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BROWNIAN PATHS AND CONES[1]

By Krzysztof Burdzy

University of California, Berkeley[2]

If $\cos(\theta/2) < 1/\sqrt{n}$ then a.s. there are times $0 \le s_1 < s_2$ such that the n-dimensional Brownian motion $Z(t)$ stays for all $t \in (s_1, s_2)$ in a cone with vertex $Z(s_1)$ and angle θ. If $\cos(\theta/2) > 1/\sqrt{n}$ then the same event has probability 0.

1. Introduction. Throughout this paper n will denote a fixed integer greater or equal to 2 and will be suppressed in the notation. For every $0 < \alpha < \pi/2$, we will consider the family $\mathscr{W}(\alpha)$ of all cones in the n-dimensional space which may be obtained by means of arbitrary translations and rotations of the following cone:

$$W_\alpha^* = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n\colon \tan^2(\alpha) \cdot x_1^2 > x_2^2 + \cdots + x_n^2\}.$$

The vertex of a cone $W \in \mathscr{W}(\alpha)$ will be denoted $v(W)$. We will write Z_t or $Z(t)$ to denote the standard n-dimensional Brownian motion. The main result of this note is contained in the following

THEOREM 1. *Consider the following event*:

$A(\alpha) = \{$*There exist times* s_1 *and* s_2, $0 \le s_1 < s_2$, *and a cone* $W \in \mathscr{W}(\alpha)$ *such that* $v(W) = Z(s_1)$ *and* $Z(t) \in W$ *for all* $t \in (s_1, s_2)\}$.

The probability of $A(\alpha)$ *is equal to*

(i) 0 *if* $\cos(\alpha) > 1/\sqrt{n}$,
(ii) 1 *if* $\cos(\alpha) < 1/\sqrt{n}$.

We will prove the above theorem in Sections 2 and 3. We will announce now a related result, the proof of which will be given elsewhere. Let $H \subset \mathbb{R}^n$ be a fixed $(n-1)$-dimensional hyperplane.

THEOREM 2. *Consider the event*

$B(\alpha) = \{$*There exist times* s_1 *and* s_2, $0 \le s_1 < s_2$, *and a cone* $W \in \mathscr{W}(\alpha)$ *such that* $W \cap H = \varnothing$, $v(W) = Z(s_1) \in H$ *and* $Z(t) \in W$ *for all* $t \in (s_1, s_2)\}$.

The probability of $B(\alpha)$ *is zero for all* $0 < \alpha < \pi/2$.

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[2] On leave from Mathematical Institute, Polish Academy of Sciences, Wrocław, Poland.
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In other words, no Brownian excursion from a hyperplane stays locally (near its starting point) in a cone. This is surprising in view of Theorem 1(ii), since the time t when an excursion from H starts may be thought of as exceptional in the following sense: Brownian motion has a considerable drift in the direction perpendicular to H just after t.

Theorem 2 has been proved in [1] in the case $n = 2$. The proof may be easily generalized to higher dimensions. The methods used in [1] may be also applied to excursions and sets other than cones.

2. Proof of Theorem 1(i). Our proof is based on ideas from [4] and [6] which are related to [5].

First we introduce some more notation. $|\cdot|$ will denote the Euclidean norm in the space $\mathbb{R}^n$. The cone which has the vertex z and which is the image of W_α^* by a suitable translation will be denoted $W_\alpha(z)$. The totality of such cones will be denoted $\mathscr{W}_1(\alpha)$. The distribution of the n-dimensional Brownian motion starting from z will be denoted by P^z and E^z will be the expectation corresponding to P^z. $P(A) = c$ means that c is the common value of $P^z(A)$ for all $z \in \mathbb{R}^n$. The vector $(0, 0, \cdots, 0) \in \mathbb{R}^n$ will be denoted 0. The boundary of the set $A \subset \mathbb{R}^n$ will be denoted as ∂A. The hitting time T_A of a set A is defined by $T_A = \inf\{t \geq 0: Z_t \in A\}$. The following lemma is derived from general results of Burkholder [2].

LEMMA 1. *If* $0 < \alpha < \beta < \pi/2$, $\cos(\alpha) > 1/\sqrt{n}$ *and* $\cos(\beta) < 1/\sqrt{n}$ *then there exists an* $\varepsilon = \varepsilon(\alpha, \beta, n) > 0$ *such that*

$$E^z(T_{\partial W_\alpha(0)})^{1+\varepsilon} < \infty \quad \text{and} \quad E^z(T_{\partial W_\beta(0)})^{1-\varepsilon} = \infty, \quad \text{for all} \quad z \in W_\alpha(0).$$

PROOF. Let F be the hypergeometric function,

$$F(a, b, c, t) = \textstyle\sum_{k=0}^{\infty} (a)_k(b)_k t^k/((c)_k k!),$$

where $(a)_0 = 1$, $(a)_1 = a$, $(a)_2 = a(a + 1)$, etc. Let $n \geq 3$ and

$$h(\theta) = F(-2, n, (n-1)/2, (1 - \cos\theta)/2).$$

A simplified formula for h is $h(\theta) = 1 - (\cos\theta + 1)(\cos\theta - 1)n/(n - 1)$ and it follows that the smallest zero of h in $(0, \pi)$ is equal to arc $\cos(1/\sqrt{n})$. Now we use Theorem 3.1 of Burkholder [2] and its special case, formula (3.10). It says that if $n \geq 3$ then $E^z(T_{\partial W_\alpha(0)}) < \infty$ iff $\cos(\alpha) > 1/\sqrt{n}$. The remarks following formula (3.10) [2] imply even more: if $\cos(\alpha) > 1/\sqrt{n}$ and $\cos(\beta) < 1/\sqrt{n}$ then there exists an $\varepsilon = \varepsilon(\alpha, \beta) > 0$ such that $E^z(T_{\partial W_\alpha(0)})^{1+\varepsilon} < \infty$ and $E^z(T_{\partial W_\beta(0)})^{1-\varepsilon} = \infty$. The case $n = 2$ is settled by formula (3.8) [2]. □

We need some more definitions and notation. If $a \in \mathbb{R}$ and $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$, then $z - a$ will denote $(z_1 - a, z_2, z_3, \cdots, z_n) \in \mathbb{R}^n$. If I is an interval, then

$A_1(\alpha, I) = \{$There exist $s \in I$, $s \geq 0$ and a cone $W \in \mathscr{W}_1(\alpha)$ such that $Z(s) = v(W)$ and $Z_t \in W$ for all $t \in (s, s + 2)\}$.

Let r_α denote the distance from $(1, 0, \cdots, 0)$ to $\partial W_\alpha(0) \cup \{|z| = 2\}$.

If $A_1(\alpha, [a, b])$ holds, $b - a < 1$, then, for geometric reasons, at least one of the following events holds:

$$A_2(\alpha, [a, b]) = \{\sup_{t\in[a,b]} | Z_t - Z_a | \geq r_\alpha (b - a)^{1/2-\varepsilon}\}$$

or

$$A_3(\alpha, [a, b]) = \{Z_t \in W_\alpha(Z_a - (b - a)^{1/2-\varepsilon}) \text{ for all } t \in (a, a + 1)\}.$$

$\varepsilon > 0$ above denotes a constant. By Brownian scaling, the probability of $A_2(\alpha, [a, b])$ is equal to the probability of

$$\{\sup_{t\in[0,1]} | Z_t - Z_0 | \geq r_\alpha (b - a)^{-\varepsilon}\} =_{\text{df}} A_4((b - a)^{-\varepsilon}).$$

It is easy to see that $P(A_4(u))$ is exponentially decreasing as a function of u when $u \to \infty$. In particular,

$$(1) \qquad k \cdot P(A_2(\alpha, [a, a + 1/k])) = k \cdot P(A_4(k^\varepsilon)) \to 0 \quad \text{when} \quad k \to \infty.$$

The same technique of scaling shows that

$$\begin{aligned} P(A_3(\alpha, [a, b])) &= P^z(T_{\partial W_\alpha(0)} > (b - a)^{-1+2\varepsilon}) \\ &= P^z((T_{\partial W_\alpha(0)})^{1/(1-2\varepsilon)} > 1/(b - a)), \end{aligned}$$

where $z = (1, 0, \cdots, 0)$. We choose now $\varepsilon = \varepsilon(\alpha) > 0$ so small that $E^z(T_{\partial W_\alpha(0)})^{1/(1-2\varepsilon)} < \infty$ (Lemma 1). With this choice of ε we have

$$(2) \qquad \begin{aligned} &k \cdot P(A_3(\alpha, [a, a + 1/k])) \\ &\qquad = k \cdot P^z((T_{\partial W_\alpha(0)})^{1/(1-2\varepsilon)} > k) \to 0 \quad \text{when} \quad k \to \infty. \end{aligned}$$

The event $A_1(\alpha, [0, m])$ is contained in the union of events

$$A_1(\alpha, [j/k, (j + 1)/k]), \quad j = 0, 1, \cdots, m \cdot k - 1.$$

We have also

$$A_1(\alpha, [0, m]) \subset \cup_{j=0}^{m\cdot k-1} (A_2(\alpha, [j/k, (j - 1)/k]) \cup A_3(\alpha, [j/k, (j + 1)/k]))$$

and

$$\begin{aligned} P(A_1(\alpha, [0, m])) \leq &\textstyle\sum_{j=0}^{m\cdot k-1} P(A_2(\alpha, [j/k, (j + 1)/k])) \\ &+ \textstyle\sum_{j=0}^{m\cdot k-1} P(A_3(\alpha, [j/k, (j + 1)/k])). \end{aligned}$$

Thus for every fixed m we have $P(A_1(\alpha, [0, m])) = 0$, by (1) and (2). It follows that $P(A_1(\alpha, [0, \infty))) = 0$. The scaling property of Brownian motion implies that the following event has probability 0:

$A_5(\alpha) = \{$There exist $0 \leq s_1 < s_2$ and a cone $W \in \mathscr{W}_1(\alpha)$ such that $Z(s_1) = v(W)$ and $Z_t \in W$ for all $t \in (s_1, s_2)\}$.

Let us fix now arbitrary $0 < \alpha < \beta < \pi/2$ with $\cos(\beta) > 1/\sqrt{n}$. For every $k = 0, 1, 2, \cdots$ we define $\mathscr{W}_1^k(\beta)$ in the same way as $\mathscr{W}_1(\beta)$ with the exception that W_β^* is replaced by a cone $W_\beta^k \in \mathscr{W}(\beta)$ which is chosen as follows. We find such a family of cones W^k_β, $k = 0, 1, \cdots, m = m(\alpha, \beta)$, that for every $W \in \mathscr{W}(\alpha)$ there exists $k \le m$ and a cone $W_1 \in \mathscr{W}^k_1(\beta)$ with $W \subset W_1$, $v(W) = v(W_1)$. Roughly speaking such choice of W^k_β's is possible, since the $(n-1)$-dimensional sphere is compact. Let now $A^k_5(\beta)$ have the same definition as $A_5(\beta)$ with $\mathscr{W}_1(\beta)$ replaced by $\mathscr{W}^k_1(\beta)$. For geometric reasons we have

$$A(\alpha) \subset \cup_{k=0}^m A^k_5(\beta).$$

Brownian motion is rotation invariant and we already know that $P(A_5(\beta)) = 0$, so $P(A^k_5(\beta)) = 0$. Thus $P(A(\alpha)) = 0$ and part (i) of Theorem 1 is proved.

3. Proof of Theorem 1(ii). We will follow Davis [4] very closely and we refer interested readers to [4] for details.

Let $k > 0$ be a fixed integer. We define a sequence of stopping times T_i, $i = 0, 1, 2, \cdots$.

$$T_0 = 0,$$

$$T_i = (T_{i-1} + 1) \wedge \inf\{t \ge T_{i-1} + 1/k: Z_t \notin W(Z(T_{i-1}))\} \quad i = 1, 2, \cdots .$$

Let U denote the random variable $\inf\{t \ge 1: Z_t \notin W_\alpha(0)\}$. We have by Brownian scaling

$$P(T_{i+1} - T_i = 1) = P^0(U \ge k) \tag{3}$$

and

$$E(T_{i+1} - T_i) = E^0(U \wedge k)/k. \tag{4}$$

Let $0 < \alpha < \pi/2$ be fixed and such that $\cos(\alpha) < 1/\sqrt{n}$. By Lemma 1 there exists $\varepsilon > 0$ such that $E^z(T_{\partial W_\alpha(0)})^{1-\varepsilon} = \infty$ for $z \in W_\alpha(0)$. The P^0-probability of $\{Z_1 \in W_\alpha(0)\}$ is greater than zero, so

$$E^0 U^{1-\varepsilon} = \infty. \tag{5}$$

Davis ([4], Lemma 2.1 and end of Section 2) has proved that (3), (4) and (5) imply that

$$P(T_{i+1} - T_i = 1 \text{ for some } i \le m \text{ such that } T_i \le 1) \ge c, \tag{6}$$

for infinitely many integers $k > 0$. The number $m = m(k)$ above is a suitable integer and $c > 0$ is a constant independent of k. Let

$$B_k(\alpha) = \{\text{There exists } 0 \le s < 1 \text{ such that } Z_t \in W(Z_s) \text{ for all } t \in (s + 1/k, s + 1)\}.$$

We see that $B_{k+1}(\alpha) \subset B_k(\alpha)$ and it follows from (6) that $P(\cap_k B_k(\alpha)) > c > 0$. We also have $\cap_k B_k(\alpha) \subset A(\alpha)$ and so $P(A(\alpha)) > 0$.

4. Remarks. i) Theorem 1 has been proved in the case of $n = 2$ in [1].

ii) Theorem 1(ii) would be obvious if we allowed for $\alpha = \pi/2$, and identified halfspace with a degenerate cone $W^*_{\pi/2}$. The following less trivial statement seems to be true.

CONJECTURE. Let $n = 2$ and let $\mathbb{R}^2$ be identified with the complex plane. For every $\pi/2 < \alpha < \infty$ the following event has probability 1:

{There exist $s \geq 0$ and $\varepsilon > 0$ such that for all $0 < \delta < \varepsilon$ the range of every continuous version of $t \to \arg(Z_t - Z_s)$, $t \in (s, s + \delta)$ is an interval of the length α}.

The analogous result is true for $n \geq 3$ and cones with the angle θ, $\pi \leq \theta < 2\pi$. □

A possible way to prove the conjecture would be to follow closely the proof of Theorem 13 of Greenwood and Perkins [6]. The necessary information concerning the tails of random variables, like $T_{\partial W_\alpha(0)}$ above, might be found in Chapter II of [1] ($n = 2$) and in Burkholder [3] ($n \geq 3$).

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We would like to thank the referee for finding an error in the original statement of the conjecture.

After submitting this paper for publication we have learned that Michio Shimura [8] proved independently the two-dimensional version of Theorem 1, extending the part (i) to the case $\cos(\alpha) = 1/\sqrt{2}$. His papers [7] and [8] contain many interesting results on Brownian excursions.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720