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A CENTRAL LIMIT THEOREM FOR DIFFUSIONS WITH PERIODIC COEFFICIENTS[1]

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It is proved that if X_t is a diffusion generated by the operator $L = \frac{1}{2} \sum a_{ij}(x)\partial^2/\partial x_i \partial x_j + \sum u_0 b_i(x)\partial/\partial x_i$ having periodic coefficients, then $\lambda^{-1/2}(X_{\lambda t} - \lambda u_0 \bar{b} t)$, $t \geq 0$, converges in distribution to a Brownian motion as $\lambda \to \infty$. Here $\bar{b}$ is the mean of $b(x) = (b_1(x), \cdots, b_k(x))$ with respect to the invariant distribution for the diffusion induced on the torus $T^k = [0, 1)^k$. The dispersion matrix of the limiting Brownian motion is also computed. In case $\bar{b} = 0$ this result was obtained by Bensoussan, Lions and Papanicolaou (1978). (See Theorem 4.3, page 401, as well as the author's remarks on page 529.) The case $\bar{b} \neq 0$ is of interest in understanding how solute dispersion in a porous medium behaves as the liquid velocity increases in magnitude.

1. The limit theorem. Let $L = \frac{1}{2} \sum_{i,j=1}^k a_{ij}(x)\partial^2/\partial x_i \partial x_j + \sum_{i=1}^k u_0 b_i(x)\partial/\partial x_i$ be a differential operator ($k \geq 1$), whose coefficients satisfy the following assumptions.

Assumptions. (1) *For each x the* ($k \times k$) *matrix* $((a_{ij}(x)))$ *is symmetric and positive definite*; (2) *the functions* $a_{ij}(x)$, $b_i(x)$ *are real valued and periodic, i.e.,* $a_{ij}(x + \nu) = a_{ij}(x)$, $b_i(x + \nu) = b_i(x)$ *for all x and all vectors* ν *with integers as coordinates* ($1 \leq i, j \leq k$); (3) *the functions* $a_{ij}(x)$ *have bounded second order derivatives, and* $b_i(x)$ *have continuous first order derivatives*; (4) u_0 *is a real parameter.*

Let $(\Omega, \mathscr{A}, P^{\pi'})$ be a probability space on which are defined (1) a random vector X_0 with values in $\mathbb{R}^k$ and distribution π', and (2) a standard k-dimensional *Brownian motion* $\{B_t = (B_t^{(1)}, \cdots, B_t^{(k)}): t \geq 0\}$ which is independent of X_0. In case $\pi'(\{x\}) = 1$, $P^{\pi'}$ will also be denoted by P^x. $E^{\pi'}$ denotes *expectation* under $P^{\pi'}$.

Let $\{X_t: t \geq 0\}$ be the solution (continuous, nonanticipative) to Itô's stochastic integral equation

$$(1.1) \quad X_t = X_0 + \int_0^t u_0 b(X_s)\, ds + \int_0^t \sigma(X_s)\, dB_s, \quad (X_t = (X_t^{(1)}, \cdots, X_t^{(k)})),$$

where $\sigma(x)$ is the positive square root of $((a_{ij}(x)))$. The $P^{\pi'}$*-distribution* of $\{X_t: t \geq 0\}$ is a probability measure on (the Borel sigmafield of) the space $C([0, \cdots): \mathbb{R}^k)$ of continuous functions on $[0, \infty)$ into $\mathbb{R}^k$, endowed with the topology of uniform convergence on compact subsets of $[0, \infty)$. Note that

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$\{X_t: t \geq 0\}$ is a Markov process. Let $p(t, x, y)$ be the transition probability density (with respect to Lebesgue measure on $\mathbb{R}^k$) of this Markov process. Because of periodicity of the coefficients one has

$$(1.2) \qquad p(t; x, y) = p(t; x + \nu, y + \nu)$$

for every vector ν with integral coordinates.

We shall write

$$(1.3) \qquad \dot{x} = (x^{(1)}(\text{mod } 1), \cdots, x^{(k)}(\text{mod } 1)) \quad (x = (x^{(1)}, \cdots, x^{(k)}) \in \mathbb{R}^k),$$

$$(1.4) \qquad \dot{X}_t = X_t(\text{mod } 1) = (X_t^{(1)}(\text{mod } 1), \cdots, X_t^{(k)}(\text{mod } 1))$$

In view of (1.2), $\dot{X}_t$ is a Markov process on the state space $T^k = [0, 1)^k$ having the transition probability density function (with respect to Lebesgue measure on $[0, 1)^k$)

$$(1.5) \qquad \dot{p}(t; x, y) = \textstyle\sum_{\nu \in \mathbb{Z}^k} p(t; x, y + \nu), \quad (x, y \in [0, 1)^k).$$

Assumptions (1)–(3) imply, by the maximum principle (Friedman, 1975, Chapter 6),

$$(1.6) \qquad \inf_{x,y \in [0,1]^k} p(t; x, y) > 0, \quad (t > 0).$$

Therefore,

$$(1.7) \qquad \inf_{x,y \in [0,1)^k} \dot{p}(t; x, y) > 0, \quad (t > 0).$$

This implies Döblin's condition and irreducibility (Doob, 1953, Theorem 2.1, page 256; Bensoussan, Lions and Papanicolaou, 1978, Theorem 3.2, page 373), and the existence of a probability density $\pi(x)$ on $[0, 1)^k$ and positive constants c, β such that

$$(1.8) \qquad \int_{[0,1)^k} \dot{p}(t; x, y)\pi(x)\, dx = \pi(y) \quad \text{a.e.} \quad (dy) \quad \text{on} \quad [0, 1)^k$$

and

$$(1.9) \qquad \sup_{x \in [0,1)^k} \int_{[0,1)^k} |\dot{p}(t; x, y) - \pi(y)|\, dy \leq ce^{-\beta t} \quad (t > 0)$$

The following proposition is easy to prove.

PROPOSITION 1. *The P^x-distribution of $\{X_t - x: t \geq 0\}$ is the same as the $P^{\dot{x}}$-distribution of $\{X_t - \dot{x}: t \geq 0\}$.*

Next consider the discrete parameter stochastic process

$$(1.10) \qquad Y_m \doteq X_m - X_{m-1} \qquad (m = 1, 2, \cdots).$$

Denote by $\mathscr{F}_t$ the sigmafield generated by $\{X_s: 0 \leq s \leq t\}$. By the Markov property and Proposition 1, the conditional distribution of the stochastic process $\{X_{m+n-1} - X_{n-1}: m = 1, 2, \cdots\}$ given $\mathscr{F}_{n-1}$ is the same as the $P^{\dot{x}}$-distribution of $\{X_m - \dot{x}: m = 1, 2, \cdots\}$ with $x = X_{n-1}$. But $\{\dot{X}_m: m = 0, 1, 2, \cdots\}$ is a *stationary* *sequence* under P^{π}. Hence the (unconditional) P^{π}-distribution of $\{X_{m+n-1} - X_{n-1}: m = 1, 2, \cdots\}$ equals the P^{π}-distribution of $\{X_m - \dot{X}_0: m = 1, 2, \cdots\}$. In particular, the P^{π}-distribution of $\{Y_{m+n-1} = (X_{m+n-1} - X_{n-1}) - (X_{m-1+n-1} - X_{n-1}): m = 1, 2, \cdots\}$ is the same as the P^{π}-distribution of $\{Y_m = (X_m - \dot{X}_0) - (X_{m-1} - \dot{X}_0): m = 1, 2, \cdots\}$. This proves that $\{Y_m: m = 1, 2, \cdots\}$ is a *stationary sequence* under P^{π}.

Now let B be a Borel subset of $(\mathbb{R}^k)^{\mathbb{Z}^+}$, where $\mathbb{Z}^+ = \{1, 2, \cdots\}$. Then

$$\begin{aligned} &P^{\pi}(\{Y_{m+n+j}: j = 1, 2, \cdots\} \in B)/\mathscr{F}_m \\ (1.11) \qquad &= E^{\pi}(P^{\dot{X}_{m+n}}(\{Y_j: j = 1, 2, \cdots\} \in B)/\mathscr{F}_m) \\ &= E^{\pi}(f(\dot{X}_{m+n})/\mathscr{F}_m) = E^{\pi}(f(\dot{X}_{m+n})/\dot{X}_m), \end{aligned}$$

where $f(x) = P^x(\{Y_j: j = 1, 2, \cdots\} \in B)$. By (1.8), (1.9)

$$\begin{aligned} &|E(f(\dot{X}_{m+n})/\dot{X}_m) - E^{\pi}f(\dot{X}_{m+n})| \\ (1.12) \qquad &= \left| \int_{[0,1)^k} f(y)\dot{p}(n; \dot{X}_m, y)\, dy - \int_{[0,1)^k} f(y)\pi(y)\, dy \right| \\ &\le c\, \|f\|_{\infty} e^{-\beta n} \le c e^{-\beta n}. \end{aligned}$$

Combining (1.11), (1.12) and recalling the definition of ϕ-mixing (Billingsley, 1968, page 166) one arrives at the following result.

PROPOSITION 2. *Under P^{π} the sequence $\{Y_m: m = 1, 2, \cdots\}$ defined by* (1.10) *is stationary and ϕ-mixing, with a ϕ-mixing coefficient which decays to zero exponentially fast.*

Consider the real Hilbert space $L^2([0, 1)^k, \pi)$ with inner product and norm

$$(1.13) \qquad \langle f, g\rangle = \int_{[0,1)^k} f(y)g(y)\pi(y)\, dy, \quad \|f\| = (\langle f, f\rangle)^{1/2}.$$

Let $\{\dot{T}_t: t > 0\}$ be the strongly continuous semigroup of contractions on this space defined by

$$(1.14) \qquad (\dot{T}_t f)(x) = \int_{[0,1)^k} \dot{p}(t; x, y,)f(y)\, dy, \quad (x \in [0, 1)^k).$$

Let $\hat{A}$ be the infinitesimal generator of this semigroup on the domain $\mathscr{D}_{\hat{A}}$. Let $\mathscr{R}_{\hat{A}}$ be the range of $\hat{A}$. Then $\mathscr{R}_{\hat{A}} = 1^{\perp}$, the set of all functions f in $L^2([0, 1)^R, \pi)$ such that $\langle f, 1\rangle = 0$, and given any $f \in 1^{\perp}$ there exists a unique element g in $\mathscr{D}_{\hat{A}} \cap 1^{\perp}$ such that (Bhattacharya, 1982, Theorem 2.1 and Remark 2.3.1)

$$(1.15) \qquad \hat{A}_g = f, \quad g(x) = -\int_0^{\infty} (\dot{T}_t f)(x)\, dt.$$

We will denote this element by $\hat{A}_{1^{\perp}}^{-1} f$:

$$(1.16) \qquad g = \hat{A}_{1^{\perp}}^{-1} f.$$

Now write

$$(1.17)\qquad \begin{aligned} \bar{b}_i &= \langle b_i, 1\rangle, \quad \bar{b} = (\bar{b}_1, \cdots, \bar{b}_k),\\ \bar{a}_{ij} &= \langle a_{ij}, 1\rangle, \quad \bar{a} = ((\bar{a}_{ij})),\\ g_i &= \hat{A}_1^{-1}(b_i - \bar{b}_i), \quad (1 \le i \le k). \end{aligned}$$

Under Assumptions (1)–(4), g_i is (equivalent to) a twice continuously differentiable function, when extended to $\mathbb{R}^k$ periodically, and $\pi(y)$ is a continuously differentiable periodic function (Bensoussan et al., 1978, 386–401).

The main result of this article may be stated as follows.

THEOREM 3. *Under Assumptions* (1)–(4), *no matter what the initial distribution* π' *is, the stochastic process*

$$(1.18)\qquad \{Z_{t,\lambda} \doteq \lambda^{-1/2}(X_{\lambda t} - \lambda u_0 t\bar{b}) : t \ge 0\}$$

converges weakly, as $\lambda \to \infty$, *to a Brownian motion with zero drift and dispersion matrix* $D = ((D_{ij}))$ *given by*

$$(1.19)\qquad \begin{aligned} D_{ij} &= -u_0^2\langle b_i, g_j\rangle - u_0^2\langle b_j, g_i\rangle + \bar{a}_{ij}\\ &+ \int_{[0,1)^k} u_0\left\{g_i(y)\sum_{r=1}^k \frac{\partial}{\partial y_r}(a_{rj}(y)\pi(y)) + g_j(y)\sum_{r=1}^k \frac{\partial}{\partial y_r}(a_{ri}(y)\pi(y))\right\} dy. \end{aligned}$$

PROOF. First let the initial distributions be $\pi(x)\,dx$. Write

$$(1.20)\qquad \begin{aligned} S_n &= Y_1 + \cdots + Y_n - nu_0\bar{b} = \textstyle\sum_{m=1}^n (Y_m - u_0\bar{b}) = X_n - X_0 - nu_0\bar{b},\\ W_{t,n} &= \frac{S[nt]}{\sqrt{n}} + \frac{(t - ([nt]/n))Y_{[nt]+1}}{\sqrt{n}}, \end{aligned}$$

where $[nt]$ is the integer part of nt. Then, in view of Proposition 2, $\{W_{t,n} : t \ge 0\}$ converges in distribution to a Brownian motion with zero drift, as $n \to \infty$. (See Billingsley, 1968, Theorem 20.1, page 174, where the result is stated for $W'_{t,n} = S_{[nt]}/\sqrt{n}$. It is easy to check that $\max\{|W_{t,n} - W'_{t,n}| : 0 \le t \le T\} \to 0$ in probability for every $T > 0$.)

Now, fix a $T > 0$ arbitrarily and note that

$$(1.21)\qquad \begin{aligned} &\max_{0\le t<T}|W_{t,n} - Z_{t,n}|\\ &\le \frac{|X_0| + |u_0\bar{b}|}{\sqrt{n}} + \frac{1}{\sqrt{n}}\max_{1\le m\le[nT]}\max_{0\le t'\le 1}|X_{m+t'} - X_m - t'u_0\bar{b}|. \end{aligned}$$

The sequence $\max\{|X_{m+t'} - X_m - t'u_0\bar{b}| : 0 \le t' \le 1\}$ is stationary. Also, the exponential martingale inequality (Friedman, 1975, page 93) may be used to prove that the common distribution of this sequence has finite moments of all orders. Chebyshev's inequality may be used now to show that the last summand on the right side of (1.21) converges to zero in probability, as $n \to \infty$. Hence $\{Z_{t,n}: t \geq 0\}$ converges in distribution to a Brownian motion.

Now let the initial distribution be π', different from π. In view of (1.12) one has

$$\lim_{m\to\infty}\sup_B | P^{\pi'}(\{Y_{m+j}: j = 1, 2, \cdots\} \in B) - P^{\pi}(\{Y_{m+j}: j = 1, 2, \cdots\} \in B)| = 0, \tag{1.22}$$

where the supremum is taken over all Borel subsets B of $(\mathbb{R}^k)^{\mathbb{Z}^+}$.

Define

$$S_{m,n} = Y_{m+1} + \cdots + Y_{m+n} - nu_0\bar{b}, \qquad W_{t,m,n} = \frac{S_{m,[nt]}}{\sqrt{n}} + \frac{(t - ([nt]/n))Y_{m+[nt]+1}}{\sqrt{n}}. \tag{1.23}$$

It follows from (1.22) that the (variation) norm distance between the measures induced by $W_{t,n}$ under P^{π} and $W_{t,m,n}$ under $P^{\pi'}$ (on $C([0, \infty): \mathbb{R}^k)$) goes to zero as $m \to \infty$, *uniformly for all n*. Also for every positive integer m and every $T > 0$, whatever the initial distribution π',

$$\begin{aligned}\max_{0\leq t\leq T} | W_{t,m,n} - W_{t,n} | &\leq \frac{1}{\sqrt{n}} \max_{1\leq r\leq [nT]} | Y_r + Y_{r+1} + \cdots + Y_{r+m-1} | \\ &\quad + \frac{m\,|\bar{b}|\,u_0}{\sqrt{n}} \to 0 \quad \text{in probability as} \quad n \to \infty,\end{aligned} \tag{1.24}$$

by the same type of moment estimates as used for (1.21). If ψ is a real bounded continuous function on $C([0, \infty): \mathbb{R}^k)$, then by (1.24),

$$\begin{aligned}\limsup_{n\to\infty} | E^{\pi'}\psi(W_{t,n}) - E^{\pi}\psi(W_{t,n}) | &\leq \limsup_{n\to\infty} | E^{\pi'}\psi(W_{t,n}) - E^{\pi'}\psi(W_{t,m,n}) | \\ &\quad + \limsup_{n\to\infty} | E^{\pi'}\psi(W_{t,m,n}) - E^{\pi}\psi(W_{t,n}) | \\ &= \limsup_{n\to\infty} | E^{\pi'}\psi(W_{t,m,n}) - E^{\pi}\psi(W_{t,n}) |.\end{aligned} \tag{1.25}$$

But the last expression in (1.25) goes to zero as $m \to \infty$. The proof of convergence to a Brownian motion is completed by observing that $\max\{| Z_{t,\lambda} - Z_{t,[\lambda]} |: 0 \leq t \leq T\}$ goes to zero for every sample point, as $\lambda \to \infty$.

It remains to compute D_{ij}. Let us first show that $\{| Z_{1,\lambda} |^2: \lambda \geq 1\}$ is uniformly integrable with respect to P^{π}. One has

$$\begin{aligned}E^{\pi} | Z_{1,\lambda}^{(i)} |^4 &= \lambda^{-2}E^{\pi}\left(u_0 \int_0^{\lambda} b_i(X_s)\, ds + \int_0^{\lambda} \textstyle\sum_{r=1}^k \sigma_{ir}(X_s)\, dB_s^{(r)} + X_0^{(i)} - u_0\lambda\bar{b}_i\right)^4 \\ &\leq \lambda^{-2}3^3\left\{E^{\pi}(X_0^{(i)})^4 + u_0^4E^{\pi}\left(\int_0^{\lambda} (b_i(X_s) - \bar{b}_i)\, ds\right)^4 \right.\\ &\qquad \left. + E^{\pi}\left(\int_0^{\lambda} \textstyle\sum_{r=1}^k \sigma_{ir}(X_s)\, dB_s^{(r)}\right)^4\right\}.\end{aligned} \tag{1.26}$$

It is known that (McKean, 1969, page 40)

$$E^{\pi}\left(\int_0^{\lambda} \textstyle\sum_{r=1}^{k} \sigma_{ir}(X_s)\, dB_s^{(r)}\right)^4 \le c_1\lambda^2 \tag{1.27}$$

for some c_1 which does not depend on λ. Also, writing $f(x) = b_i(x) - \bar{b}_i$, one has

$$\begin{aligned} E^{\pi}\left(\int_0^{\lambda} (b_i(X_s) - \bar{b}_i)\, ds\right)^4 &= E^{\pi}\left(\int_0^{\lambda} f(X_s)\, ds\right)^4 \\ &= 24\int_0^{\lambda}\int_{s_1}^{\lambda}\int_{s_2}^{\lambda}\int_{s_3}^{\lambda} E^{\pi} f(\dot{X}_{s_1})f(\dot{X}_{s_2})f(\dot{X}_{s_3})f(\dot{X}_{s_4})\, ds_4\, ds_3\, ds_2\, ds_1 \\ &= 24\int_0^{\lambda}\int_{s_1}^{\lambda}\int_{s_2}^{\lambda}\int_{s_3}^{\lambda} \langle f, \dot{T}_{s_2-s_1} f(\dot{T}_{s_3-s_2}(f(\dot{T}_{s_4-s_3}f)))\rangle\, ds_4\, ds_3\, ds_2\, ds_4 \\ &\le 24\int_0^{\lambda}\int_{s_1}^{\lambda}\int_{s_2}^{\lambda}\int_{s_3}^{\lambda} 2c^2 \|f\| \cdot \|f\|_{\infty}^3 e^{-\beta(s_2-s_1)} e^{-\beta(s_4-s_3)}\, ds_4\, ds_3\, ds_2\, ds_1. \end{aligned} \tag{1.28}$$

The last inequality follows from (1.12) (with n replaced by s):

$$\begin{aligned} \|\dot{T}_s f\|_{\infty} &= \sup_x |\dot{T}_s f(x)| \le c\|f\|_{\infty} e^{-\beta s}, \\ |\langle f, \dot{T}_s g\rangle| &= |\langle f, \dot{T}_s(g - E^{\pi}g)\rangle| \le \|f\|\ \|\dot{T}_s(g - E^{\pi}g)\| \\ &\le c\|f\| \cdot 2\|g\|_{\infty} e^{-\beta s}, \end{aligned} \tag{1.29}$$

applied first to $g = f(\dot{T}_{s_3-s_2}(f(\dot{T}_{s_4-s_3})))$ and $s = s_2 - s_1$, and then to $g = f$ and $s = s_4 - s_3$. A straightforward evaluation of the last multiple integral in (1.28) yields

$$E^{\pi}\left(\int_0^{\lambda} (b_i(X_s) - \bar{b}_i)\, ds\right)^4 \le c_2\lambda^2, \tag{1.30}$$

where c_2 does not depend on λ. Using (1.27), (1.30) in (1.26) one gets the desired uniform integrability. It now follows that

$$\begin{aligned} D_{ij} &= \lim_{\lambda\to\infty} E^{\pi} Z_{1,\lambda}^{(i)} Z_{1,\lambda}^{(j)} \\ &= \lim_{\lambda\to\infty} \frac{1}{\lambda}\left[u_0^2 E^{\pi}\left(\int_0^{\lambda} (b_i(\dot{X}_s) - \bar{b}_i)\, ds \cdot \int_0^{\lambda} (b_j(\dot{X}_s) - \bar{b}_j)\, ds\right)\right. \\ &\quad + u_0 E^{\pi}\left(\int_0^{\lambda} (b_i(X_s) - \bar{b}_i)\, ds \cdot \int_0^{\lambda} \textstyle\sum_{r=1}^{k} \sigma_{jr}(X_s)\, dB_s^{(r)}\right) \\ &\quad + u_0 E^{\pi}\left(\int_0^{\lambda} (b_j(X_s) - \bar{b}_j)\, ds \cdot \int_0^{\lambda} \textstyle\sum_{r=1}^{k} \sigma_{ir}(X_s)\, dB_s^{(r)}\right) \\ &\quad \left. + E^{\pi}\left(\int_0^{\lambda} \textstyle\sum_{r=1}^{k} \sigma_{ir}(X_s)\, dB_s^{(r)} \cdot \int_0^{\lambda} \textstyle\sum_{r=1}^{k} \sigma_{jr}(X_s)\, dB_s^{(r)}\right)\right] \end{aligned} \tag{1.31}$$

By Theorem 2.1 and Remark 2.3.1 in Bhattacharya (1982) and relation (1.30) one has

$$\begin{aligned}
&\lim_{\lambda\to\infty}\frac{u_0^2}{\lambda}E^{\pi}\left(\int_0^{\lambda}(b_i(\dot{X}_s)-\bar{b})\,ds\cdot\int_0^{\lambda}(b_j(\dot{X}_s)-\bar{b}_j)\,ds\right)\\
&\quad= u_0^2(\langle -b_i-\bar{b}_i, g_j\rangle+\langle -b_j-\bar{b}_j, g_i\rangle) = -u_0^2(\langle b_i, g_i\rangle+\langle b_j, g_i\rangle).
\end{aligned}\tag{1.32}$$

Also, from a standard result in stochastic integrals (Friedman, 1975, Chapter 4),

$$\begin{aligned}
&E^{\pi}\left(\int_0^{\lambda}\textstyle\sum_{r=1}^{k}\sigma_{ir}(X_s)\,dB_s^{(r)}\cdot\int_0^{\lambda}\sum_{r=1}^{k}\sigma_{jr}(X_s)\,dB_s^{(r)}\right)\\
&\quad=\int_0^{\lambda}E^{\pi}(\textstyle\sum_{r=1}^{k}\sigma_{ir}(\dot{X}_s)\sigma_{jr}(\dot{X}_s))\,ds=\int_0^{\lambda}\bar{a}_{ij}\,ds=\lambda\bar{a}_{ij}.
\end{aligned}\tag{1.33}$$

It remains to estimate the second and third terms in (1.31). For this estimation we make use of the definition of stochastic integrals as limits (at least in L^2 w.r.t. the product measure $P^{\pi}\times ds$ on $\Omega\times[0,\lambda]$). One has

$$\begin{aligned}
&\textstyle\sum_{r=1}^{k}E^{\pi}\left(\int_0^{\lambda}(b_i(X_s)-\bar{b}_i)\,ds\cdot\int_0^{\lambda}\sigma_{jr}(X_s)\,dB_s^{(r)}\right)\\
&\quad=\textstyle\sum_{r=1}^{k}E^{\pi}\left(\int_0^{\lambda}\left((b_i(X_t)-\bar{b}_i)\int_0^{t}\sigma_{jr}(X_s)\,dB_s^{(r)}\right)dt\right),
\end{aligned}\tag{1.34}$$

using the orthogonality of $b_i(X_t)-\bar{b}_i$ and the stochastic integral over $[t,\lambda]$. Fix $\varepsilon>0$, sufficiently small. Then

$$\begin{aligned}
&E^{\pi}\left(\int_0^{\lambda}\left((b_i(X_t)-\bar{b}_t)\left(\int_{(t-\varepsilon)\vee 0}^{t}\sigma_{jr}(X_s)\,dB_s^{(r)}\right)dt\right)\right)^2\\
&\quad\le E^{\pi}\left(2\sup\{|b_i(x)|: x\in\mathbb{R}^k\}\cdot\left(\int_0^{\lambda}\left|\int_{(t-\varepsilon)\vee 0}^{t}\sigma_{jr}(X_s)\,dB_s^{(r)}\right|dt\right)\right)^2\\
&\quad\le c_3E^{\pi}\left(\int_0^{\lambda}\left|\int_{(t-\varepsilon)\vee 0}^{t}\sigma_{jr}(X_s)\,dB_s^{(r)}\right|dt\right)^2\\
&\quad\le c_3\lambda E^{\pi}\left(\int_0^{\lambda}\left(\int_{(t-\varepsilon)\vee 0}^{t}\sigma_{jr}(X_s)\,dB_s^{(r)}\right)^2dt\right)\\
&\quad= c_3\lambda\int_0^{\lambda}\left(\int_{(t-\varepsilon)\vee 0}^{t}(E^{\pi}\sigma_{jk}^2(X_s))\,ds\right)dt\le c_4\lambda^2\varepsilon
\end{aligned}\tag{1.35}$$

for appropriate positive constants c_2, c_4. It is, therefore, enough to evaluate the last expression in (1.34) with t replaced by $t-\varepsilon$. Now, writing $E^xf(X_t)=T_tf(x)$, one has

$$\sum_{r=1}^{k} E^{\pi}\left(\int_0^{\lambda}\left((b_i(X_t) - \bar{b}_i)\int_0^{t-\varepsilon}\sigma_{jr}(X_s)\, dB_s^{(r)}\right) dt\right)$$

$$= \sum_{r=1}^{k} \lim_{h\downarrow 0} E^{\pi}\left(\int_0^{\lambda} (b_i(X_t) - \bar{b}_i)\{\sum_{m=0}^{[(t-\varepsilon)/h]} \sigma_{jr}(X_{mh})\right.$$

$$\left. (B_{(m+1)h}^{(r)} - B_{mh}^{(r)})\}\, dt\right) \tag{1.36}$$

$$= \sum_{r=1}^{k} \lim_{h\downarrow 0} E^{\pi}\left(\int_0^{\lambda} \{\sum_{m=0}^{[(t-\varepsilon)/h]} T_{t-(m+1)h}(b_i - \bar{b}_i)(X_{(m+1)h})\right.$$

$$\left. \cdot\, \sigma_{jr}(X_{mh})(B_{(m+1)h}^{(r)} - B_{mh}^{(r)})\}\, dt\right).$$

By Itô's lemma (Friedman, 1975, page 90) one has

$$T_{t(m+1)h}(b_i - \bar{b}_i)(X_{(m+1)h})$$

$$= T_{t-(m+1)h}(b_i - \bar{b}_i)(X_{mh})$$

$$+ \int_{mh}^{(m+1)h} LT_{t-(m+1)h}(b_i - \bar{b}_i)(X_s)\, ds \tag{1.37}$$

$$+ \int_{mh}^{(m+1)h} \operatorname{grad} T_{t-(m+1)h}(b_i - \bar{b}_i)(X_s)\sigma(X_s)\, dB_s.$$

Now $(t', x) \to LT_{t'}(b_i - \bar{b}_i)(x)$ is bounded on $[\varepsilon/2, \lambda] \times \mathbb{R}^k$, since this function is continuous on $[\varepsilon/2, \lambda] \times \mathbb{R}^k$ (Friedman, 1975, Chapter 6) and periodic in x. Hence, for $h < \varepsilon/2$, the first integral in (1.37) is bounded above by $c_5 h$. Also,

$$x \to \operatorname{grad} T_{t'}(b_i - \bar{b}_i)(x)$$

is differentiable with a derivative which is continuous on $[\varepsilon/2, \lambda] \times \mathbb{R}^k$. Since this derivative is also periodic, the second integrand in (1.37) differs from

$$\operatorname{grad} T_{t-(m+1)h}(b_i - \bar{b}_i)(X_{mh})\sigma(X_{mh})$$

by a quantity smaller than $c_6 h$. Therefore, the second integral differs from

$$\operatorname{grad} T_{t-(m+1)h}(b_i - \bar{b}_i)(X_{mh})\sigma(X_{mh})(B_{(m+1)h} - B_{mh})$$

by a quantity whose square has expectation less than $c_7 h^2$. In view of this it is easy to check that (1.36) equals

$$\lim_{h\downarrow 0} \textstyle\sum_{r=1}^{k} \displaystyle\int_0^{\lambda} E^{\pi}\Big\{\textstyle\sum_{m=0}^{[(t-\varepsilon)/h]} \Big[\textstyle\sum_{r'=1}^{k} \Big(\dfrac{\partial}{\partial x_{r'}} T_{t-(m+1)h}(b_i - \bar{b}_i)\Big)(X_{mh})$$

$$\times \Big(\textstyle\sum_{r''=1}^{k} \sigma_{r'r''}(X_{mh})(B^{(r'')}_{(m+1)h} - B^{(r'')}_{mh})\Big)\Big]\sigma_{jr}(X_{mh})(B^{(r)}_{(m+1)h} - B^{(r)}_{mh})\Big\}\, dt$$

$$= \lim_{h\downarrow 0} \textstyle\sum_{r=1}^{k} \displaystyle\int_0^{\lambda} E^{\pi}\Big\{\textstyle\sum_{m=0}^{[(t-\varepsilon)/h]} \Big[\textstyle\sum_{r'=1}^{k} \Big(\dfrac{\partial}{\partial x_{r'}} T_{t-(m+1)h}(b_i - \bar{b}_i)\Big)(X_{mh})$$

$$\times \sigma_{r'r}(X_{mh})\sigma_{jr}(X_{mh})(B^{(r)}_{(m+1)h} - B^{(r)}_{mh})^2\Big]\Big\}\, dt$$

$$(1.38)\quad = \int_0^{\lambda}\int_0^{t-\varepsilon} \Big(\textstyle\sum_{r'=1}^{k} E^{\pi}\Big\{\Big(\dfrac{\partial}{\partial x_{r'}} T_{t-s}(b_i - \bar{b})\Big)(X_s)a_{jr'}(X_s)\Big\}\Big)\, ds\, dt$$

$$= \int_0^{\lambda}\int_0^{t-\varepsilon} \Big[\textstyle\sum_{r'=1}^{k} \displaystyle\int_{[0,1)^k} \Big(\dfrac{\partial}{\partial x_{r'}} T_{t-s}(b_i - \bar{b}_i)\Big)(x)a_{jr'}(x)\pi(x)\, dx\Big]\, ds\, dt$$

$$= -\int_0^{\lambda}\int_0^{t-\varepsilon} \Big[\textstyle\sum_{r'=1}^{k} \displaystyle\int_{[0,1)^k} T_{t-s}(b_i - \bar{b}_i)(x)\, \dfrac{\partial}{\partial x_{r'}}(a_{jr'}(x)\pi(x))\, dx\Big]\, ds\, dt$$

$$= -\int_0^{\lambda}\Big\{\int_{[0,1)^k}\Big(\int_0^{t-\varepsilon} T_{t-s}(b_i - \bar{b}_i)(x)\, ds\Big)\Big(\textstyle\sum_{r'=1}^{k} \dfrac{\partial}{\partial x_{r'}}(a_{jr'}(x)\pi(x))\Big)\, dx\Big\}\, dt.$$

Since

$$(1.39)\quad \int_0^{t-\varepsilon} T_{t-s}(b_i - \bar{b}_i)(x)\, ds = \int_{\varepsilon}^{t} T_s(b_i - \bar{b}_i)(x)\, ds = \int_{\varepsilon}^{t} \dot{T}_s(b_i - \bar{b}_i)(x)\, ds \to -g_i(x) - \int_0^{\varepsilon} \dot{T}_s(b_i - \bar{b}_i)(x)\, ds,$$

uniformly in $x \in [0, 1)^k$ as $t \to \infty$, (1.38), (1.35), (1.34) yield

$$(1.40)\quad \lim_{\lambda\to\infty} \frac{1}{\lambda} \textstyle\sum_{r=1}^{k} E^{\pi}\Big(\displaystyle\int_0^{\lambda} (b_i(X_s) - \bar{b}_i)\, ds \cdot \int_0^{\lambda} \sigma_{jr}(X_s)\, dB^{(r)}_s\Big) = \int_{[0,1)^k} g_i(x)\Big(\textstyle\sum_{r'=1}^{k} \dfrac{\partial}{\partial x_{r'}}(a_{jr'}(x)\pi(x))\Big)\, dx.$$

Using (1.32), (1.33), and (1.40) in (1.31) one obtains the desired result (1.19). Q.E.D.

Extensions. I. The functional central limit theorem proved above holds if Assumption (3) is replaced by (3′): *$a_{ij}(x)$ are continuous and $b_i(x)$ are Borel* *measurable and bounded.* The proof of convergence to a Brownian motion may be carried out as above. An alternative proof in this case may also be given by the renewal method (see Bhattacharya and Ramasubramanian, 1982). In computing D_{ij} all the steps can be justified, except (1.40). This is the main reason why the smoothness assumption (3) was made.

II. Let B_ν be an open ball of radius r with center at the (lattice) point ν, where $0 < r < \frac{1}{2}$. Let $B = UB_\nu$, the union being over all ν in $\mathbb{Z}^k$. Consider the diffusion on $\mathbb{R}^k \backslash B$ whose transition probability density function $p(t;\, x, y)$ satisfies the equation $\partial p/\partial t = Lp$ in the interior and a Neumann boundary condition on ∂B (e.g., vanishing of the conormal derivative $\sum_{i,j=1}^k (x_i - \nu_i)a_{ij}(x)\partial p/\partial x_j$ on ∂B). If $\{X_t\colon t \geq 0\}$ is this diffusion, then $\{\dot{X}_t\colon t \geq 0\}$ is a diffusion on $T^k \backslash \dot{B}$, where $\dot{B}$ is the image of B under the map $x \to \dot{x}$. The diffusion on the torus is ergodic and its transition probability density $p(t;\, x, y)$ is bounded away from zero (for each $t > 0$). Therefore, Propositions 1, 2 carry over to this case, and as a consequence so does the first part of the proof of Theorem 3. Hence $\{\lambda^{-1/2}(X_{\lambda t} - \lambda u_0 t\bar{b})\colon t \geq 0\}$ converges to a Brownian motion with zero drift, as $\lambda \to \infty$. Here $\bar{b}_i = \int_{T^k \backslash \dot{B}} b_i(x)\pi(x)dx$, $\pi(x)\, dx$ being the invariant probability for $\dot{p}$.

2. Concluding remarks. Suppose the diffusion matrix is αI, where α is a positive constant and I is the $k \times k$ identity matrix. Suppose also that $\operatorname{div} b(x) = 0$ for all x. In this case $\pi(x) \equiv 1$, i.e., the invariant distribution is the uniform distribution on the torus. In various examples of this type numerical computation of the diagonal elements D_{ii} of the dispersion matrix D, using (1.19), shows that D_{ii} increase with u_0; at first approximately quadratically, and then at higher values possibly at a linear rate. These computations as well as their significance in modelling solute transport in porous media will appear in Bhattacharya and Gupta (1984). However, D can be explicitly computed for the case $k = 1$ for general periodic functions $b(x)$ and $a(x) > 0$; this computation shows that *D goes to zero* as $u_0 \to \infty$. This is not really a great surprise. For, in the one dimensional case, as u_0 increases $\dot{X}_t$ winds around the same path (the circle) faster; the fluctuations become less important and $X_\lambda - u_0\lambda\bar{b}$ is close to zero for large λ. In two or higher dimensions this does not happen unless the coordinates are separated. Detailed computations will appear in the article mentioned above.

Note also that in case $((a_{ij}(x))) = \alpha I$, and $\operatorname{div} b(x) = 0$ for all x, the last two terms in (1.19) vanish. In particular, one has

$$D_{ii} = -2u_0^2\langle b_i, g_i\rangle + \alpha. \tag{2.1}$$

In problems of interest in solute dispersion the first term dominates (and goes to infinity as u_0 goes to infinity).

Check also that in this model the first term in (2.1) remains the same if the period is taken to be u_0, while the factor u_0 in the drift is taken to be 1. Thus asymptotic steady increase in dispersion with respect to the magnitude of the mean (liquid) velocity is equivalent to its asymptotic steady increase with respect to the spatial scale of heterogeneity. This is the so-called *scale effect* which has also been observed repeatedly in hydrological experiments (Molinary et al., 1977).

Of course, in the context of solute dispersion in porous media, the assumption of a periodic liquid velocity field is much too idealized. Unfortunately, this seems to be the only broad class of drift functions with nonzero mean (or large scale average of some sort) for which the central limit theorem has been proved. There is a novel central limit theorem type result due to Papanicolaou and Varadhan (1979) for the case of an L in divergence form:

$$L = \tfrac{1}{2} \sum_{i=1}^{k} (\partial/\partial x_i)(\sum_{j=1}^{k} a_{ij}(x)\partial/\partial x_j),$$

with the functions $a_{ij}(x)$ *almost periodic* in the sense of Bohr. But the divergence theorem shows that in this case the large scale volume average of the drift is zero, which (in the context of solute transport) says that the higher scale velocity of the liquid is zero. This makes the result inapplicable to the present context. An appropriate extension of the result together with a perturbation expansion of the dispersion coefficients in a parameter like u_0 would be of much interest.

An entirely different type of model has been considered by Gelhar and Axness (1983) and independently by Winter, Newman and Neuman (1983). The results of Winter et al. (1983) are somewhat more general. In their model they take $((a_{ij}(x))) = I$, and the drift as $\mu + \varepsilon U(x)$, where μ is a constant (mean) vector, ε is a small parameter and $U(x)$ is a *mean zero stationary ergodic random field* (indexed by the spatial parameter x). Assuming that the central limit theorem does hold, Winter et al. (1983) obtain a perturbation expansion of the dispersion matrix of the limiting Gaussian distribution (or Brownian motion). It would be important to prove such a central limit theorem. For the case $\mu = 0$, div $b = 0$, Papanicolaou and Pironeau (1981) have proved that $\{\varepsilon X_{t/\varepsilon^2} : t \geq 0\}$ converges to a Brownian motion as $\varepsilon \downarrow 0$, and have computed the dispersion matrix of the limiting Brownian motion. The case of nonzero mean velocity, however, is the one of importance for solute dispersion in porous media, and this case remains open.

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