The Annals of Probability
1985, Vol. 13, No. 3, 1018–1021

A NOTE ON A LIMIT THEOREM FOR DIFFERENTIABLE MAPPINGS

BY K. A. BOROVKOV

Steklov Mathematical Institute

The main purpose of this paper is to draw attention to a simple and useful general "continuity theorem" type result, from which a great deal of limit theorems follow as almost immediate consequences. As an example, we give a new very short and transparent proof of the recent result by H. Teicher and C. Hagwood (A multidimensional CLT for maxima of normed sums); in fact, a much more general assertion is proved here. Another application of the main result establishes a correspondence between the convergence of empirical and quantile processes. (A similar result holds for the renewal and partial sums processes.)

In a recent paper [4], a multidimensional analog of the following assertion was proved. If $X_1, X_2, \cdots$ is a sequence of i.i.d. random variables, $EX_1 = \mu > 0$, $\sigma^2 = \mathrm{Var}(X_1) < \infty$, and $S_n = \sum_{k=1}^{n} X_k$, then for any $\alpha \in [0, 1)$

$$Y_n = n^{\alpha - 1/2}(\max_{k \le n} S_k k^{-\alpha} - \mu n^{1-\alpha}) \Rightarrow N_{0,\sigma^2} \quad \text{as} \quad n \to \infty$$

(here $\Rightarrow$ stands for weak convergence of distributions, N_{0,σ^2} is the normal distribution with parameters 0, σ^2). At first sight this result looks strange and even intriguing. However, after rewriting Y_n in the form

$$n^{-1/2}(\max_{k \le n}(S_k^0 + \mu k)(n/k)^\alpha - \mu n),$$

S_k^0 are sums of $X_k^0 = X_k - \mu$, it becomes clear that the maximum is attained on the value of k, which is very close to n and is equal approximately to S_n. (In fact, the only needed property of $\{X_n^0\}$ is that the weak invariance principle holds for this sequence, i.e., $S_n(t) = n^{-1/2} S_{[nt]}^0 \Rightarrow \sigma w(t)$ in Skorokhod space $D[0, 1]$, where $w(t)$ is the standard Wiener process.) Now it is easy to find a straightforward proof of Theorem 1 in [4], which is very short and transparent (unlike the original one). Similar proofs are valid not only for the case of norming factors $k^{-\alpha}$ (this restriction apparently is due to the method of proof in [4]), but also for a wide class of norming sequences. The best understanding of the idea of all these proofs can be achieved when using the following approach.

We formulate below a simple general assertion for random elements of normed spaces, from which the main result of [4] and many similar ones follow readily as consequences.

Let $\mathscr{X}, \mathscr{Y}$ be arbitrary normed spaces endowed with σ-fields of Borel sets, and let a separable subspace $\mathscr{S} \subset \mathscr{X}$. Suppose that a measurable mapping $F: \mathscr{X} \to \mathscr{Y}$ satisfies a Lipshitz condition in a neighbourhood of some point $x_0 \in \mathscr{X}$ and

Received May 1984; revised January 1985.
AMS 1980 *subject classifications.* Primary 60F17; secondary 60F05.
Key words and phrases. Weak convergence and mappings, weak invariance principle.

$F(x_0 + x)$, $x \in \mathscr{S}$, has a Gateaux differential at 0, i.e.,

(i) $\exists \varepsilon > 0, K < \infty$: $\| F(x_0 + h) - F(x_0) \| \le K \| h \|$ if $\| h \| < \varepsilon$;
(ii) $\forall h \in \mathscr{S}$ $\exists$ $\lim_{t\to 0} t^{-1}(F(x_0 + th) - F(x_0)) = U(h)$.

It is easy to see that these conditions will be met if F has Frechet derivative at x_0.

PROPOSITION. *Let $\{u_n, a_n\}_{n\ge 0}$ be a sequence of random elements $u_n \in \mathscr{X}$, $a_n \in R$, $a_n \ne 0$ a.s., such that $u_n \Rightarrow u_0$, $a_n \Rightarrow a_0$ as $n \to \infty$, and $P(u_0 \in \mathscr{S}, a_0 = 0) = 1$. Then $a_n^{-1}(F(x_0 + a_n u_n) - F(x_0)) \Rightarrow U(u_0)$.*

PROOF. Since $(u_n, a_n) \Rightarrow (u_0, a_0)$, it follows from a variant of the Skorokhod theorem (see [5]) that this sequence can be redefined on a common probability space so that $(u_n, a_n) \to (u_0, a_0)$ a.s. To complete the proof it suffices to note that

$$a_n^{-1}(F(x_0 + a_n u_n) - F(x_0)) = a_n^{-1}(F(x_0 + a_n u_0) - F(x_0)) + a_n^{-1}(F(x_0 + a_n u_n) - F(x_0 + a_n u_0)),$$

where the first term in the right-hand side converges a.s. to $U(u_0)$ and the norm of the second one for sufficiently large n does not exceed $2K \| u_n - u_0 \| \to 0$ a.s. (This Proposition can be easily derived, e.g., from [1, Theorem 5.5] as well.)

REMARK. A similar assertion (proved in an entirely different way) was used in [2, Section 1.8], where the convergence of differentiable functionals of empirical distribution functions was considered. Note also that instead of weak differentiability of F we could assume that, say, for some $\alpha \in (0, 1)$ this function belongs to Lip_α-class and that there exists $\lim_{t\to 0} t^{-\alpha}(F(x_0 + th) - F(x_0)) = U(h)$, or we could consider more general spaces, etc. But for our present purposes it is enough to use the Proposition in its present form.

We could have shown how Theorem 1 of [4] follows from the Proposition, but we would rather prove a more general assertion.

THEOREM 1. *Let $X_1, X_2, \cdots$ be i.i.d. random vectors in R^d with mean μ and covariance matrix σ^2, and let $y_i \in R^d$, $f_i, g_i \in C[0, 1]$, $g_i(0) = 0$, $i = 1, \cdots, \ell$. Suppose that for all i*

$$m_i = \max_{0\le t\le 1}(f_i(t) + (y_i, \mu)g_i(t)) > 0,$$

and let $\inf M_i > 0$, *where* $M_i = \{t: f_i(t) + (y_i, \mu)g_i(t) = m_i\}$ *and* $(\cdot, \cdot)$ *denotes an inner product. Then*

$$n^{1/2}\{\max_{k\le n}(f_i(k/n) + g_i(k/n)k^{-1}(y_i, S_k)) - m_i\}_{i=1}^{\ell} \Rightarrow \{\max_{t\in M_i} g_i(t)t^{-1}(y_i, \sigma w(t))\}_{i=1}^{\ell}, \quad n \to \infty,$$

where w is the standard d-dimensional Wiener process.

The main result of [4] is a particular case of Theorem 1 with $\ell = d$, $\{y_i\}$ being an orthonormal basis in R^d, $f_i \equiv 0$, $g_i(t) = t^{1-\alpha_i}$, $\alpha_i \in [0, 1)$, $\mu_i = (y_i, \mu) > 0$.

PROOF. Since the argument in the general case does not differ from that in the one-dimensional case, we consider here only the latter one ($d = \ell = y_1 = 1$).

First note that there exists an $\varepsilon > 0$ such that $P(\max_{\varepsilon n \le k \le n} A_{k,n} = \max_{k \le n} A_{k,n}) \to 1$, where

$$A_{k,n} = f(k/n) + g(k/n)k^{-1}S_k = f(k/n) + \mu g(k/n) + g(k/n)k^{-1}S_k^0.$$

Indeed,

$$\sup_{k \le n} g(k/n)k^{-1}S_k^0 \le \sup_{t \le \delta} g(t) \sup_{k \ge 1} k^{-1}S_k^0 + \sup_{t \le 1} g(t) \sup_{k \ge n\delta} k^{-1}S_k^0,$$

and the first term in the right-hand side can be made (for almost every elementary event ω) less than an arbitrary small number by the appropriate choice of $\delta = \delta(\omega)$, and the second term a.s. vanishes as $n \to \infty$ for any $\delta > D$ (we used here the fact that by the strong law of large numbers $\lim_{n\to\infty} \sup_{k \ge n} k^{-1}S_k^0 = 0$ a.s.). Hence $\sup_{k \le n} g(k/n)k^{-1}S_k^0 \to 0$ a.s., and our assertion follows from the continuity of $f + \mu g$ and the condition $\inf M > 0$.

Further,

$$n^{1/2}(\max_{\varepsilon n \le k \le n} A_{k,n} - m) = a_n^{-1}(F(x_0 + a_n u_n) - F(x_0)) + O(n^{-1/2}),$$

where

$$a_n = n^{-1/2}, \quad F(x) = \sup_{\varepsilon \le t \le 1} x(t), \quad x_0(t) = f(t) + \mu g(t),$$

and

$$u_n(t) = g(n^{-1}[nt])n[nt]^{-1}s_n(t) \Rightarrow u_0(t) = \sigma g(t)t^{-1}w(t)$$

in $\mathscr{X} = D[\varepsilon, 1]$ (with the uniform norm) by the invariance principle. Now a simple calculation shows that the Lipshitz function $F: \mathscr{X} \to \mathscr{Y} = R$ has at the point x_0 Gateaux differential $U(h) = \max_{t \in M} h(t)$, $h \in \mathscr{S} = C[\varepsilon, 1]$, and to complete the proof it remains to apply our Proposition.

As we mentioned above, assertions of this type proved to be useful in statistics, and here is another result in this field.

Let $\{X_1^{(n)}, \cdots, X_n^{(n)}\}$, $n \ge 1$, be a sequence of increasing samples of random variables ranging in $[0, 1]$. Suppose that

$$u_n(t) = n^{1/2}(G^{(n)}(t) - t) \Rightarrow u_0(t), \quad n \to \infty, \tag{1}$$

in the space $D[0, 1]$, where $G^{(n)}$ is the empirical distribution function for the nth sample and u_0 is some continuous process (this is the case, e.g., when $X_i^{(n)} = X_i$ and X_i form the sequence of independent uniformly distributed in $[0, 1]$ random variables; in this case u_0 is a Brownian bridge). Define the mapping F by

$$(F(x))(p) = 1 \wedge \inf\{t \ge 0: x(t) \ge p\}, \quad p \in [0, 1],$$

for $x \in D[0, 1]$; obviously, $(F(G^{(n)}))(p) = Q^{(n)}(p) \equiv X_{([np]+1)}^{(n)}$, where $X_{(k)}^{(n)}$ is the

kth order statistic in the nth sample. It is easy to see that $\| F(x_0 + h) - F(x_0) \| \le \| h \|$, where $x_0(t) \equiv t$, $\| x \| = \sup_{0 \le t \le 1} | x(t) |$, and that $F(x_0 + x)$, $x \in \mathscr{S} = C_0 = C[0, 1] \cap \{x\colon x(0) = x(1) = 0\}$, has the Gateaux differential $(U(h))(p) = -h(p)$ at $x = 0$. Now it follows from the Proposition that (1) implies convergence $v_n(p) = n^{1/2}(Q^{(n)}(p) - p) \Rightarrow -u_0(p)$; the same argument shows that the converse implication is true as well. (Note that in the general case $G^{(n)}$ is not the random element of $\mathscr{X} = D = [0, 1]$ with sup-norm, but this difficulty with measurability can be easily overcome, e.g., by smoothing $G^{(n)}$.) Therefore, we come to the following result, relating convergence of empirical and quantile processes (here $\Rightarrow$ means weak convergence in the space $D[0, 1]$ with Skorokhod metric).

THEOREM 2. *Assume that* $u_0 \in C_0$. *Then* $u_n \Rightarrow u_0$ *iff* $v_n \Rightarrow -u_0$.

If u_0 is a Brownian bridge, as in the standard situation, $u_0 = -u_0$ in distribution, and it is not difficult to show that in this case the difference between the rates of convergence of u_n and v_n to u_0 (in minimal metric, corresponding to the uniform distance in $D[0, 1]$) does not exceed $n^{-1/4}$ (up to a logarithmic factor).

Using a function F of the same type it is easy to establish a similar correspondence between the convergence of renewal processes and the convergence of partial sums processes in the invariance principle type theorems (both weak and strong). For other approaches to this probem see, e.g., [3] and [1, Theorem 17.3].

REFERENCES

[1] BILLINGSLEY, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[2] BOROVKOV, A. A. (1984). *Mathematical Statistics.* Nauka, Moscow. (In Russian.)

[3] BOROVKOV, K. A. (1982). On the rate of convergence in the invariance principle for generalized renewal processes. *Theory Probab. Appl.* **27** 461–471.

[4] HAGWOOD, C., and TEICHER, H. (1983). A multidimensional CLT for maxima of normed sums. *Ann. Probab.* **11** 1048–1050.

[5] WICHURA, M. J. (1970). On the construction of almost uniformly convergent r.v.'s with given weakly convergent image laws. *Ann. Math. Statist.* **41** 284–291.

STEKLOV MATHEMATICAL INSTITUTE
VAVILOVA 42
117966 MOSCOW GSP-1
USSR