]$ denotes the greatest integer in λ. Also let

$$I_\lambda^0 = \lambda I \backslash [\lambda] I$$

Then

$$X(\lambda I) = X(I_\lambda^0) + \sum_{\mathbf{n} \in D_\lambda} Z_{\mathbf{n}}.$$

Moreover,

$$X(\lambda I) - EX(\lambda I) = X(I_\lambda^0) - EX(I_\lambda^0) + \sum_{\mathbf{n} \in D_\lambda} (Z_{\mathbf{n}} - EZ_{\mathbf{n}}).$$

However, by Lemma 6.1, since $I_\lambda^0 \subset ([\lambda] + 1)I \setminus [\lambda]I$ and $\mathrm{Cov}(X(A), X(B)) \geq 0$,

$$\mathrm{Var}\, X(I_\lambda^0) = 0(\lambda^{d-1}) \quad \text{as } \lambda \to \infty.$$

In particular, it follows from Chebyshev's inequality that

$$\frac{X(I_\lambda^0) - EX(I_\lambda^0)}{\lambda^{d/2}} \to 0 \text{ in probability as } \lambda \to \infty.$$

Since $[\lambda]^d \sim \lambda^d$ as $\lambda \to \infty$ the result follows from Newman's central limit theorem in the case of the marginal distribution of $X(\lambda I)$, centered and scaled. For arbitrary disjoint unit cubes $I_1, \dots, I_m$ the same considerations may be applied to the random vector $(X(\lambda I_1), \dots, X(\lambda I_m))$ centered and scaled. □

Proof of Theorem 5.3. In view of Theorems (5.2) and (4.1), it suffices to show that

$$\sum_{\mathbf{k} \in Z^d} \mathrm{Cov}(X(I), X(I + \mathbf{k})) = \rho\xi < \infty.$$

To see this simply note

$$\mathrm{Cov}(X(I), X(I + \mathbf{k})) = \int_{\mathbb{R}^d} E\{V(I + \mathbf{x}) V(I + \mathbf{k} + \mathbf{x})\} \rho\, d\mathbf{x}$$

so that

$$\begin{aligned}\sum_{\mathbf{k}} \mathrm{Cov}(X(I), X(I + k)) &= \sum_{\mathbf{k}} \int_{\mathbb{R}^d} E\{V(I + \mathbf{x}) V(I + \mathbf{k} + \mathbf{x})\} \rho\, d\mathbf{x} \\ &= \int_{\mathbb{R}^d} E\{V(I + \mathbf{x}) V(\mathbb{R}^d)\} \rho\, d\mathbf{x} \\ &= \rho E V^2(\mathbb{R}^d) = \rho\xi < \infty. \quad \square\end{aligned}$$

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