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ON THE COVERAGE OF k-DIMENSIONAL SPACE BY k-DIMENSIONAL SPHERES

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Let n k-dimensional spheres, each of content a_n, be distributed within a k-dimensional cube according to density f. We derive necessary and sufficient conditions on a_n in order that the probability that the cube is completely covered at least ℓ times by the spheres, tend to one as $n \to \infty$. (Here ℓ is an arbitrary positive integer.) In the special case $f \equiv$ const., we obtain upper and lower bounds of the same order of magnitude for the probability of incomplete coverage.

1. Introduction and results. Let $\mathscr{S}_k$ denote the surface of a sphere in $[k + 1]$ (i.e., $k + 1$ dimensions), and let f be a density on $\mathscr{S}_k$. Distribute n points independently on $\mathscr{S}_k$ according to f, and at each point construct a circular "cap" of fractional angular radius $a = a_n$. What properties must a_n have in order that the probability of complete coverage of $\mathscr{S}_k$ at least $\ell(\geq 1)$ times, tend to one as $n \to \infty$?

Previous treatment of this problem has been confined largely to the case where f is the uniform density, and to small values of k and ℓ. For example, when $f \equiv$ const. and $k = \ell = 1$, it may be deduced from work of Stevens (1939), Fisher (1940), Siegel (1979) and others, that a necessary and sufficient condition for ultimate complete coverage, is $na_n - \log n \to +\infty$. See also Shepp (1972a, 1972b). When $f \equiv$ const., $k = 2$ and $\ell = 1$, it follows from Moran and Fazekas de St. Groth (1962) and Gilbert (1965) that a necessary and sufficient condition is $na_n - \log n - \log \log n \to +\infty$. See also Miles (1969). Exact coverage probabilities have been derived by Glaz and Naus (1979) in the case determined by $f \equiv$ const., $k = 1$ and $\ell \geq 1$. (Their formulae are highly algebraically complex, and seem difficult to apply to produce asymptotic results.) Moran (1973) has provided an approximation to the probability of coverage in the case $f \equiv 1$, $k = 3$ and $\ell = 1$. For related recent work, principally in the case $k = 1$, see Siegel (1978a, 1978b, 1979), Holst (1980, 1981) and Hüsler (1982). Davy (1982) has surveyed several types of coverage problem.

Our main aim in the present paper is to give a complete solution to the coverage problem stated in the first paragraph, for a very general class of densities f and for *all* $k \geq 1$ and $\ell \geq 1$. As by-products of our investigation we derive new upper and lower bounds, of the same order of magnitude, for the probability of incomplete coverage in the case $f \equiv 1$; see Theorem 1 below.

The problem of covering a $[k + 1]$ sphere with circular "caps" is the same in all essential respects as that of filling a $[k]$ cube with $[k]$ spheres, provided we introduce an appropriate convention to take care of edge effects. Since the latter

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problem can be visualised in $k = 1$, 2 or 3 dimensions, whereas the former is really only recognisable in $k = 1$ and 2 dimensions, we shall confine ourselves to the spheres-in-a-cube problem. Let $\mathscr{C}_k$ denote the unit cube $[0, 1]^k$, and let f be a density on $\mathscr{C}_k$. Distribute n points independently throughout $\mathscr{C}_k$ according to f, and at each point construct a $[k]$ sphere of radius δ and $[k]$ content $a = a_n$. Define $V_\ell = V_\ell(n, a_n)$ to be the ℓth order vacancy in $\mathscr{C}_k$, or the amount of content in $\mathscr{C}_k$ not covered by at least ℓ of the random spheres. Then $\mathscr{C}_k$ is completely covered at least ℓ times, if and only if $V_\ell = 0$.

We interpret distances, integrals and coverage problems on $\mathscr{C}_k$ as though $\mathscr{C}_k$ were topologically a torus. See Miles (1969, pages 673–674) for a discussion of this method of disposing of edge effects.

Our main result in the case $f \equiv 1$ is the following pair of inequalities.

THEOREM 1. *Suppose $f \equiv 1$. There exist constants K_1 and K_2, depending only on k and ℓ, such that*

$$K_1 \min[n\{1 + (na_n)^{k+\ell-2}\}(1 - a_n)^n, 1] \le P(V_\ell > 0)$$
$$\le K_2 \min[n\{1 + (na_n)^{k+\ell-2}\}(1 - a_n)^n, 1]$$

uniformly in $n \ge \max(k + \ell - 1, 2\ell - 1)$ and $0 \le \delta \le \frac{1}{2}$.

The condition $0 \le \delta \le \frac{1}{2}$ ensures that no spheres self-intersect.

The following corollary provides necessary and sufficient conditions for ultimate ℓth order coverage, and is easily deduced from Theorem 1.

COROLLARY 1. *Suppose $f \equiv 1$. Then a necessary and sufficient condition for $P(V_\ell = 0) \to 1$ as $n \to \infty$, is that*

$$na_n - \log n - (k + \ell - 2)\log \log n \to +\infty$$

as $n \to \infty$.

It does not seem possible to derive a completely general version of Theorem 1, valid for arbitrary f. However, the method used to derive Theorem 1 may be adapted to yield a version of Corollary 1 for a very broad class of nonuniform densities. Our next result shows that it is behaviour in the vicinity of the *minimum* of f which determines coverage properties.

Assume that f is bounded on $\mathscr{C}_k$ and has a unique, nonzero minimum on $\mathscr{C}_k$, occurring at the interior point $\mathbf{m} \in \mathscr{C}_k$:

$$\text{for each} \quad \varepsilon > 0, \quad \inf_{|\mathbf{x}-\mathbf{m}|>\varepsilon} f(\mathbf{x}) > f(\mathbf{m}) > 0.$$

Suppose that as $\mathbf{x} \to \mathbf{m}$, $f(\mathbf{x})$ decreases to $f(\mathbf{m})$ in the usual second-order way. That is, there exists a neighbourhood $\mathscr{N} = \{\mathbf{x}: |\mathbf{x} - \mathbf{m}| \le \varepsilon\}$ of $\mathbf{m}$, such that all the first- and second-order derivatives of f exist on $\mathscr{N}$, the first derivatives vanish at $\mathbf{m}$, the second derivatives are continuous at $\mathbf{m}$, and the $k \times k$ matrix of second derivatives,

$$\mathbf{D} = (\partial^2 f(\mathbf{x})/\partial x_i \partial x_j \,|_{\mathbf{x}=\mathbf{m}}),$$

is positive definite.

THEOREM 2. *Under the above conditions on f, a necessary and sufficient condition for* $P(V_\ell = 0) \to 1$ *as* $n \to \infty$ *is that*

$$na_n f(\mathbf{m}) - \log n - (k/2 + \ell - 2)\log\log n \to +\infty \tag{1.1}$$

as $n \to \infty$.

An interesting feature of Theorems 1 and 2 is the subtle way in which the coefficient of log log n depends on the nature of f. Hall (1983) has shown that for *any* density f bounded away from zero and infinity on $\mathscr{C}_k$, there exists a constant $K(f)$ such that $P(V_1 = 0) \to 1$ provided $na_n > K(f)\log n$ for all sufficiently large n. A similar result may be proved for V_ℓ.

All the results described above have analogues in the case of a Poisson distribution of spheres in $\mathscr{C}_k$. For example, suppose points are distributed in $\mathbb{R}^k$ according to a Poisson random field with constant intensity λ per unit content. At each point, construct a $[k]$ sphere of radius δ and content $a = a_\lambda$. Let V denote the vacancy within $\mathscr{C}_k$. On this occasion it is convenient to measure distance in $\mathbb{R}^k$ in the usual way, without using the torus topology. With this convention, the following analogues of Theorem 1 and Corollary 1 hold.

THEOREM 3. *In the Poisson* (λ) *case, there exist constants* K_1 *and* K_2, *depending only on k and* ℓ, *such that*

$$K_1\min[\lambda\{1 + (\lambda a_\lambda)^{k+\ell-2}\}\exp(-\lambda a_\lambda), 1] \le P(V_\ell > 0)$$
$$\le K_2\min[\lambda\{1 + (\lambda a_\lambda)^{k+\ell-2}\}\exp(-\lambda a_\lambda), 1]$$

uniformly in $\lambda > 1$ *and* $0 < a_\lambda < \frac{1}{2}$.

COROLLARY 2. *In the Poisson* (λ) *case, a necessary and sufficient condition for* $P(V_\ell = 0) \to 1$ *as* $\lambda \to \infty$, *is that*

$$\lambda a_\lambda - \log\lambda - (k + \ell - 2)\log\log\lambda \to +\infty$$

as $\lambda \to \infty$.

Theorem 3 may be derived by following the lines of the proof of Theorem 1, and so is not proved here. It is also possible to derive an analogue of Theorem 2 for a Poisson field of variable intensity $\lambda(\mathbf{x})$, $\mathbf{x} \in \mathscr{C}_k$. All the results above may be extended to the problem of covering a rectangular box in $[k]$ by $[k]$ spheres, albeit with slightly more complicated notation.

2. Proofs.

If $na_n < \ell$ then

$$V_\ell \ge 1 - (na_n/\ell) > 0,$$

and so $P(V_\ell > 0) = 1$. This identity also holds if $na_n = \ell$. It now follows easily that if $na_n \le \ell$, the two inequalities in Theorem 1 hold. Therefore we may assume for the future that $na_n > \ell$. But $na_n > \ell$ implies that at least one point of $\mathscr{C}_k$ is covered at least ℓ times by spheres. This fact will be used without further comment in the work which follows. For example, it implies that at least two spheres intersect, and hence that the variable N_ℓ introduced just before (2.1) is well-defined.

We first derive an upper bound to $P(V_\ell > 0)$. Our proof is inspired by Gilbert (1965).

Let S_1 denote any of the n $[k]$ spheres of radius δ placed into $\mathscr{C}_k$ according to density f, and suppose S_1 has centre $\mathbf{x}_1 = (x_{11}, \cdots, x_{1k})^T$. Let $\mathbf{x}_2$ be any point distant r from $\mathbf{x}_1$, where $0 \le r \le 2\delta$. The expected number of spheres (other than S_1) centred within a rectangle of side lengths dx_{2i} about the point $\mathbf{x}_2 = (x_{21}, \cdots, x_{2k})^T$, equals $(n-1)f(\mathbf{x}_2)d\mathbf{x}_2$. Let S_2 denote such a sphere, and let $T_2(\mathbf{x}_1, \mathbf{x}_2)$ be the intersection of the *surfaces* of S_1 and S_2. Then $T_2(\mathbf{x}_1, \mathbf{x}_2)$ is the *surface* of a $[k-1]$ sphere of radius $(\delta^2 - r^2/4)^{1/2}$, whose centre and orientation in $\mathscr{C}_k$ are completely determined by $\mathbf{x}_1$ and $\mathbf{x}_2$. Let $p_2(\mathbf{x}_1, \mathbf{x}_2)$ denote the probability that the other $n-2$ spheres do *not* completely envelop $T_2(\mathbf{x}_1, \mathbf{x}_2)$ at least ℓ times.

Observe that with probability one, $\mathscr{C}_k$ is completely filled at least ℓ times by the n random $[k]$ spheres, if and only if each set formed by the intersection of the surfaces of every pair of spheres, is completely enveloped at least ℓ times by the remaining $n-2$ spheres. That is, writing N_ℓ for the number of intersections of the surfaces of two random spheres which are not enveloped at least ℓ times,

$$P(V_\ell > 0) = P(N_\ell \ge 1) \le E(N_\ell), \tag{2.1}$$

using Markov's inequality. Now, the expected number of spheres centred within a rectangle of side lengths dx_{1i} about the point $\mathbf{x}_1$, equals $nf(\mathbf{x}_1)d\mathbf{x}_1$. Therefore

$$E(N_\ell) \le \int_{\mathscr{C}_k} nf(\mathbf{x}_1)\, d\mathbf{x}_1 \int_{\mathscr{C}_k} (n-1)f(\mathbf{x}_2)p_2(\mathbf{x}_1, \mathbf{x}_2)\, d\mathbf{x}_2. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$P(V_\ell > 0) \le (nb)^2 \int_{\mathscr{C}_k^2} p_2(\mathbf{x}_1, \mathbf{x}_2)\, d\mathbf{x}_1\, d\mathbf{x}_2, \tag{2.3}$$

where $b = \sup_{\mathbf{x}\in\mathscr{C}_k} f(\mathbf{x})$, and where we define $p_2(\mathbf{x}_1, \mathbf{x}_2) = 0$ if $|\mathbf{x}_1 - \mathbf{x}_2| > 2\delta$.

If $T_2(\mathbf{x}_1, \mathbf{x}_2)$ is not completely enveloped at least ℓ times by the other $n-2$ random spheres, then one of the following occurs:

(*i*) no sphere surface intersects $T_2(\mathbf{x}_1, \mathbf{x}_2)$, and the number of single spheres which wholly contain $T_2(\mathbf{x}_1, \mathbf{x}_2)$ is fewer than ℓ; or

(*ii*) at least one sphere surface intersects $T_2(\mathbf{x}_1, \mathbf{x}_2)$, and the intersection of some sphere surface with $T_2(\mathbf{x}_1, \mathbf{x}_2)$ is not completely covered at least ℓ times by other random spheres.

Denote these events by $E_{(i)}(\mathbf{x}_1, \mathbf{x}_2)$ and $E_{(ii)}(\mathbf{x}_1, \mathbf{x}_2)$, respectively. We shall handle the two possibilities in turn.

For each $\mathbf{z} \in \mathscr{C}_k$, let $p(\mathbf{z})$ equal the probability that a single sphere placed at random into $\mathscr{C}_k$ according to density f covers $\mathbf{z}$. Let $\mathbf{z}_2 = \mathbf{z}_2(\mathbf{x}_1, \mathbf{x}_2)$ denote that point of $T_2(\mathbf{x}_1, \mathbf{x}_2)$ whose first coordinate is greatest. (Indeed, any single point of $T_2(\mathbf{x}_1, \mathbf{x}_2)$ will do.) Then $1 - p(\mathbf{z}_2)$ dominates the probability that a single sphere placed at random into $\mathscr{C}_k$ according to density f does not intersect $T_2(\mathbf{x}_1, \mathbf{x}_2)$. Therefore the probability that exactly m out of $n - 2$ spheres wholly contain $T_2(\mathbf{x}_1, \mathbf{x}_2)$, and none of the other $n - 2 - m$ spheres even intersect $T_2(\mathbf{x}_1, \mathbf{x}_2)$, is dominated by

$$\binom{n-2}{m}(a_n b)^m \{1 - p(\mathbf{z}_2)\}^{n-m-2}.$$

Consequently,

$$\begin{aligned}(2.4)\quad &\int_{\mathscr{C}_k^2 \cap \{|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta\}} P\{E_{(i)}(\mathbf{x}_1, \mathbf{x}_2)\}\, d\mathbf{x}_1\, d\mathbf{x}_2 \\ &\le \sum_{m=0}^{\ell-1} \binom{n-2}{m}(a_n b)^m \int_{\mathscr{C}_k^2 \cap \{|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta\}} \{1 - p(\mathbf{z}_2)\}^{n-m-2}\, d\mathbf{x}_1\, d\mathbf{x}_2 \\ &\le \ell b^{\ell-1}\{1 + (na_n)^{\ell-1}\} \int_{\mathscr{C}_k^2 \cap \{|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta\}} \{1 - p(\mathbf{z}_2)\}^{n-\ell-1}\, d\mathbf{x}_1\, d\mathbf{x}_2.\end{aligned}$$

Next we estimate

$$\int_{\mathscr{C}_k^2 \cap \{|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta\}} P\{E_{(ii)}(\mathbf{x}_1, \mathbf{x}_2)\}\, d\mathbf{x}_1\, d\mathbf{x}_2.$$

Let $\mathbf{x}_3$ be any point in $\mathscr{C}_k$ distant no more than δ away from at least one point of $T_2(\mathbf{x}_1, \mathbf{x}_2)$. The expected number of $[k]$ spheres (other than S_1 and S_2) centred within a rectangle of side lengths dx_{3i} about $\mathbf{x}_3$, equals $(n-2)f(\mathbf{x}_3)d\mathbf{x}_3$. Let S_3 denote such a sphere, and let $T_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ be the intersection of $T_2(\mathbf{x}_1, \mathbf{x}_2)$ with the surface of S_3. Then $T_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is the surface of a $[k-2]$ sphere whose radius does not exceed δ, and whose radius, centre and orientation in $\mathscr{C}_k$ are completely determined by $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Write $p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ for the probability that $T_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is not completely enveloped at least ℓ times by the remaining spheres. Using the argument leading to (2.1) and (2.2), we may deduce that

$$P\{E_{(ii)}(\mathbf{x}_1, \mathbf{x}_2)\} \le \int_{\mathscr{C}_k} nf(\mathbf{x}_3)p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\, d\mathbf{x}_3,$$

where $p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is defined to be zero if $|\mathbf{x}_1 - \mathbf{x}_2| > 2\delta$ or if $|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta$ but $\mathbf{x}_3$ is distant more than δ away from all points of $T_2(\mathbf{x}_1, \mathbf{x}_2)$. Thus,

$$\begin{aligned}(2.5)\quad &\int_{\mathscr{C}_k^2 \cap \{|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta\}} P\{E_{(ii)}(\mathbf{x}_1, \mathbf{x}_2)\}\, d\mathbf{x}_1\, d\mathbf{x}_2 \\ &\le nb \int_{\mathscr{C}_k^3} P_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\, d\mathbf{x}_1\, d\mathbf{x}_2\, d\mathbf{x}_3.\end{aligned}$$

Combining (2.3), (2.4) and (2.5), we obtain:

$$P(V_\ell > 0) \le (nb)^2 \int_{\mathscr{C}_k^2 \cap \{|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta\}} [P\{E_{(i)}(\mathbf{x}_1, \mathbf{x}_2)\} + P\{E_{(ii)}(\mathbf{x}_1, \mathbf{x}_2)\}]\, d\mathbf{x}_1\, d\mathbf{x}_2$$

$$\le n^2 \ell b^{\ell+1}\{1 + (na_n)^{\ell-1}\} \int_{\mathscr{C}_k^2 \cap \{|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta\}} \{1 - p(\mathbf{z}_2)\}^{n-\ell-1}\, d\mathbf{x}_1\, d\mathbf{x}_2$$

$$+ (nb)^3 \int_{\mathscr{C}_k^3} p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\, d\mathbf{x}_1\, d\mathbf{x}_2\, d\mathbf{x}_3.$$

Another iteration of this argument would produce the estimate,

$$\int_{\mathscr{C}_k^3} p_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\, d\mathbf{x}_1\, d\mathbf{x}_2\, d\mathbf{x}_3$$

$$\le \ell b^{\ell-1}\{1 + (na_n)^{\ell-1}\} \int_{\mathscr{R}_3} \{1 - p(\mathbf{z}_3)\}^{n-\ell-2}\, d\mathbf{x}_1\, d\mathbf{x}_2\, d\mathbf{x}_3$$

$$+ nb \int_{\mathscr{R}_4} p_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\, d\mathbf{x}_1\, d\mathbf{x}_2\, d\mathbf{x}_3\, d\mathbf{x}_4,$$

where $\mathbf{z}_3 = \mathbf{z}_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ denotes that point of $T_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ whose first coordinate is greatest; $\mathscr{R}_3$ is the set of triples $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ such that $|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta$ and $\mathbf{x}_3$ is distant no more than δ from some point of $T_2(\mathbf{x}_1, \mathbf{x}_2)$; $\mathscr{R}_4$ is the set of quadruples $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ such that $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathscr{R}_3$ and $\mathbf{x}_4$ is distant no more than δ from some point of $T_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$; and $p_4(\cdot)$ denotes the probability that $T_4(\cdot)$ is not completely enveloped at least ℓ times by the remaining spheres. Reasoning in this way, we conclude that

$$P(V_\ell > 0)$$

$$\le L_1 b^{k+\ell}\{1 + (na_n)^{\ell-1}\} \sum_{i=2}^{k-1} n^i \int_{\mathscr{R}_i} \{1 - p(\mathbf{z}_i)\}^{n-k-\ell+1}\, d\mathbf{x}_1 \cdots d\mathbf{x}_i \tag{2.6}$$

$$+ (nb)^k \int_{\mathscr{R}_k} p_k(\mathbf{x}_1, \cdots, \mathbf{x}_k)\, d\mathbf{x}_1 \cdots d\mathbf{x}_k,$$

where the constant L_1 depends only on ℓ; $\mathbf{z}_i(\mathbf{x}_1, \cdots, \mathbf{x}_i)$ equals that point of $T_i(\mathbf{x}_1, \cdots, \mathbf{x}_i)$ whose first coordinate is greatest; $p_k(\mathbf{x}_1, \cdots, \mathbf{x}_k)$ denotes the probability that the surface of a certain sphere $T_k(\mathbf{x}_1, \cdots, \mathbf{x}_k)$, of $k - (k-1) = 1$ dimension, is not completely enveloped at least ℓ times by $n - k$ spheres whose centres are placed into $\mathscr{C}_k$ according to density f; and $\mathscr{R}_i$ denotes that subset of $\mathscr{C}_k^i$ consisting of i-tuples $(\mathbf{x}_1, \cdots, \mathbf{x}_i)$ satisfying the condition

$$\mathbf{x}_1 \in \mathscr{C}_k; \quad |\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta; \quad \text{and} \quad \mathbf{x}_j \in \mathscr{C}_k \text{ is distant}$$

$$\text{at most } \delta \text{ from } T_{j-1}(\mathbf{x}_1, \cdots, \mathbf{x}_{j-1}), \quad \text{for} \quad 3 \le j \le i.$$

The surface of a [1] sphere consists of just two points. In the case of the [1] sphere surface $T_k(\mathbf{x}_1, \cdots, \mathbf{x}_k)$, let these points be $\mathbf{z}^{(1)}$ and $\mathbf{z}^{(2)}$. Then

$$p_k(\mathbf{x}_1, \cdots, \mathbf{x}_k) \le \sum_{i=1}^{2} \sum_{j=0}^{\ell-1} \binom{n-k}{j} p^j(\mathbf{z}^{(i)})\{1 - p(\mathbf{z}^{(i)})\}^{n-k-j}$$

$$\le \ell b^{\ell-1}\{1 + (na_n)^{\ell-1}\} \sum_{i=1}^{2} \{1 - p(\mathbf{z}^{(i)})\}^{n-k-\ell+1}.$$

Substituting this estimate into (2.6), and writing $\mathbf{z}_k$ for either $\mathbf{z}^{(1)}$ or $\mathbf{z}^{(2)}$ (precise choice does not matter), we obtain:

$$P(V_\ell > 0) \le L_2 b^{k+\ell}\{1 + (na_n)^{\ell-1}\} \sum_{i=2}^{k} n^i \int_{\mathscr{R}_i} \{1 - p(\mathbf{z}_i)\}^{n-k-\ell+1}\, d\mathbf{x}_1 \cdots d\mathbf{x}_i, \tag{2.7}$$

where the constant L_2 depends only on ℓ.

At this stage, the proofs in the cases $f \equiv 1$ and $f \not\equiv 1$ part company. Let us suppose first that $f \not\equiv 1$, and that $na_n \to \infty$ but $na_n \le (\log n)^2$ for large n. Choose $\eta > 0$ so small that all the first and second derivatives of $f(\mathbf{x})$ exist and are bounded within the neighbourhood $|\mathbf{x} - \mathbf{m}| \le 3\eta$. If $|\mathbf{x}_1 - \mathbf{m}| > 2\eta$ then, for all sufficiently large n, $|\mathbf{z}_i(\mathbf{x}_1, \cdots, \mathbf{x}_i) + \mathbf{y} - \mathbf{m}| > \eta$ uniformly in $1 \le i \le k$ and points $\mathbf{x}_1, \cdots, \mathbf{x}_i$ and $\mathbf{y}$ such that $(\mathbf{x}_1, \cdots, \mathbf{x}_i) \in \mathscr{R}_i$ and $|\mathbf{y}| \le \delta$. Therefore

$$\begin{aligned}\int_{\mathscr{R}_i \cap \{|\mathbf{x}_1 - \mathbf{m}| > 2\eta\}} &\{1 - p(\mathbf{z}_i)\}^{n-k-\ell+1}\, d\mathbf{x}_1 \cdots d\mathbf{x}_i \\ &\le \{1 - a_n \inf_{|\mathbf{x}-\mathbf{m}|>\eta} f(\mathbf{x})\}^{n-k-\ell+1} \int_{\mathscr{R}_i} d\mathbf{x}_1 \cdots d\mathbf{x}_i \\ &\le C_1 \exp\{-na_n(1+\varepsilon) f(\mathbf{m})\} \int_{\mathscr{R}_i} d\mathbf{x}_1 \cdots d\mathbf{x}_i\end{aligned} \tag{2.8}$$

for some $\varepsilon > 0$, where (here and below) C_j denotes a generic positive constant not depending on n. Note that

$$p(\mathbf{z}) = \int_{|\mathbf{y}| \le \delta} f(\mathbf{z} + \mathbf{y})\, d\mathbf{y}.$$

For each $\mathbf{x}_1, \cdots, \mathbf{x}_j$, the set of all points $\mathbf{x}_{j+1} \in \mathscr{C}_k$ distant no more than δ from $T_j(\mathbf{x}_1, \cdots, \mathbf{x}_j)$, has Lebesgue measure at most $(4\delta)^k$, *for* $2 \le j \le k - 1$. Furthermore, the set of points $\mathbf{x}_2$ distant between u and $u + du$ from $\mathbf{x}_1$, has Lebesgue measure $s_k u^{k-1} du$ where s_k denotes the surface $[k-1]$ content of a unit $[k]$ sphere. Therefore

$$\begin{aligned}\int_{\mathscr{R}_i} d\mathbf{x}_1 \cdots d\mathbf{x}_i &\le \{(4\delta)^k\}^{i-2} s_k \int_0^{2\delta} u^{k-1}\, du \\ &= C_2(\delta^k)^{i-1} = C_3 a_n^{i-1}.\end{aligned} \tag{2.9}$$

Combining this estimate with (2.8), we obtain:

$$(2.10)\quad \int_{\mathscr{R}_i\cap\{|\mathbf{x}_1-\mathbf{m}|>2\eta\}} \{1 - p(\mathbf{z}_i)\}^{n-k-\ell+1}\, d\mathbf{x}_1 \cdots d\mathbf{x}_i \le C_4 a_n^{i-1}\exp\{-na_n(1+\varepsilon)f(\mathbf{m})\}.$$

Now assume that $|\mathbf{x}_1 - \mathbf{m}| \le 2\eta$. Then for all sufficiently large n, $|\mathbf{z}_i(\mathbf{x}_1, \cdots, \mathbf{x}_i) + \mathbf{y} - \mathbf{m}| \le 3\eta$ and $|\mathbf{x}_1 + \mathbf{y} - \mathbf{m}| \le 3\eta$ uniformly in i and points $\mathbf{x}_1, \cdots, \mathbf{x}_i$ and $\mathbf{y}$ such that $(\mathbf{x}_1, \cdots, \mathbf{x}_i) \in \mathscr{R}_i$ and $|\mathbf{y}| \le \delta$. Furthermore, $|\mathbf{z}_i(\mathbf{x}_1, \cdots, \mathbf{x}_i) - \mathbf{x}_1| \le 2\delta$. Therefore

$$\left| \int_{|\mathbf{y}|\le\delta} f(\mathbf{z}_i + \mathbf{y})\, d\mathbf{y} - \int_{|\mathbf{y}|\le\delta} f(\mathbf{x}_1 + \mathbf{y})\, d\mathbf{y} \right| \le C_5\delta \int_{|\mathbf{y}|\le\delta} d\mathbf{y} = C_5\delta a_n$$

and

$$\left| \int_{|\mathbf{y}|\le\delta} f(\mathbf{x}_1 + \mathbf{y})\, d\mathbf{y} - a_n f(\mathbf{x}_1) \right| \le C_5\delta a_n,$$

whence

$$\begin{aligned} &\int_{\mathscr{R}_i\cap\{|\mathbf{x}_1-\mathbf{m}|\le 2\eta\}} \{1 - p(\mathbf{z}_i)\}^{n-k-\ell+1}\, d\mathbf{x}_1 \cdots d\mathbf{x}_i \\ (2.11)\quad &\le \int_{\mathscr{R}_i\cap\{|\mathbf{x}_1-\mathbf{m}|\le 2\eta\}} \{1 - a_n f(\mathbf{x}_1) + 2C_5\delta a_n\}^{n-k-\ell+1}\, d\mathbf{x}_1 \cdots d\mathbf{x}_i \\ &\le C_6 a_n^{i-1} \int_{|\mathbf{x}-\mathbf{m}|\le 2\eta} \exp\{-na_n f(\mathbf{x})\}\, d\mathbf{x}, \end{aligned}$$

using the argument leading to (2.9). (Note that, under our assumptions on a_n, $n\delta a_n \to 0$.) Since

$$f(\mathbf{x}) = f(\mathbf{m}) + (1/2)(\mathbf{x} - \mathbf{m})^T\mathbf{D}(\mathbf{x} - \mathbf{m}) + r(\mathbf{x}),$$

where $r(\mathbf{x}) = o(|\mathbf{x} - \mathbf{m}|^2)$ as $\mathbf{x} \to \mathbf{m}$ and $\mathbf{D}$ is the positive definite matrix of second derivatives defined just before Theorem 2, then

$$\begin{aligned} &\int_{|\mathbf{x}-\mathbf{m}|<2\eta} \exp\{-na_n f(\mathbf{x})\}\, d\mathbf{x} \\ (2.12)\quad &\sim \exp\{-na_n f(\mathbf{m})\} \int_{|\mathbf{x}-\mathbf{m}|<2\eta} \exp\left\{-\left(\frac{na_n}{2}\right)(\mathbf{x} - \mathbf{m})^T\mathbf{D}(\mathbf{x} - \mathbf{m})\right\} d\mathbf{x} \\ &\sim \exp\{-na_n f(\mathbf{m})\}(2\pi/na_n)^{k/2}|\mathbf{D}|^{-1/2}. \end{aligned}$$

Substituting the estimates (2.10)–(2.12) into (2.7), we obtain:

$$\begin{aligned} P(V_\ell > 0) &\le C_7\{1 + (na_n)^{\ell-1}\} \textstyle\sum_{i=1}^k n^i[a_n^{i-1}\exp\{-na_n(1+\varepsilon)f(\mathbf{m})\} \\ (2.13)\quad &\qquad + n^{k/2}a_n^{i-1-k/2}\exp\{-na_n f(\mathbf{m})\}] \\ &\le C_8 n(na_n)^{k/2+\ell-2}\exp\{-na_n f(\mathbf{m})\}. \end{aligned}$$

The right-hand side of this expression converges to zero if and only if (1.1) holds, and so (1.1) is a sufficient condition for $P(V_\ell = 0) \to 1$. (Note that $P(V_\ell > 0)$ is a monotone decreasing function of a_n, and so our restriction that $na_n \to \infty$ and $na_n \le (\log n)^2$ causes no difficulty.)

When $f \equiv 1$, the argument following (2.7) can be simplified considerably. In that case it follows from (2.7) and (2.9) that for $n \ge k + \ell - 1$ and $0 \le a_n \le 1/2$,

$$P(V_\ell > 0) \le D_1\{1 + (na_n)^{\ell-1}\}(1 - a_n)^n \sum_{i=2}^{k} n^i \int_{\mathscr{R}_i} d\mathbf{x}_1 \cdots d\mathbf{x}_i$$

$$\le D_2 n(na_n)^{k-1}\{1 + (na_n)^{\ell-1}\}(1 - a_n)^n,$$

where (here and below) D_j denotes a positive constant depending only on k and ℓ. Since $P(V_\ell > 0) \le 1$, the upper inequality in Theorem 1 follows immediately. (Note that we are assuming $na_n > \ell$.)

The remainder of the proof consists in deriving lower bounds to $P(V_\ell > 0)$. Since $V_\ell = V_\ell I(V_\ell > 0)$, then by the Cauchy-Schwartz inequality, $(EV_\ell)^2 \le E(V_\ell^2)P(V_\ell > 0)$, or

$$P(V_\ell > 0) \ge (EV_\ell)^2/E(V_\ell^2). \tag{2.14}$$

(See also Shepp, 1972a.) Let I denote the indicator function given by

$$I(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \text{ is not covered at least } \ell \text{ times} \\ 0 & \text{otherwise,} \end{cases}$$

for $\mathbf{x} \in \mathscr{C}_k$. Then

$$V_\ell = \int_{\mathscr{C}_k} I(\mathbf{x})\, d\mathbf{x}, \tag{2.15}$$

and if $n \ge \ell$,

$$E\{I(\mathbf{x})\} = \sum_{j=0}^{\ell-1} \binom{n}{j} \left\{\int_{|\mathbf{y}|\le\delta} f(\mathbf{x} + \mathbf{y})\, d\mathbf{y}\right\}^j \left\{1 - \int_{|\mathbf{y}|\le\delta} f(\mathbf{x} + \mathbf{y})\, d\mathbf{y}\right\}^{n-j}$$

$$\ge D_3\{1 + (na_n \inf f)^{\ell-1}\}\left\{1 - \int_{|\mathbf{y}|\le\delta} f(\mathbf{x} + \mathbf{y})\, d\mathbf{y}\right\}^n.$$

From this inequality and (2.15) we may deduce that if $f \equiv 1$,

$$E(V_\ell) \ge D_3\{1 + (na_n)^{\ell-1}\}(1 - a_n)^n, \tag{2.16}$$

and if $f \not\equiv 1$, if $na_n \to \infty$ and $na_n \le (\log n)^2$,

$$E(V_\ell) \ge C_{10}(na_n)^{l-k/2-1}\exp\{-na_n f(\mathbf{m})\}. \tag{2.17}$$

(For (2.17), use the argument leading to (2.13).)

Next observe that if $|\mathbf{x}_1 - \mathbf{x}_2| > 2\delta$,

$$
\begin{aligned}
&E\{I(\mathbf{x}_1)I(\mathbf{x}_2)\} \\
(2.18)\quad &= \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} \frac{n!}{i!j!(n-i-j)!}\left\{\int_{|\mathbf{y}|\le\delta} f(\mathbf{x}_1+\mathbf{y})\,d\mathbf{y}\right\}^i \\
&\times\left\{\int_{|\mathbf{y}|\le\delta} f(\mathbf{x}_2+\mathbf{y})\,d\mathbf{y}\right\}^j\left\{1-\sum_{\alpha=1}^{2}\int_{|\mathbf{y}|\le\delta} f(\mathbf{x}_\alpha+\mathbf{y})\,d\mathbf{y}\right\}^{n-i-j},
\end{aligned}
$$

which is dominated by

$$(2.19)\quad D_4\{1+(na_n)^{2(\ell-1)}\}(1-2a_n)^{n-2(\ell-1)} \le D_5\{1+(na_n)^{\ell-1}\}^2(1-a_n)^{2n}$$

if $f \equiv 1$, $n \ge 2\ell - 1$ and $0 < a_n \le 1/2$, and by

$$(2.20)\quad C_{11}(na_n)^{2(\ell-1)-k}\exp\{-2na_n f(\mathbf{m})\}$$

if $f \not\equiv 1$, $na_n \to \infty$ and $na_n \le (\log n)^2$. (For (2.20), use the argument leading to (2.13).) Now suppose $|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta$, and let B denote the set of $\mathbf{x} \in \mathscr{C}_k$ which satisfy both $|\mathbf{x}_1 - \mathbf{x}| \le \delta$ and $|\mathbf{x}_2 - \mathbf{x}| \le \delta$. Write A_i for the set of $\mathbf{x}$ satisfying $|\mathbf{x}_i - \mathbf{x}| \le \delta$, but which are not in B ($i = 1, 2$). Then

$$
\begin{aligned}
&E\{I(\mathbf{x}_1)I(\mathbf{x}_2)\} \\
&= \sum_{h=0}^{\ell-1}\sum_{i=0}^{\ell-1-h}\sum_{j=0}^{\ell-1-h} \frac{n!}{h!i!j!(n-h-i-j)!}\left\{\int_B f(\mathbf{y})\,d\mathbf{y}\right\}^h \\
(2.21)\quad &\times\left\{\int_{A_1} f(\mathbf{y})\,d\mathbf{y}\right\}^i\left\{\int_{A_2} f(\mathbf{y})\,d\mathbf{y}\right\}^j\left\{1-\int_{A_1\cup A_2\cup B} f(\mathbf{y})\,d\mathbf{y}\right\}^{n-h-i-j} \\
&\le b^{2(\ell-1)}\sum_{h=0}^{\ell-1}\sum_{i=0}^{\ell-1-h}\sum_{j=0}^{\ell-1-h} n^{h+i+j}a_n^h\{v(A_1)\}^{i+j} \\
&\cdot\left\{1-\int_{A_1\cup A_2\cup B} f(\mathbf{y})\,d\mathbf{y}\right\}^{n-h-i-j},
\end{aligned}
$$

where $v(A_1)$ denotes the content of A_1. Note that $v(A_1) = v(A_2)$.

It follows from (2.18)–(2.20) that

$$
\begin{aligned}
(2.22)\quad &\int_{\mathscr{C}_k^2\cap\{|\mathbf{x}_1-\mathbf{x}_2|>2\delta\}} E\{I(\mathbf{x}_1)I(\mathbf{x}_2)\}\,d\mathbf{x}_1\,d\mathbf{x}_2 \\
&\le \begin{cases} D_5\{1+(na_n)^{\ell-1}\}^2(1-a_n)^{2n} & \text{if } f\equiv 1 \\ C_{11}(na_n)^{2(\ell-1)-k}\exp\{-2na_n f(\mathbf{m})\} & \text{otherwise.}\end{cases}
\end{aligned}
$$

Next we shall use (2.21) to derive a similar estimate for the integral over $\mathscr{C}_k^2 \cap \{|\mathbf{x}_1 - \mathbf{x}_2| \le 2\delta\}$. Suppose first that $f \equiv 1$, and observe that if $|\mathbf{x}_1 - \mathbf{x}_2| = 2u\delta$, where $0 \le u \le 1$, then

$$v(A_1) = a_n\left(\frac{2v_{k-1}}{v_k}\right)\int_0^u (1-w^2)^{(k-1)/2}\,dw \le D_6 a_n u,$$

where v_k is the content of a $[k]$ unit sphere. Furthermore,

$$\int_{A_1\cup A_2\cup B} f(\mathbf{y})\, d\mathbf{y} = a_n\left\{1 + 2\left(\frac{v_{k-1}}{v_k}\right)\int_0^u (1-w^2)^{(k-1)/2}\, dw\right\} \geq a_n(1 + D_7 u).$$

Therefore if $0 < a_n \leq \frac{1}{2}$ and $n \geq 2\ell - 1$,

$$\int_{\mathscr{C}_k^2\cap\{|\mathbf{x}_1-\mathbf{x}_2|\leq 2\delta\}} \{v(A_1)\}^{i+j}\left\{1 - \int_{A_1\cup A_2\cup B} f(\mathbf{y})\, d\mathbf{y}\right\}^{n-h-i-j} d\mathbf{x}_1\, d\mathbf{x}_2$$

$$\text{(2.23)} \qquad \leq D_8 a_n^{i+j+1}\int_0^1 u^{i+j+k-1}\{1 - a_n(1 + D_7 u)\}^{n-h-i-j}\, du$$

$$\leq D_9 a_n^{i+j+1}(1-a_n)^n \int_0^1 u^{i+j+k-1}\exp(-na_nD_{10}u)\, du$$

$$\leq D_{11} a_n^{i+j+1}(1-a_n)^n (na_n)^{-(i+j+k)}.$$

Substituting this estimate into (2.21), we see that

$$\text{(2.24)} \qquad \int_{\mathscr{C}_k^2\cap\{|\mathbf{x}_1-\mathbf{x}_2|\leq 2\delta\}} E\{I(\mathbf{x}_1)I(\mathbf{x}_2)\}\, d\mathbf{x}_1\, d\mathbf{x}_2$$

$$\leq D_{12} a_n (na_n)^{-k}\{1 + (na_n)^{\ell-1}\}(1-a_n)^n.$$

Assume next that $f \not\equiv 1$, and $na_n \to \infty$ and $na_n \leq (\log n)^2$. Using the argument leading to (2.13) we see that in this case, the far left-hand side of (2.23) is dominated by

$$C_{12} a_n^{i+j+1}\int_{\mathscr{C}_k} d\mathbf{x}\int_0^1 u^{i+j+k-1}\exp\left\{-nf(\mathbf{x})\int_{A_1\cup A_2\cup B} d\mathbf{y}\right\} du$$

$$\leq C_{13} a_n^{i+j+1}(na_n)^{-(i+j+k)}\int \exp\{-na_nf(\mathbf{x})\}\, d\mathbf{x}$$

$$\leq C_{14} a_n^{i+j+1}(na_n)^{-(i+j+3k/2)}\exp\{-na_nf(\mathbf{m})\}.$$

Substituting into (2.21) we find that

$$\text{(2.25)} \qquad \int_{\mathscr{C}_k^2\cap\{|\mathbf{x}_1-\mathbf{x}_2|\leq 2\delta\}} E\{I(\mathbf{x}_1)I(\mathbf{x}_2)\}\, d\mathbf{x}_1\, d\mathbf{x}_2$$

$$\leq C_{15} a_n (na_n)^{\ell-3k/2-1}\exp\{-na_nf(\mathbf{m})\}.$$

From (2.15), (2.22) and (2.24) we obtain

$$E(V_\ell^2) \leq D_{13}\max[\{1 + (na_n)^{\ell-1}\}(1-a_n)^n,\ a_n(na_n)^{-k}]\{1 + (na_n)^{\ell-1}\}(1-a_n)^n$$

in the case $f \equiv 1$, and from (2.15), (2.22) and (2.25),

$$E(V_\ell^2) \leq C_{16}\max[(na_n)^{2(\ell-1)-k}\exp\{-2na_nf(\mathbf{m})\},\ a_n(na_n)^{\ell-3k/2-1}\exp\{-na_nf(\mathbf{m})\}]$$

in the case $f \not\equiv 1$. Combining these estimates with (2.14), (2.16) and (2.17), we deduce that

$$(2.26) \qquad P(V_\ell > 0) \geq D_{14}\min[n(na_n)^{k-1}\{1 + (na_n)^{\ell-1}\}(1 - a_n)^n, 1]$$

if $f \equiv 1$, and

$$(2.27) \qquad P(V_\ell > 0) \geq C_{17}\min[n(na_n)^{k/2+\ell-2}\exp\{-na_n f(\mathbf{m})\}, 1]$$

if $f \not\equiv 1$. Since we are assuming $na_n > \ell$ (note the first paragraph of Section 2), then inequality (2.26) is equivalent to the lower bound in Theorem 1. The right-hand side of (2.27) tends to zero if and only if condition (1.1) holds, and so (1.1) is necessary for $P(V_\ell = 0) \to 1$.

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